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Chapter 1

Introduction

1.1 Natural Units & Conversion Factors

In Quantum Mechanics we know that $E = hf = \hbar\omega$ and $p = \frac{h}{\lambda} = \hbar k$ travelling in quantum waves of the form $\psi \sim e^{\frac{i}{\hbar}(kx - t\omega)}$ or $\psi \sim e^{\frac{i}{\hbar}(px - Et)}$ & for light $c = f\lambda = \frac{\omega}{k}$.

For Natural Units set $\hbar = c = 1$ using this consider the units of c & $\hbar$:

$$\begin{aligned} c\hbar &= [L][T]^{-1}[E][T] \\ &= [L][E] \end{aligned}$$

$$\begin{aligned} \hbar &= 6.6 \times 10^{-25} \text{ GeV s} \\ \Rightarrow \frac{1}{\text{GeV}} &= 6.6 \times 10^{-25} \text{ s} \\ c &= 3 \times 10^{10} \text{ cm s}^{-1} \\ \Rightarrow c\hbar &= 1.97 \times 10^{-14} \text{ GeV cm} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{1}{\text{GeV}^{-2}} &= 3.89 \times 10^{-25} \text{ cm}^2 \\ &= 0.398 \text{ mb} \end{aligned}$$

Chapter 2

Experiments of the last 50 years

Until the 1950s particle physics was studied by observing cosmic rays in cloud chambers and nuclear emulsion. After 1950 nucleon scattering experiments were carried out at cyclotrons and energies became high enough to produce pions which led to pion-nucleon scattering experiments.

In 1952 $\pi^+ + p \to \Delta^{++} \to \pi^+ + p$ gave rise to theories of colour. Also from electron beams, photons could be used $\gamma + p \to \Delta^{++} \to \gamma + p$. Although this has a much lower rate due to the strength of the EM coupling compared to the strong coupling.

Other processes were also observed:

$$\begin{aligned}
&\pi^+ \to \mu^+ + \nu \\
&\mu^+ \to e^+ + \overline{\nu}_\mu + \nu_e
\end{aligned}$$

where the purely leptonic former process was of great interest.

In 1956 parity violation in the weak interaction was discovered (Lee and Yang received the Nobel prize). The experiment was the Beta decay of polarized cobalt 60. An asymmetry was discovered in the electron spectrum with respect to ^{60}Co nuclear spin.

key $\Uparrow$: spin $\uparrow$: movement

$$\begin{array}{llll}
J = 5 & J = 4 & J = \frac{1}{2} & J = \frac{1}{2} \\
\Uparrow & \to \Uparrow & + \Uparrow\uparrow & + \Uparrow\downarrow \\
{}^{60}Co & \to {}^{60}Ni* & + \overline{\nu}_{e,R} & + e^-_L
\end{array}$$

By the 1960s kaon beams were made at cyclotrons and this confirmed the discovery strangeness made in cosmic ray experiments, establishing the quark substructure of hadrons i.e. hadrons are made of $q\overline{q}$ pairs (mesons) or as qqq systems (baryons) where q is a quark. Theorist regard the evidence for strangeness as establishing the SU(3) flavour symmetry which we now know to be accidental. SU(3) was thought to work because of the u d and s quarks it is in fact because of the colour carried by them and the gluons (R G or B).

2.1 Neutrino Experiments

2.1.1 Gargamelle Experiment

Using the CERN proton synchrotron protons were extracted from the accelerator and impinged upon a think Beryllium target within a 'neutrino horn'. In the target, π and K were created and the horn partially selected either positive or negative charges so the partially focused π^+ decayed to μ^+ and ν_μ. An iron shield filtered out the remaining hadrons and muons, measurement of the muons enabled the neutrino spectrum to be determined. Then the remainder was a neutrino beam. The neutrinos passed in to a large heavy liquid bubble chamber, gargamelle. It was expected that charged current interactions would be seen (2.1) but neutral current interactions mediated by z^0 were also seen (2.2). The ν_μs are invisible to detectors so only the hadrons are visible in the neutral interaction and the hadrons and muon in the charged.

This was the discovery of the neutral current in 1974

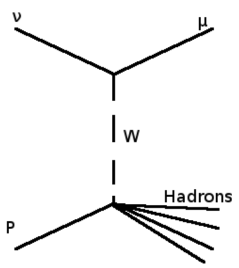

Figure 2.1: Charged Current t-channel scattering $\nu_\mu + p \rightarrow \mu^+ + \text{Hadrons}$

2.1.2 Underground Experiments

Solar neutrinos are produced primarily by:

$$P + P \rightarrow d + e^+ + \nu_e$$

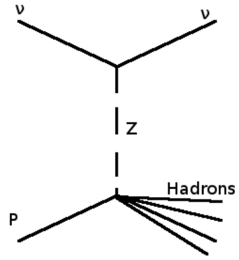

Figure 2.2: Charged Current t-channel scattering $\nu_\mu + p \to \nu_\mu +$ Hadrons

Atmospheric neutrinos are produced primarily by proton bombardment of the atmosphere:

$$P + N \to \pi^+(K^+) + H$$
$$\pi^+ \to \mu^+ + \nu_\mu$$
$$\mu^+ \to e^+ + \overline{\nu}_\mu + \nu_e$$

Crudely we would expect N$\frac{\nu_e}{\nu_\mu} \sim \frac{1}{2}$ This has been measured to be closer to 1 that would be expected. This ratio was measured to be closer to 1 (at Super Kamiokande) demonstrating that ν_μ were missing. An azimuthal variation i.e. N (ν_μ)Vs N (ν_e) was measured between the above atmosphere and the other side of the earth (trans-geo neutrinos). About $\frac{1}{2}$ of the trans-geo neutrinos were lost suggesting that the neutrinos had oscillated into ν_τ. Oscillations imply that neutrinos have a mass and must then have a sub-light speed velocity.

Super Kamiokande was a large water detector looking for:

$$\nu_\mu + N \to \mu^- + H\text{hydrogen}$$
$$\nu_e + N \to e^- + H$$

both μ^- and e^- were detected by ∼5,000 PMTs by considering their characteristic Cherenkov light (e^- produce 'fuzzy' Cherenkov rings due while μ^- have much clearer version).

2.1.3 Solar Neutrinos

In the experiment by Ray Davis, mainly 'high' energy neutrinos (∼14 MeV) were used from the process

$$p +^7 Be \to^8 Be + \gamma$$
$$\hookrightarrow^8 Be + e^+ + \nu_e$$

The looked for reaction was:

$$\nu_e +^{37} Cl \rightarrow e^- +^{37} Ar$$

This was for neutrinos impinging upon a tank of C_2Cl_4, there were not as many reactions as expected (the so-called 'solar neutrino problem').

To detect lower energy neutrinos tanks of Gallium are used:

$$\nu_e + Ga \rightarrow Ge + e^-$$

These were also produced at a lower rate than expected. In the Sudbury neutrino observatory (SNO) a tank of heavy water was used and the following reaction detected:

$$\nu_e + n \rightarrow e^- + p$$

Again a deficit in the of ν_e was seen, $\frac{1}{3}$ of the expectation value. Combined with the Super Kamiokande results this explained the solar neutrino problem: we only see $\frac{1}{3}$ of the expected ν_3 because the other $\frac{2}{3}$ oscillate into ν_μ and ν_τ. Further confirmation from SNO was gained when salt was added to the water increasing the sensitivity to ν_μ and ν_τ:

$$\nu_{e/\mu/\tau} + n \rightarrow \nu_{e/\mu/\tau} + n_{scattered}$$
$$n_{scattered} +^{35} Cl \rightarrow^{36} Cl + \gamma$$

This interaction gave a measure of the total flux which was measured to be consistent with the (originally) predicted solar flux, hence the neutrinos were oscillating.

2.2 Beam experiments (colliding and fixed target)

There are various different types of colliding beams, which have different properties and can probe different phenomena. We can classify them into 3 broad types:

- e^+e^- collider: purely leptonic these, therefore, are very 'clean' with a controllable centre of mass energy. These have a large discovery potential but is limited by synchrotron radiation. Any further e^+e^- colliders will either be linear or changed to $\mu^-\mu^+$.
- NN(pp) These collide nucleons, either protons or anti-protons (sometimes one of each). These machines access the highest possible energies but create 'messy' interactions due to the quark composition of the protons.

- lN: mixed colliders use a lepton and a nucleon, giving higher energies but with a cleaner interaction obviously not as clean or as high energy as pure leptonic or hadron colliders.

2.2.1 Lepton-nucleon scattering

In the late 1960s and early 1970s deep inelastic scattering experiments using electrons, neutrinos and also muons probed the structure of the proton and neutron. It looked as if scattering occured on point like objects in the nucleon and about 50% of the nucleon interacted in this way. The remaining 50% was carried by the gluons. This was the beginning of Q.C.D.

At HERA this was advanced further in e^-p collisions e^- (and some e^+) at an energy ∼27.5 GeV collided with p (∼920 GeV) yielding a centre of mass energy ∼320 GeV. There were 2 multi purpose colliding beams experiments which measured a wide range of phenomena: photon and proton structure; other aspects of QCD; electroweak physics and searches for effects beyond SM (eg leptoquarks).

Main measurements from HERA: Structure of the proton, measured over a vast kinematic range of the the first measurements in the 1960s

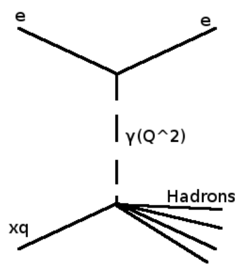

Figure 2.3: Deep inelastic scattering of electrons from a parton within a proton carrying momentum x

These measurements give an idea that as x changes you probe different partons of the proton:

- high x ∼ valence quarks
- low x ∼ 'sea' of quarks

These measurements give us precise knowledge of the structure of matter. Also practically many colliders use protons which requires a good physical knowledge of the proton in order to carry out detailed analysis.

There are two possible propagator bosons for the neutral current electron Deep Inelastic Scattering (DIS 2.3) as well as the photon (shown) measurements of the cross section of this process reveal that the Z^0 also contributes.

As well as neutral current process there is also the charged current process mediated by the W boson (2.5). Comparison of neutral current against charged current cross sections are shown in (2.6)

2.2.2 e^-e^+ Colliders

There have been a multitude of e^-e^+ experiments with a centre of mass energies in the range $\sim$ few GeV to $\sim$ 200 GeV. There are plans for a linear e^-e^+ collider that would probe into the 1 TeV (maybe up to 3 TeV) ranges.

e^-e^+ Discoveries:

- Charm quark discovered in 1974 at SLAC (also in proton-Be at BNL) via detection of the decay of the ground state of J/Ψ meson $M_{J/\Psi} \sim 3.6$ GeV

- in 1979 the gluon was discovered by the experiments at PETRA collider at DESY with up to $\sqrt{s} = 35$ GeV. Although e^+e^- is a clean leptonic environment they can be a powerful probe of QCD through considerations of the products. Eg discovery of the gluon through the observation of tri-jet events. A naive view of $e^+e^- \rightarrow p\overline{p}$ would be that of 2.8 but through first order corrections where gluons are radiated by the final state quarks we get tri-jet events (2.9)

- in 1989 LEP (large e^-e^+ collider) turned on embarking (along with SLD, a linear collider running at the Z^0 mass) on a new era of precision physics. Initially running at $\sim$91 GeV (M_z) then moved through, in $\sim$2 M_w (160 GeV) and finally moved to $\sim$200 GeV (looking for the Higgs). There were 4 multi-purpose experiments (ALEPH, DELPH, LE and OPAL) most famous for the precision measurement of Electro-weak parameters (eg M_z and M_w). The measurement of σ as a function of $\sqrt{s}$ was fundamental to measuring M_z and constraining the number of neutrinos (2.10), it measured Z^0 mass as:

  $$M_z = 91.1876 \pm 0.0021 \text{ GeV}$$

  with $\sqrt{s}$ at $\sim$160 GeV $W^\pm$ was measured:

  $$M_w = 80.398 \pm 0.025 \text{ GeV}$$

  in the absence of direct measurements, precise determination of known parameters constrain new physics phenomena eg. Higgs, SUSY etc.

- In its final throes LEP also searched for the Higgs via so-called 'Higgs-stralung' where a virtual z results in a real z and a Higgs. The search energy was constant increased as $\sqrt{s} > M_H + M_z$ is required.

- The next, planned, major leptonic collider is the ILC. This will be a large linear e^+e^- collider which will act as a $t\bar{t}$ factory for Higgs-stralung etc.

2.2.3 Hadron-hadron colliders

- Discovery of the bottom quark in 1977 by production of γ mesons decaying to $\mu^+\mu^-$ in p-Be collisions at FNAL (2.11.
- Discovery of the $W^{\pm}$ and Z^0 bosons in 1984 at the SPPS collider at CERN. Looked for W or Z decaying leptonicaly, which are low background relative to the hadronic decays $\sqrt{s} \simeq 540$ GeV in 1995
- Discovery of the top quark by CDF and D0 at $\sqrt{s} = 1800$ GeV in 1995.
- Measurements of transverse jet energy, $P_{\perp}^{jet}$ 2.12

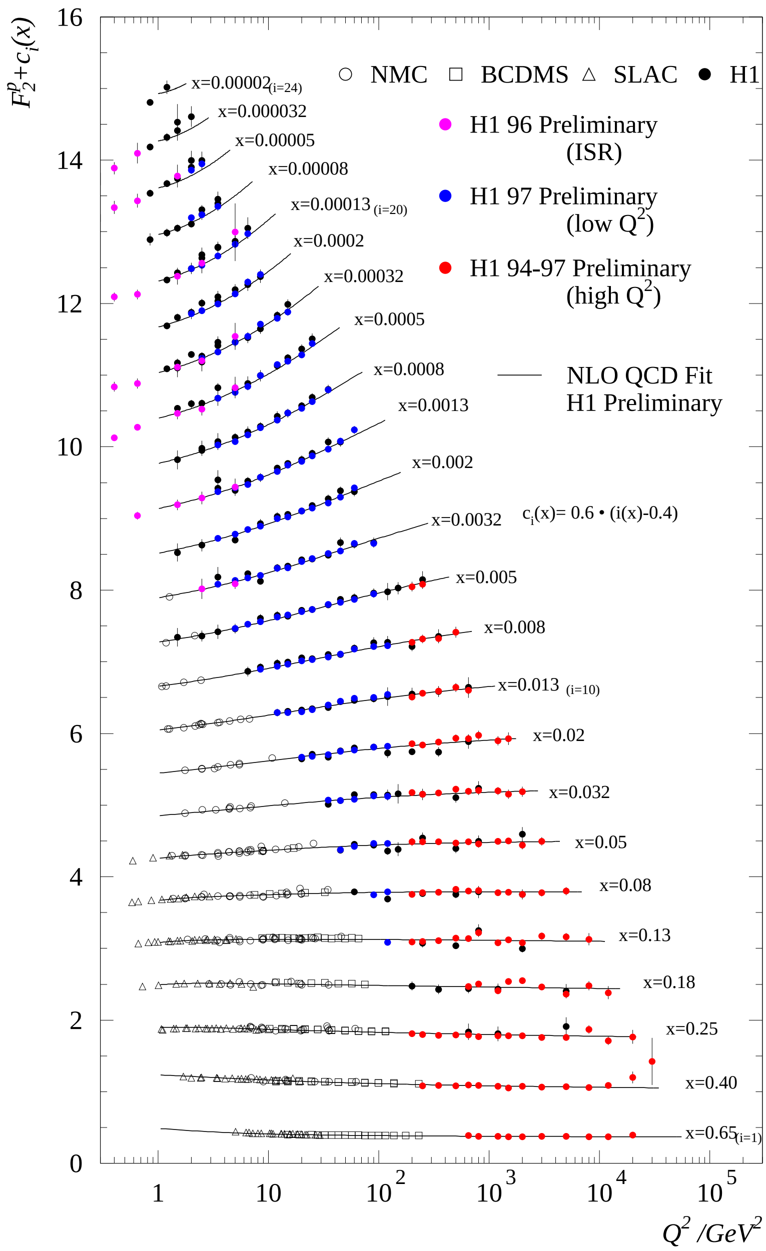

Figure 2.4: Variance of structure function F_2 with Q^2 at fix values of x

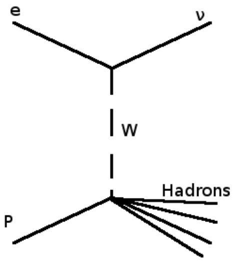

Figure 2.5: electron-proton DIS mediated by the W boson

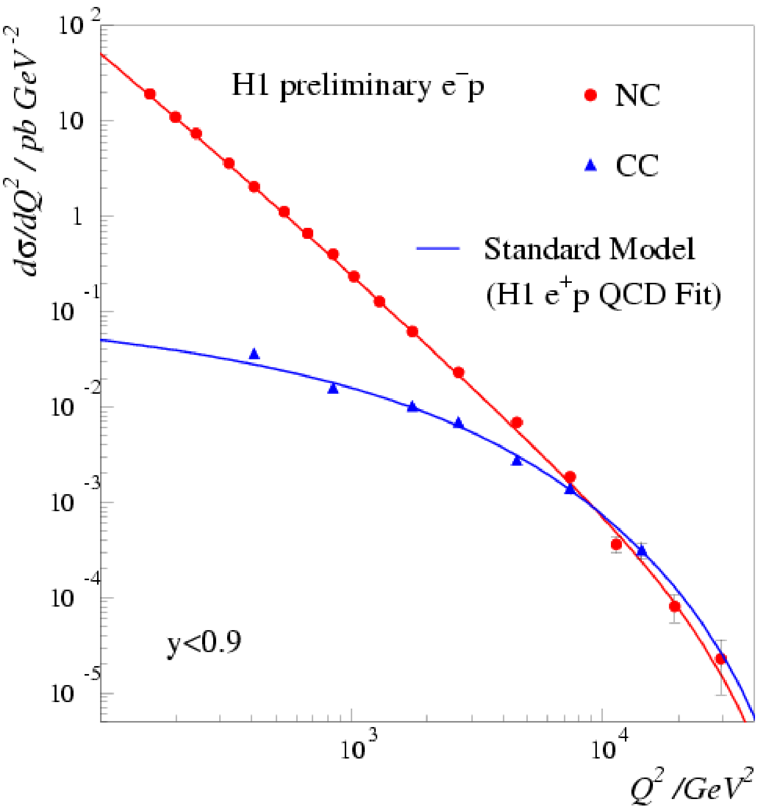

Figure 2.6: Comparison of CC and NC cross-sections against exchange boson momentum

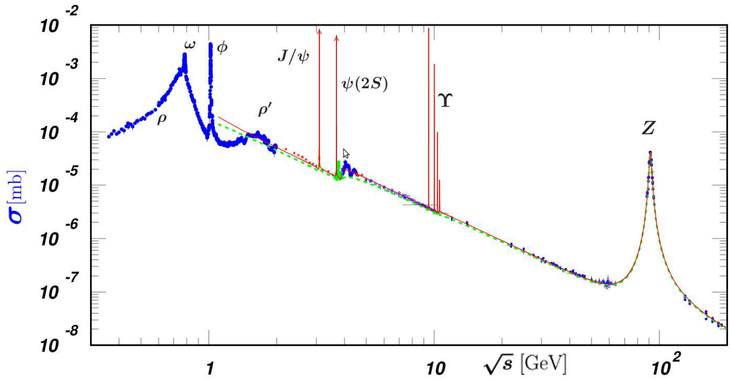

Figure 2.7: Total cross-section of $e^+e^- \to$ hadrons for a range of energies

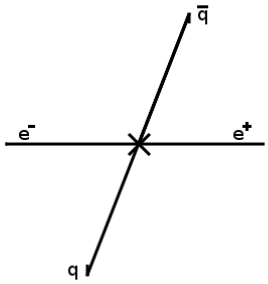

Figure 2.8: Production of di-jets from e^+e^- collision

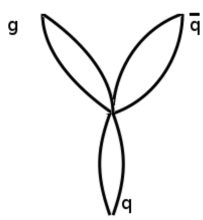

Figure 2.9: end-on view of a tri-jet event produced through e^+e^- collisions

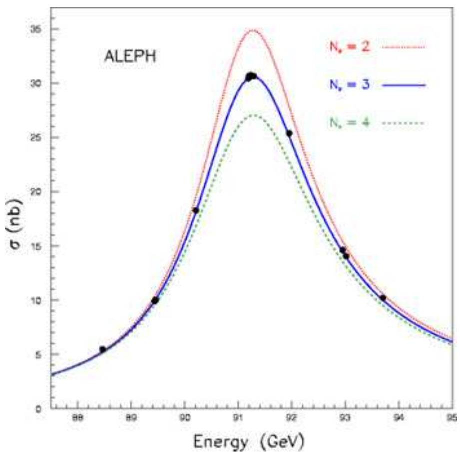

Figure 2.10: Cross sections of Z^0 with various numbers of neutrinos

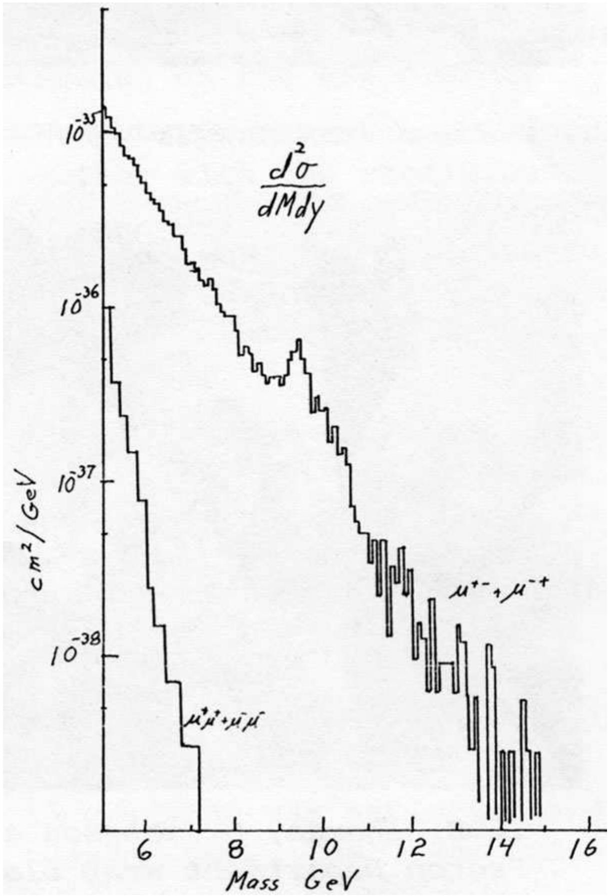

Figure 2.11: The upsilon resonance in p-Be cross section

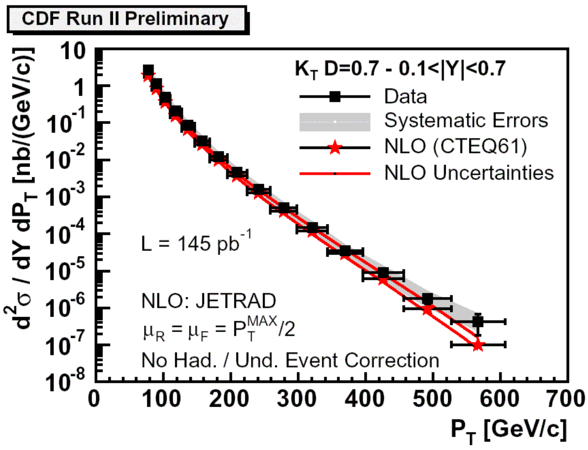

Figure 2.12: transverse momentum of jets against cross section. note: size of cross section scale 9 orders of magnitude

Chapter 3

Non-relativistic Quantum Mechanics

3.1 Shrödinger picture & probability current

For a free particle of mass, m, the classical energy-momentum relation is:

$$E = \frac{p^2}{2m}$$

in QM p and E become differential operators (in natural units).

$$E \rightarrow i\frac{d}{dt} \text{ and p} \rightarrow -\text{i}\nabla$$

these operate on the wavefunction

$$\begin{aligned}
\frac{(-i)^2}{2m}\nabla^2\Psi &= i\frac{d\Psi}{dt} \\
-\frac{1}{2m}\nabla^2\Psi &= i\frac{d\Psi}{dt} \\
(\times\Psi^*)\ -\frac{1}{2m}\Psi^*\nabla^2\Psi &= i\Psi^*\frac{d\Psi}{dt} \\
\text{Conjugate}:\ -\frac{1}{2\text{m}}\Psi\nabla^2\Psi^* &= -i\Psi\frac{d\Psi^*}{dt}
\end{aligned}$$

Sum these terms

$$\Rightarrow\ i(\Psi^*\frac{d\Psi}{dt} + \Psi\frac{d\Psi^*}{dt}) = \frac{-1}{2m}(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*)$$

$$(\times i)\ \frac{d}{dt}(\Psi^*\Psi) + \frac{i}{2m}(\Psi\nabla^2\Psi^* - \Psi^*\nabla^2\Psi) = 0$$

integrate over a volume V:

$$\frac{\partial}{\partial t}\int \Psi^*\Psi dV + \frac{i}{2m}\int_v \nabla.(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)dV = 0$$

the above looks like a conservation equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0$$

where:

$$\begin{aligned}\rho &= |\Psi|^2 \\ \vec{j} &= \frac{i}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)\end{aligned}$$

what are ρ and $\vec{j}$ for a plane Q wave?

$$\begin{aligned}\Psi &= Ne^{i(px-Et)} \\ \rho &= NN^* = |N|^2 \\ \vec{j} &= \frac{i}{2m}(Ne^{i(px-Et)}\nabla(e^{-i(px-Et)}N) - N^*e^{-i(px-Et)}\nabla Ne^{i(px-Et)}) \\ &= \frac{i}{2m}(NN^*(-ip) - N^*N(ip)) \\ &= \frac{1}{2m}2NN^*\vec{p} \\ &= \frac{\vec{p}}{m}|N|^2\end{aligned}$$

In the SE picture the operators are time independent where as the wave functions are not. In classical mechanics the operators momentum and energy are time dependent. Can we go to a formulation of Quantum Mechanics where the operators are time dependent?

3.2 The Heisenberg picture

Starting from the SE:

$$i\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$$

recall that the expectation of an observable, $\hat{A}$, is given by:

$$\langle\hat{A}\rangle = \int \Psi^*\hat{A}\Psi d^3x$$

solving the SE:

$$\begin{aligned}
i\frac{\partial\Psi(x,t)}{\Psi(x,t)} &= \hat{H}\partial t \\
i\int\frac{\partial\Psi(x,t)}{\Psi(x,t)} &= \int_0^t \hat{H}\partial t \\
\ln[\Psi(x,t)] - \ln[(x,0)] &= -i\hat{H}t \\
\Rightarrow\ \Psi(x,t) &= \Psi(x,0)e^{-i\hat{H}t}
\end{aligned}$$

now consider $\langle\hat{A}\rangle$:

$$\langle\hat{A}\rangle = \int \Psi^*(x,0)e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}d^3x$$

define

$$\begin{aligned}
A_H &= e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t} \\
\Rightarrow\ \frac{dA_H}{dt} &= i\hat{H}e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t} + e^{i\hat{H}t}\hat{A}(-i\hat{H})e^{-i\hat{H}t} \\
&= i(\hat{H}\hat{A} - \hat{A}\hat{H}) \\
&= i[\hat{H}\hat{A}]
\end{aligned}$$

Remember in the previous equations both $\hat{H}$ and $\hat{A}$ are time independent. In the interaction picture, time dependent perturbation theory

$$\hat{H} = \hat{H}_0 + \hat{H}'$$

then define

$$\begin{aligned}
\hat{H}'_I &= e^{i\hat{H}_0}\hat{H}'e^{-i\hat{H}_0} \\
\Rightarrow \frac{d\hat{H}'_I}{dt} &= i[\hat{H}_0, \hat{H}']
\end{aligned}$$

3.3 The harmonic Oscillator

This is a mechanical system which we use to introduce the concept of creation and annihilation operators:

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}\omega^2 mq^2$$

let

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\hat{p}) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p})
\end{aligned}$$

How do $\hat{a}$ and $\hat{a}^\dagger$ commute? given:

$$\begin{aligned}
[\hat{q}, \hat{p}] &= i \\
[\hat{a}, \hat{a}^\dagger] &= \frac{1}{2}(m\omega[\hat{q}, \hat{q}] + \frac{i}{\sqrt{m\omega}}[\hat{p}, \hat{q}]m\omega - i\sqrt{m\omega}[\hat{q}, \hat{p}]\frac{1}{\sqrt{m\omega}} + \frac{1}{m\omega}[\hat{p}, \hat{p}]) \\
&= 1
\end{aligned}$$

Can write $\hat{H}$ as:

$$\begin{aligned}
\hat{H} &= \frac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\omega \\
\hat{a}^\dagger\hat{a} &= \frac{1}{2}(m\omega\hat{q}^2 + \frac{\hat{p}^2}{m\omega} + i(\hat{q}\hat{p} - \hat{p}\hat{q})) \\
\hat{a}\hat{a}^\dagger &= \frac{1}{2}(m\omega\hat{q}^2 + \frac{\hat{p}^2}{m\omega} + i(\hat{p}\hat{q} - \hat{q}\hat{p})) \\
\Rightarrow \ \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger &= m\omega\hat{q}^2 + \frac{\hat{p}^2}{m\omega}
\end{aligned}$$

Recall:

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 \\
&= [\frac{\hat{p}^2}{m\omega} + m\omega\hat{q}^2]\frac{1}{2}\omega \\
&= [\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger]\frac{1}{2}\omega \\
&= [\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}]\frac{1}{2}\omega \\
&= \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})
\end{aligned}$$

Why can we interpret $\hat{a}$ as an annihilation operator and $\hat{a}^\dagger$ as creation? What are $[\hat{H}, \hat{a}]$ and $[\hat{H}, \hat{a}^\dagger]$?

$$\begin{aligned}
[\hat{H}, \hat{a}] &= [(\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega, \hat{a}] \\
&= (\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a})\omega \\
&= [\hat{a}^\dagger, \hat{a}]\hat{a}\omega \\
&= -\hat{a}\omega
\end{aligned}$$

Similarly $[\hat{H}, \hat{a}] = \hat{a}^\dagger\omega$.

Now consider the state $|n\rangle$ such that $\hat{H}|n\rangle = E_n|n\rangle$ what is E_n for state $\hat{a}^\dagger|n\rangle$?

$$\begin{aligned}
\hat{H}\hat{a}^\dagger|n\rangle &= (\hat{a}^\dagger\omega + \hat{a}^\dagger\hat{H})|n\rangle \\
&= \hat{a}^\dagger(\omega + E_n)|n\rangle \\
&= (\omega + E_n)\hat{a}^\dagger|n\rangle
\end{aligned}$$

Similarly $\hat{H}\hat{a}|n\rangle = (E_n - \omega)\hat{a}|n\rangle$ So the energy of the state $\hat{a}^\dagger|n\rangle$ or $\hat{a}|n\rangle$ is ω greater (or smaller) than that of the state $|n\rangle$. There must be a state containing no quanta of mechanical oscillation such that $\hat{a}|n\rangle = 0$

Now apply the creation operators to this $|0\rangle$ (ground) state:

$$\hat{a}^\dagger|0\rangle = |1\rangle$$
$$\frac{1}{\sqrt{2}}(\hat{a}^\dagger)^2|0> = |2\rangle$$

generally:

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

apply the $\hat{H}$ operator to the ground state:

$$\begin{aligned}\hat{H}|0\rangle &= (\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega|0\rangle \\ &= \frac{\omega}{2}|0\rangle\end{aligned}$$

Now apply $\hat{H}$ to $\hat{a}^\dagger|0\rangle$

$$\begin{aligned}\hat{H}\hat{a}^\dagger|0\rangle &= (\hat{a}^\dagger\omega + \hat{H}\hat{a}^\dagger)|0\rangle \\ &= \hat{a}^\dagger(\omega + \frac{\omega}{2})|0\rangle \\ &= \frac{3}{2}\omega\hat{a}^\dagger|0\rangle\end{aligned}$$

Generally:

$$\hat{H}_n\hat{a}^\dagger|0\rangle = (n + \frac{1}{2})\omega\hat{a}^\dagger|0\rangle$$

But,

$$\begin{aligned}\hat{H} &= \hat{a}^\dagger\hat{a} + \frac{\omega}{2} \\ \Rightarrow \hat{a}^\dagger\hat{a} &= n\end{aligned}$$

cf,

$$\begin{aligned}\hat{H}_n|n\rangle &= E_n|n\rangle \\ \Rightarrow E_n &= (n + \frac{1}{2})\omega \\ |n\rangle &= (n!)^{-\frac{1}{2}}(\hat{a}^\dagger)^n|0\rangle \\ \Rightarrow |n+1\rangle &= ((n + \frac{1}{2})!)^{-\frac{1}{2}}(\hat{a}^\dagger)^{n+1}|0\rangle \\ &= ((n + \frac{1}{2})!)^{-\frac{1}{2}}\hat{a}^\dagger(\hat{a}^\dagger)^n|0\rangle \\ &= ((n + \frac{1}{2})!)^{-\frac{1}{2}}\hat{a}^\dagger\sqrt{n!}|0\rangle \\ \therefore\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle\end{aligned}$$

likewise

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \therefore\ \hat{a}^\dagger\hat{a}|n\rangle &= n|n\rangle \end{aligned}$$

3.4 An-harmonic Oscillator

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 + \lambda\hat{q}^2 \\ &= \hat{H}_0 + \lambda\hat{H}' \end{aligned}$$

3.4.1 Rayleigh-Schrödinger Perturbation theory

we have

$$\begin{aligned} \hat{H}_0|n\rangle^{(0)} &= E_n^{(0)}|n\rangle^{(0)} \\ E_n &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots \\ \text{and } |\mathrm{n}\rangle &= |n\rangle^{(0)} + \lambda|n\rangle^{(1)} + \lambda^2|n\rangle^{(2)} + \ldots \end{aligned}$$

where $|n\rangle^{(1)}$,$|n\rangle^{(2)}$ are $\perp$ to $|n\rangle^{(0)}$

$$\begin{aligned} \therefore \hat{H}|n\rangle &= (\hat{H}_0 + \lambda\hat{H}')\{|n\rangle^{(0)} + \lambda|n\rangle^{(1)} + \lambda^2|n\rangle^{(2)} + \ldots\} \\ &= E_n|n\rangle \\ &= (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots)\{|n\rangle^{(0)} + \lambda|n\rangle^{(1)} + \lambda^2|n\rangle^{(2)} + \ldots\} \end{aligned}$$

λ^0 terms:

$$\hat{H}_0|n\rangle^{(0)} = E_n^{(0)}|n\rangle^{(0)}$$

λ^1 terms:

$$\begin{aligned} \hat{H}_0|n\rangle^{(1)} - E_n^{(0)}|n\rangle^{(1)} + \hat{H}'|n\rangle^{(0)} - E_n^{(1)}|n\rangle^{(0)} &= 0 \\ \text{(i.e.) } (\hat{\mathrm{H}}_0 - \mathrm{E}_\mathrm{n}^{(0)})|\mathrm{n}\rangle^{(1)} + (\hat{\mathrm{H}}' - \mathrm{E}_\mathrm{n}^{(1)})|\mathrm{n}\rangle^{(0)} &= 0 \end{aligned}$$

multiply by ${}^{(0)}\langle n|$

$$\begin{aligned} \rightarrow 0 +^{(0)} \langle n|(\hat{H}' - E_n^{(1)})|n\rangle^{(0)} &= 0 \\ \therefore \langle E_n^{(1)}\rangle &=^{(0)} \langle n|\hat{H}'|n\rangle^{(0)} \end{aligned}$$

can also multiply by ${}^{(0)}\langle m|$ which is orthogonal to $|n\rangle^{(0)}$

$$\begin{aligned} 0 &=^{(0)} \langle m|\hat{H}_0|n\rangle^{(1)} -^{(0)} \langle m|E_n^0|n\rangle^{(1)} +^{(0)} \langle m|\hat{H}'|n\rangle^{(0)} -^{(0)} \langle m|E_n^1|n\rangle^{(0)} \\ \Rightarrow 0 &=^0 \langle m|E_m^0 - E_n^0|n\rangle^1 +^0 \langle m|\hat{H}'|n\rangle^0 - 0 \end{aligned}$$

$$\because \ {}^{(0)}\langle m|\hat{H}_0 =^{(0)} \langle m|E_m^0$$

$$\begin{aligned} \Rightarrow \ {}^0\langle m|n\rangle^1 &= \frac{{}^0\langle m|\hat{H}'|n\rangle^0}{-E_m^0 + E_n^0} \\ \therefore \ |n\rangle &= |n\rangle^0 + \lambda|n\rangle^1 \dots \\ &= |n\rangle^0 + \lambda \sum_m |m\rangle^0 \frac{{}^0\langle m|\hat{H}'|n\rangle^0}{-E_m^0 + E_n^0} \end{aligned}$$

where we have used the identity:

$$|n\rangle' = \sum_m |m\rangle^{00}\langle m|n\rangle^0$$

so what is $\lambda\hat{q}^3$ (sometimes x^3)

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\hat{p}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p}) \\ \hat{a} + \hat{a}^\dagger &= \frac{1}{\sqrt{2}}(2\sqrt{m\omega}\hat{q}) \\ \therefore \hat{q} &= \frac{1}{\sqrt{2m\omega}}(\hat{a} + \hat{a}^\dagger) \\ \Rightarrow \hat{q}^3 &= \frac{1}{2\sqrt{2}(m\omega)^{\frac{3}{2}}}(\hat{a} + \hat{a}^\dagger)^3 \end{aligned}$$

so we need:

$$\frac{\lambda}{2\sqrt{2}(m\omega)^{\frac{3}{2}}} \sum_m |m\rangle^0 \frac{{}^0\langle m|(\hat{a} + \hat{a}^\dagger)^3|n\rangle^0}{-E_m^0 + E_n^0}$$

$${}^0\langle m|(\hat{a} + \hat{a}^\dagger)^3|n\rangle^0 =^0 \langle m|(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^2)(\hat{a} + \hat{a}^\dagger)|n\rangle^0$$

given:

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1 \\ \hat{a}^\dagger\hat{a} &= n \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned}$$

result:

$$
\begin{aligned}
&{}^0\langle m|(\hat{a}+\hat{a}^\dagger)^3|n\rangle^0 = \\
&\qquad {}^0\langle m|\sqrt{n(n-1)(n-2)}|n-3\rangle^0 \\
&\qquad +^0 \langle m|\sqrt{(n+1)(n+2)(n+3)}|n+3\rangle^0 \\
&\qquad +^0 \langle m|(3n+2)\sqrt{n)}|n-1\rangle^0 \\
&\qquad +^0 \langle m|(3n+1)\sqrt{n+1)}|n+1\rangle^0
\end{aligned}
$$

3.5 Lagrangian

Physical systems evolve such that the action, s:

$$S = \int \mathcal{L}(q,\dot{q})dt$$

is a minimum, if:

$$S = \int_{t2}^{t1} \mathcal{L}(q,\dot{q})dt$$

then:

$$
\begin{aligned}
\delta S &= \int_{t2}^{t1} (\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q})dt = 0 \\
\delta \dot{q} &= \frac{d}{dt}\delta q \\
\delta S &= \int_{t2}^{t1} (\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d}{dt}\delta q)dt = 0 \\
&= \int_{t2}^{t1} (\frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q)dt + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t1}^{t2} \\
\text{where} : 0 &= \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t1}^{t2} \\
\text{as} : \delta\text{q}(\text{t}_1) &= \delta q(t_2) \\
\therefore 0 &= \frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q \\
\Rightarrow \frac{\partial \mathcal{L}}{\partial q}\delta q &= \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q \text{ Euler} - \text{Lagrange}
\end{aligned}
$$

for the harmonic oscillator:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \\ \frac{\partial \mathcal{L}}{\partial \dot{q}} &= m\dot{q} \\ \frac{\partial \mathcal{L}}{\partial q} &= -m\omega^2 q \\ \therefore m\ddot{q} &= -m\omega^2 q\end{aligned}$$

in Quantum Mechanics variables (eg momentum) are operators which in general do not commute. Specifically $\hat{q}$ and $\hat{p}$: $[\hat{q}, \hat{p}] = i$ where $\hat{p}$ is the generalisation:

$$\hat{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

The Heisenberg equation of motion for an operator, $\hat{a}$, is:

$$\frac{d\hat{A}}{dt} = i[\hat{H}, \hat{A}]$$

The Hamiltonian defined in terms of $\mathcal{L}$ is:

$$\hat{H} = \hat{p}\hat{\dot{q}} - \mathcal{L}$$

so in the classical oscillator case we had:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\hat{\dot{q}}^2 - \frac{1}{2}m\omega^2\hat{q}^2 \\ \text{As } \hat{\mathrm{H}} &= \hat{p}\hat{\dot{q}} - \mathcal{L} \\ &= \hat{p}\hat{\dot{q}} - \frac{1}{2}m\hat{\dot{q}}^2 + \frac{1}{2}m\omega^2\hat{q}^2 \\ &= \hat{p}\frac{\hat{p}}{m} - \frac{1}{2}m\frac{\hat{p}^2}{m^2} + \frac{1}{2}m\omega^2\hat{q}^2 \\ &= \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2\end{aligned}$$

3.6 Dirac-Delta function

This may be thought of as a function of height $\frac{1}{\Delta x}$ and width Δx around a value $x = x_0$ in the limit $\delta x \to 0$ (3.1) Area of Dirac-delta function:

$$A = \int_{-\infty}^{\infty} \delta(x - x_0)dx = 1$$

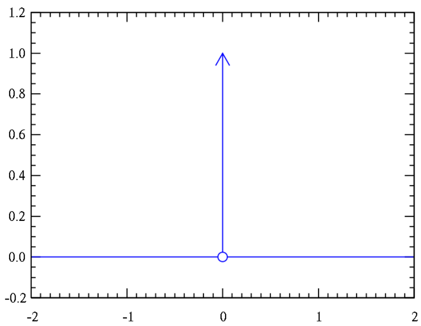

Figure 3.1: Schematic of the Dirac delta

Consider the function $f(x)$ to be split into elements Δx wide:

$$\begin{aligned} \int f(x)dx &= \sum_i f(x_i)\Delta x \\ \Rightarrow \int f(x)\delta(x-x_0) &= \sum_i f(x_i)\delta(x_i - x_0)\Delta x \\ &= f(x_0) \end{aligned}$$

Some useful expressions for the delta function:

1. $\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}$ for $\frac{-\epsilon}{2} < x < \frac{\epsilon}{2}$
2. $\delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon}{\pi(x^2+\epsilon^2)}$ Breit-Wigner resonance formula
3. $\int \delta(x)dx = 1$ at $x = 0$, it has value $\frac{1}{\pi\epsilon}$ at what value of x does this function fall to $\frac{1}{2}$ of its height?

   $$\begin{aligned} \frac{1}{2\pi\epsilon} &= \frac{\epsilon}{\pi(x^2+\epsilon^2)} \\ x_{half\ height} &= \pm\epsilon \end{aligned}$$

   so $2\epsilon = \Gamma$ is the width at half height ,Γ

4. $$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}dk$$

   Consider Fourier analysis. Recall for a well behaved function of x between $\frac{-\ell}{2}$ and $\frac{\ell}{2}$, it can be expressed as a Fourier series of functions with wavelengths ℓ, $\frac{\ell}{2}$, $\frac{\ell}{3}$ etc.

   $$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left\{\frac{-i2\pi nx}{\ell}\right\}\Delta n$$

to find a_n integrate over both sides:

$$\int_{\frac{-\ell}{2}}^{\frac{\ell}{2}} dx f(x) \exp\left\{\frac{-i2\pi nx}{\ell}\right\} = \int_{\frac{-\ell}{2}}^{\frac{\ell}{2}} dx a_n$$
$$\Rightarrow a_n = \frac{1}{\ell}\int_{\frac{-\ell}{2}}^{\frac{\ell}{2}} dx f(x) \exp\left\{\frac{-i2\pi nx}{\ell}\right\}$$

Consider the limit as $\ell \to \infty$:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left\{\frac{-i2\pi nx}{\ell}\right\} \Delta n$$

define:

$$\begin{aligned} k &= \frac{2\pi n}{\ell} \\ dk &= \frac{2\pi\Delta n}{\ell} \\ f(x) &= \sum_{n=-\infty}^{\infty} a_n \exp(ikx)\frac{\ell}{2\pi}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} g(k) dk \end{aligned}$$

where:

$$\begin{aligned} g(k) &= \ell a_n \\ &= \int_{-\infty}^{\infty} f(k)e^{-ikx}dk \\ \Rightarrow f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k)e^{-ikx'}e^{ikx}dkdk' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{ik(x-x')}dkdk' \end{aligned}$$

so

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk$$

5. $\delta(ax) = \frac{1}{a}\delta(x)$ this can be seen by::

$$\begin{aligned} \text{let } ax &= y \\ adx &= dy \\ \to \int \delta(ax)dx &= \int \delta(y)\frac{dy}{a} \\ &= \frac{1}{a} \end{aligned}$$

6. $\delta(x) = \delta(-x)$: see above

7.

$$\delta(f(x)) = \sum_i \frac{\delta(x - a_i)}{\frac{df}{dx}|_{x=a_i}}$$

where a_i are the roots of $f(x) = 0$

At each place where $f(x) = 0$, then:

$$\begin{aligned} f(x) &= f(a_i) \\ &= (x - a_i)\frac{df}{dx}|_{x=a_i} \cdots \end{aligned}$$

the delta function has non-zero contributions from each of the roots a_i of the form:

$$\begin{aligned} \delta & \left(f(x) = \sum_i (x - a_i)\frac{df}{dx}|_{x=a_i} \right) \\ & \Rightarrow \delta(f(x)) = \sum_i \frac{\delta(x - a_i)}{\frac{df}{dx}|_{x=a_i}} \end{aligned}$$

3.7 The Heaviside unit step function

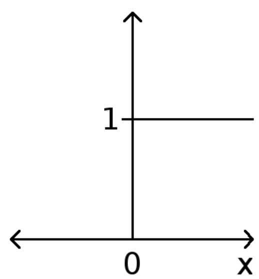

Figure 3.2: The Heaviside function

Heaviside unit step function has the following definition:

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{aligned}
\frac{\partial\theta}{\partial x} &= \delta(x) \\
&= \frac{1}{2\pi}\int e^{-i\omega x} d\omega \\
\Rightarrow \theta &= \frac{1}{2\pi}\iint e^{-i\omega x} d\omega dx \\
&= \frac{1}{2\pi}\int \frac{e^{-i\omega x}}{-i\omega} d\omega \\
&= \frac{1}{2\pi}\int \frac{e^{-i\omega x}}{\omega + i\epsilon} d\omega
\end{aligned}$$

using Cauchy's Theorem:

$$\begin{aligned}
\oint_c \frac{f(z)}{z-a} dz &= 2\pi i f(a) \\
\Rightarrow \theta &= \frac{-1}{2\pi i}.2\pi i e^{-ix,-i\epsilon} \\
&= e^{-\epsilon x} \\
\text{for } \epsilon \to 0,\ \theta \to 1 &
\end{aligned}$$

3.8 Special Relativity

Brief recap as we will be calculating processes for which v̆c.

4-vectors are defined as:

$$\begin{aligned}
x^\mu &= (t, \vec{x}) \qquad & x_\mu &= (t, -\vec{x}) \\
p^\mu &= (E, \vec{p}) \qquad & p_\mu &= (E, -\vec{p})
\end{aligned}$$

scalar products are:

$$\begin{aligned}
x.y &= x_\mu y^\mu = t_x t_y - \vec{x}.\vec{y} \\
p.q &= p_\mu q^\mu = E_p E_q - \vec{p}.\vec{q}
\end{aligned}$$

also

$$\begin{aligned}
x_\mu &= g_{\mu\nu} x^\nu \\
x^\nu &= g^{\nu\mu} x_\mu
\end{aligned}$$

where $g_{\mu\nu}$ (or $g^{\mu\nu}$) is the metric tensor defined as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{aligned} g_{\mu\nu}g^{\mu\nu} &= \sum_\mu \sum_\nu g_{\mu\nu}g^{\mu\nu} \\ &= \sum_\mu g_{\mu\mu}g^{\mu\mu} \\ &= 4 \end{aligned}$$

3.8.1 Quantum operators as 4-vectors

$$\begin{aligned} \text{originally: } E &= i\frac{\partial}{\partial t} \\ \text{and: } p &= i\nabla \\ \text{combine as a 4-vector: } p^\mu &= i(\frac{\partial}{\partial t}, -\nabla) \\ &= i\frac{\partial}{\partial x_\mu} \\ &= i\partial^\mu \end{aligned}$$

conversely (note sub/super script positioning):

$$\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} \\ &= (\frac{\partial}{\partial t}, \nabla) \end{aligned}$$

From these two we can form the invariant operator:

$$\begin{aligned} \partial_\mu \partial^\mu &= \Box^2 \\ &= \frac{\partial^2}{\partial t^2} - \nabla^2 \end{aligned}$$

3.9 Lorentz Transformation

A Lorentz transformation relates co-ordinates in two frames. Under a Lorentz transformation, where the x^1 axis has a velocity, v, causes x^μ to transform as:

$$\begin{aligned} t' &= \gamma(t - vx^1) \\ x^{1\prime} &= \gamma(-vt + x^1) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned}$$

3.10 The light cone

let x^μ be the 4-vector, $(x^0, \vec{x})$. Suppose we have someone at $y^\mu = (y^0, \vec{y})$ who sends out a light signal. Now consider the difference of x and y squared:

$$(x^\mu - y^\mu)^2 = (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2$$

if this is 0, i.e. $(x^0 - y^0)^2 = (\vec{x} - \vec{y})^2$, then this is the equation for a light beam and defines a light cone.

if $(x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 > 0$, then the separation is time like, we are within the forward light cone and causally connected.

if $(x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 < 0$, then the separation is space like and there is no causal connection.

3.11 Relativistic Kinematics

Usually consider either A→B+C (decay) or A+B→C+D (scattering). For decays the centre of mass energy is the mass of the particle, A. For a scattering process it is the invariant mass:

$$s = (P_A^\mu + P_B^\mu)^2 = (P_C^\mu + P_D^\mu)^2$$

For a fixed target experiment B is at rest:

$$\begin{aligned} s &= (E_A + E_B)^2 - (\vec{p}_A + \vec{p}_b)^2 \\ &= (E_A + m_B)^2 - (\vec{p}_A + 0)^2 \\ &= E_A^2 + m_B^2 + 2E_A m_B - \vec{p}_A^2 \\ &= m_A^2 + m_B^2 + 2E_A m_B \end{aligned}$$

in the case that $E_A \gg m_A, m_B$

$$\sqrt{s} = \sqrt{2E_A m_B}$$

for a collider in the CMS where $\vec{p}_A = -\vec{p}_B$ and in the massless limit the energy becomes:

$$\begin{aligned} s &= (E_A + E_B)^2 - (\vec{p}_A + \vec{p}_b)^2 \\ &= m_A^2 + m_b^2 + 2E_A E_B - 2\vec{p}_A.\vec{p}_B \\ &= 2E_A E_B + 2E_A E_B \qquad\qquad (m \to 0, \vec{p}_A = \vec{p}_B) \\ \Rightarrow \sqrt{s} &= 2\sqrt{E_A E_B} \end{aligned}$$

eg at HERA $E_P = 920$ GeV and $E_e = 27.5$ GeV giving $\sqrt{s}$ as 318.1 GeV

To transform between the centre of mass frame and the laboratory frame we need to calculate β and γ

$$\begin{aligned}
\beta &= \frac{\text{3-momenta part of the vector}}{\text{Energy part of the vector}} \\
&= \frac{\vec{p}_A + \vec{p}_B}{E_A + E_B} \\
\gamma &= \frac{E_A + E_B}{\sqrt{s}}
\end{aligned}$$

Energy and momentum in the centre of mass frame can be determined using invariance rather than a Lorentz transform.

$$\begin{aligned}
P_A^\mu + P_B^\mu &= (\sqrt{s}, 0) \\
P_{A\mu}(P_A^\mu + P_B^\mu) &= E_A^{CMS}\sqrt{s} \\
\Rightarrow m_A^2 + P_{A\mu}P_B^\mu &= E_A^{CMS}\sqrt{s} \qquad (1) \\
\text{as } s &= (P_A^\mu + P_B^\mu)(P_{A\mu} + P_{B\mu}) \\
&= P_A.P_A + P_B.P_B + 2P_A.P_B \\
\Rightarrow P_A.P_B &= \frac{1}{2}(s - m_A^2 - m_B^2) \qquad (2)
\end{aligned}$$

substitute 2 into 1

$$\begin{aligned}
E_A^{CMS} &= \frac{2m_A^2 + s - m_A^2 - m_B^2}{2\sqrt{s}} \\
&= \frac{s + m_A^2 - m_B^2}{2\sqrt{s}} \\
(P_A^{CMS})^2 &= \frac{s + m_A^2 - m_B^2}{2\sqrt{s}} - m_A^2 \qquad (E^2 = P^2 + m^2) \\
&= \frac{1}{4s}\left[\, s - (m_A^2 + m_B^2)^2 \,\right]\left[\, s - (m_A^2 - m_B^2)^2 \,\right]
\end{aligned}$$

3.12 Mandelstam Variables

The Mandelstam variables for a scattering process:

$$\begin{aligned}
s &= (P_A + P_B)^2 \\
t &= (P_A - P_C)^2 \\
&= (P_B - P_D)^2 \\
&= q^2 \qquad \text{(momenta of the exchange particle)} \\
u &= (P_A - P_D)^2 \\
&= (P_C - P_B)^2
\end{aligned}$$

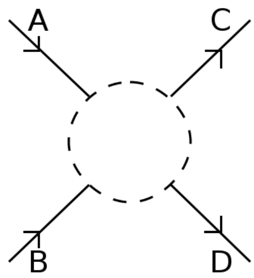

Figure 3.3: Basic form of scattering process: two particles enter, two particles leave

s-channel: annihilation process t-channel scattering: momentum transfered $P_A \rightarrow$

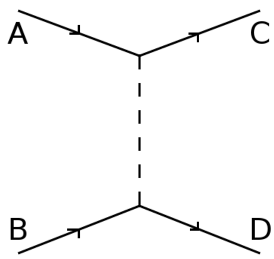

Figure 3.4: s-channel: annihilation of A and B producing C and D

P_C u-channel scattering: momentum transfered $P_A \rightarrow P_D$

A useful product of the variables is:

$$\begin{aligned}
s + t + u &= (P_A + P_B)^2 + (P_A - P_C)^2 + (P_A - P_D)^2 \\
&= P_A^2 + P_B^2 + 2P_A.P_B + P_A^2 + P_C^2 - 2P_A.P_C + P_A^2 + P_D^2 - 2P_A.P_D \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2P_A.(P_B - P_C - P_D) \\
\text{as :} P_A + P_B &= P_C + P_D \\
\Rightarrow -P_A = P_B - P_C - P_D & \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2P_A.(-P_A) \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - 2m_A^2 \\
&= m_A^2 + m_B^2 + m_C^2 + m_D^2
\end{aligned}$$

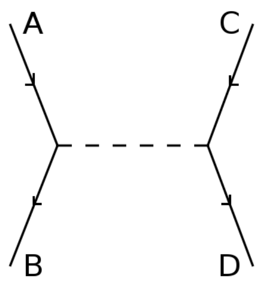

Figure 3.5: t-channel scattering: A scatters to C and B scatters to D

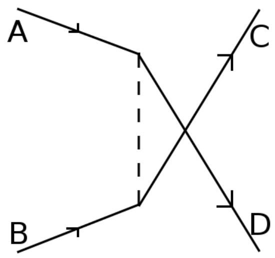

Figure 3.6: u-channel scattering: A scatters to D and B scatters to C

Chapter 4

Relativistic spin-0 Particles

4.1 The Klein-Gordon equation

The quantum wave function for a free scalar particle propagating in the x-direction is

$$\begin{aligned} \phi &\sim e^{i(\vec{p}.\vec{x}-Et)} \\ \Rightarrow \phi &\sim e^{ip^\mu x_\mu} \end{aligned}$$

Now we repeat the procedure which yielded the Shrödinger equation, but using $E^2 = P^2 + m^2$ transform this relation into an equation active on ϕ

$$-\frac{\partial^2 \phi}{\partial t^2} = -\nabla^2 \phi + m^2 \phi \tag{1}$$

This is the Klein-Gordon equation (or relativistic Shrödinger) equation. The complex conjugate of it is:

$$-\frac{\partial^2 \phi^*}{\partial t^2} = -\nabla^2 \phi^* + m^2 \phi^* \tag{2}$$

$$\begin{aligned} &[\phi^* \times (1) - \phi \times (2)] * -i \\ &i\left[\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2}\right] = i\left[\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^*\right] \qquad (m^2 \text{ terms cancel}) \\ &0 = i\frac{\partial}{\partial t}\left[\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right] - i\nabla\left[\phi^* \nabla \phi - \phi \nabla \phi^*\right] \end{aligned}$$

This is of the form $\frac{\partial \rho}{\partial t} + \nabla.j = 0$ where ρ is the probability density and j is the density flux.

What is ρ if $\phi = Ne^{i(p.x-Et)}$?

$$\begin{aligned}\rho &= i\left[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right] \\ &= i\left(N^*e^{-i(p.x-Et)}.(-iE)Ne^{i(p.x-Et)} - Ne^{i(p.x-Et)}.(iE)N^*e^{-i(p.x-Et)}\right) \\ &= 2ENN^*\end{aligned}$$

similarly we can work out j

$$\vec{j} = 2NN^*\vec{p}$$

This approach (beginning with $E^2 = P^2 + m^2$) at first seemed problematic due to the negative solutions of $E = \pm\sqrt{P^2 + m^2}$ which leads to negative probability densities ($\rho < 0$).

Note that this problem could have been anticipated, under a Lorentz boost of velocity, v, a volume element undergoes contraction:

$$d^3x \to d^3x\sqrt{1 - v^2}$$

therefore to maintain the invariance of ρd^3x ρ transforms as:

$$\rho \to \frac{\rho}{\sqrt{1 - v^2}}$$

4.2 The ‘problem’ in the Klein-Gordon equation

There are two steps with which the negative probability densities are removed. These steps work for scalar particles. In 1934 Pauli and Weisskopf revived the Klein-Gordon equation by multiplying $j^\mu[= (\rho, \vec{j})]$ by the charge of the particle, so that qj^μ becomes

$$j^\mu_{EM} = -ie(\left[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right]; [\phi^*\nabla\phi - \phi\nabla\phi^*])$$

Now $\rho = j^0_{EM}$ is a charge density which can be negative.

4.3 Feynman-Stückelberg interpretation of E<0

The idea is that negative E solutions describe a negative energy particle propagating backwards in time, or equivalently, a positive energy anti-particle propagating forwards in time.

Consider an electron: energy, E, momentum, $\vec{p}$ and charge, -e

$$j^\mu_{EM}(e^-) = -2eN^*N(E, \vec{p})$$

for a positron

$$
\begin{aligned}
j^{\mu}_{EM}(e^{+}) &= 2eN^{*}N(E,\vec{p}) \\
&= -2eN^{*}N(-E,-\vec{p})
\end{aligned}
$$

This is the same as $j^{\mu}_{EM}(e^{-})$ but with $-E$ and $-\vec{p}$ so the emission of a positron with energy, E, is the same as the absorption of an electron energy, -E. (see 4.1 and 4.2)

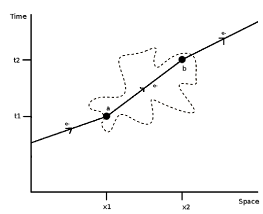

Figure 4.1: an electron scatters at a then again at b, all particles move forward in time

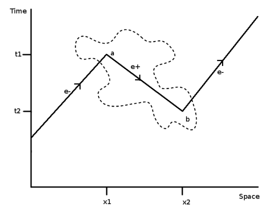

Figure 4.2: at ‘b’ an $e^{+}e^{-}$ pair is created, the e^{-} leaves the volume, the e^{+} propagates forward in time to ‘a’ where it annihilates with another e^{-}

4.4 The Propagator Approach

we want to know the quantum wave $\Psi(\vec{x}', t')$ given the wavefunction at initial co-ordinates $\Psi(\vec{x}, t)$

$$
\Psi(\vec{x}', t') = i \int d^{3}x G(\vec{x}', t'; \vec{x}, t)\Psi(\vec{x}, t)
$$

where $G(\vec{x}', t'; \vec{x}, t)$ is Greene’s function.

For $t' > t$ the wavefunction has been propagated by G from x to x'

Now consider a scattering process. An incident particle described by the plane quantum wave $\phi(\vec{x},t)$ is incident on a potential $V(\vec{x},t)$. Shroödinger's equation should describe what happens:

$$(\hat{H}_0 + V)\Psi = i\frac{\partial\Psi}{\partial t}$$
$$i\partial\Psi(\vec{x},t) - \hat{H}_0\Psi(\vec{x},t)dt = V(\vec{x},t)\Psi(\vec{x},t)dt$$

Suppose the potential acts at $\vec{x}, t$ for a short interval Δt

$$\begin{aligned}
\therefore\ i\partial\Psi(\vec{x}_1,t_1) - \int_{t_1}^{t_1+\Delta t_1}\hat{H}_0\Psi(\vec{x}_1,t_1)dt_1 &= \int_{t_1}^{t_1+\Delta t_1} V(\vec{x}_1,t_1)\Psi(\vec{x}_1,t_1)dt_1 \\
\Delta\Psi(\vec{x}_1,t_1) &= -iV(\vec{x}_1,t_1)\Psi(\vec{x}_1,t_1)\Delta t_1 \\
\because \hat{H}_0 &\approx 0 \ \ \text{(for soft scatters)} \\
\Rightarrow \Delta\Psi(\vec{x}_1,t_1) &\approx -iV(\vec{x}_1,t_1)\phi(\vec{x}_1,t_1)\Delta t_1 \\
(where)\ \Psi(\vec{x}_1,t_1) &\approx \phi(\vec{x}_1,t_1) + \Delta\Psi(\vec{x}_1,t_1)
\end{aligned}$$

Thus we have:

$$\begin{aligned}
\Delta\Psi(\vec{x}_1',t_1') &= i\int d^3xG(\vec{x}',t';\vec{x}_1,t_1)\Delta\Psi(\vec{x}_1,t_1) \\
\Psi(\vec{x}_1',t_1') &= \phi(\vec{x}_1',t_1') + \Delta\Psi(\vec{x}_1',t_1') \\
\Psi(\vec{x}_1',t_1') &= \phi(\vec{x}_1',t_1') + \int d^4x_1G_0(x';1)\phi(1) \\
\text{where}\ \ x' &= (\vec{x}',t') \\
\text{and}\ \ 1 &= (\vec{x}_1,t_1)
\end{aligned}$$

turn on the potential at $\vec{x}_2, t_2$ for Δt_2

$$\begin{aligned}
\Psi(x') = \phi(x') &+ \int d^3x_1G_0(x';1)V(1)\phi(1)\Delta t_1 \\
&+ \int d^3x_2G_0(x';2)V(2)\phi(2)\Delta t_2 \\
&+ \int\int d^3x_1d^3x_2G_0(x';2)V(2)G_0(2;1)V(1)\phi(1)\Delta t_1\Delta t_2
\end{aligned}$$

limit of continous interaction and integrating over $\Delta t_1, \Delta t_2$

$$\begin{aligned}
\Psi(x') = \phi(x') &+ \int d^4x_1G_0(x';1)V(1)\phi(1) \\
&+ \int\int d^4x_1d^4x_2G_0(x';2)V(2)G_0(2;1)V(1)\phi(1)
\end{aligned}$$

want to find $G_0(2;1)$ i.e. the propagator i.e. the propagator for all intermediate states

$$\Psi(x') = i\int_{t'>t} d^3x\, G(\vec{x}',t';\vec{x},t)\Psi(\vec{x},t)$$

This can be written in a form valid for all time:

$$\theta(t'-t)\Psi(x') = i\int d^3x\, G(\vec{x}';\vec{x})\Psi(\vec{x})$$
$$\text{where}\ \ \theta(t'-t) = \begin{cases} 1 & \text{for } t' > t \\ 0 & \text{otherwise} \end{cases}$$

Using SE:

$$\begin{aligned}
\left[i\frac{\partial}{\partial t} - \hat{H}(x')\right]\theta(t'-t)\Psi(x') &= i\delta(t'-t)\Psi(x') + \theta(t'-t)i\frac{\partial}{\partial t}\Psi(x') - \hat{H}(x')\theta(t'-t)\Psi(x') \quad \text{(LHS)} \\
&= i\delta(t'-t)\Psi(x') \\
&= i\int d^3x \left\{i\frac{\partial}{\partial t} - \hat{H}(x')\right\} G(x';x)\Psi(x) \quad \text{(RHS)}
\end{aligned}$$

Consider a particle in the absence of a potential i.e. V=0 then solve explicitly for the free particle propagator

$$i\int d^3x \left\{E - \frac{p^2}{2m}\right\} G_0(x';x)\Psi(x) \qquad \text{(RHS)}$$

use a Fourier transform to go into 4-momenta space

$$\begin{aligned}
&= i\int \frac{d^3p}{(2\pi)^3}\frac{dE}{2\pi}\left\{E - \frac{p^2}{2m}\right\} G_0(E,p)e^{ip(x'-x)}e^{iE(x'-x)}\Psi(x) \quad \text{(RHS)} \\
&= i\delta(t'-t)\Psi(x') \quad \text{(LHS)} \\
&= i\delta(t'-t)\int d^3x\Psi(x)\delta^3(\vec{x}'-\vec{x}) \\
&= i\delta^4(x'-x)\Psi(x) \\
\text{LHS} &= \text{RHS} \\
\Rightarrow \delta^4(x'-x)\Psi(x) &= i\int \frac{d^3p}{(2\pi)^3}\frac{E}{2\pi}e^{ip(x'-x)}e^{iE(x'-x)}\Psi(x) \\
\text{if:}\quad G_0 &= \frac{1}{E - \frac{p^2}{2m}}
\end{aligned}$$

The free particle propagator for real or virtual particles in momentum space is the inverse of the free SE

We shall assume that the propagator of the Klein-Gordon, Dirac and Proca equation are obtained by inverting the appropriate equation

Chapter 5

Calculations

5.1 Spin-less Electron Muon Scattering

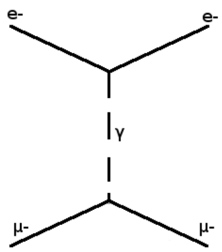

Figure 5.1: electron muon scattering in the t channel

We start with Klien-Gordon equation include the electromagnetic interaction

In electrodynamics the motion of a particle of charge, -e, in an EM potential, A^μ, is obtained by substition of $p^\mu \rightarrow p^\mu + eA^\mu$ so:

$$
\begin{aligned}
E^2 - p^2 &= m^2 \\
\text{or:} \quad p_\mu p^\mu &= m^2 \\
\text{becomes:} \quad (p_\mu + eA_\mu)(p^\mu + eA^\mu) &= m^2
\end{aligned}
$$

In QM this is:

$$
\begin{aligned}
(i\partial_\mu + eA_\mu)(i\partial^\mu + eA^\mu)\phi &= m^2\phi \\
(-\partial_\mu\partial^\mu + ie\partial_\mu A^\mu + ieA_\mu\partial^\mu + e^2 A_\mu A^\mu)\phi &= m^2\phi \\
\Rightarrow \partial_\mu\partial^\mu\phi + m^2\phi &= -V\phi
\end{aligned}
$$

where V is the EM perturbation.

The transition amplitude:

$$\begin{aligned} T_{fi} &= -i \int d^4x \phi_f^*(x) V(x) \phi_i(x) \\ &= -i \int d^4x \phi_f^*(x)(-ie)(\partial_\mu A^\mu + A_\mu \partial^\mu)\phi_i(x) \end{aligned}$$

Where the second order term in V(x) has been dropped, transform the first term so that ∂_μ acts upon ϕ_f^*

$$\begin{aligned} \int \phi_f^* \partial_\mu A^\mu \phi_i d^4x &= \left[\phi_f^* A^\mu \phi_i\right]_{-\infty}^{\infty} - \int d^4x \partial_\mu \phi_f^* A^\mu \phi_i \\ \Rightarrow T_{fi} &= -e \int d^4 \left\{\phi_f^* A_\mu \partial^\mu \phi_i - \partial_\mu \phi_f^* A_\mu \phi_i\right\} \\ \left[\phi_f^* A^\mu \phi_i\right]_{-\infty}^{\infty} &= 0 \text{ as } \lim_{\to\infty} A^\mu \to 0 \\ T_{fi} &= -i \int d^4x A^\mu j_\mu^{fi} \end{aligned}$$

at top vertex we have particle A described by: $\phi_A = N_A e^{-ip_A x}$ changing to particle C: $\phi_C = N_C e^{-ip_C x}$

$$\begin{aligned} \Rightarrow j_\mu^{CA}(x) &= -ie\left\{N_C^* e^{ip_C x} N_A e^{-ip_A x}(-ip_A)_\mu - N_C^* e^{ip_C x}(ip_C)_\mu N_A e^{-ip_A x}\right\} \\ &= -eN_C^* N_A (p_A + p_C)_\mu e^{i(p_C - p_A)x} \\ j_\mu^{CA} &= j_\mu^{fi} \end{aligned}$$

For A^μ use Maxwell's equations $\Box^2 A^\mu(x) = j^\mu(x)$ we can find a solution to this by inspection. The A^μ field arises from the EM current at the bottom vertex

$$j_{DB}^\mu = -eN_D^* N_B (p_B + p_D)^\mu e^{i(p_D - p_B)x}$$

We can guess the solution:

$$\begin{aligned} A^\mu &= \frac{-g_{\mu\nu} j_\nu^{DB}}{q^2} \\ &\text{where: } q = p_D - p_B \end{aligned}$$

Check:

$$\begin{aligned}
\partial_\mu \partial^\mu \frac{-1}{q^2} j^\mu_{DB} &= \frac{-1}{q^2} j^\mu_{DB} (i(p_D - p_B))^2 \\
&= \frac{-1}{q^2} j^\mu_{DB} (-q^2) \\
&= j^\mu_{DB} \\
\Rightarrow T_{fi} &= -i \int d^4x \, j^{CA}_\mu (x) A^\mu \\
&= -i \int d^4x \, \left\{ -e N^*_C N_A (p_A + p_C)_\mu e^{i(p_C - p_A)x} \right\} \left\{ \frac{-1}{q^2} (-e) N^*_D N_B (p_D + p_B)^\mu e^{i(p_D - p_B)x} \right\} \\
&= \frac{ie^2}{q^2} \int d^4x N^*_C N_A N^*_D N_B (p_A + p_C)_\mu (p_D + p_B)^\mu e^{i(p_C + p_D - p_B - p_A)}
\end{aligned}$$

Chapter 6

Definition of cross-section

We imagine this happening in an interaction volume V. We normalise such that there are 2E particles of each kind in the volume. If:

$$\begin{aligned}
\phi_A &= N_A e^{-ip_A x} \\
\Rightarrow \int \rho d^3x &= \int 2E\phi_A^* \phi_A \\
&= 2EN_A^* N_A V \\
&= 2E \\
\text{where: } N_A &= \frac{1}{\sqrt{V}}
\end{aligned}$$

The number of transitions per unit time per unit volume, w_{fi} is given by:

$$w_{fi} = \frac{T_{fi}^* T_{fi}}{tV}$$

the cross section is given by:

$$\begin{aligned}
\sigma &= \frac{w_{fi} \times (\text{number of final states})}{\text{initial flux}} \\
T_{fi}^* T_{fi} &= \frac{e^4}{q^4} \iint \frac{1}{V^4} \left\{(p_A + p_C)_\mu (p_D + p_B)^\mu\right\}^2 e^{i(p_C + p_D - p_B - p_A)} e^{-i(p_C + p_D - p_B - p_A)} d^4x \, d^4x' \\
\Rightarrow W_{fi} &= \frac{e^4}{q^4} \frac{1}{V^4} \left\{(p_A + p_C)_\mu (p_D + p_B)^\mu\right\}^2 \left\{ \frac{\delta^4(p_C + p_D - p_B - p_A)(2\pi)^4 \cancel{tV}}{\cancel{tV}} \right\}
\end{aligned}$$

What is the number of final states? Each particle in the final state has a 3-momentum between (p_c and $p_c + d^3p_c$) and (p_D and $p_D + d^3p_D$) as well as an energy between (E_C and $E_C + dE_C$) and (E_D and $E_D + dE_D$).

These particles are imagined to be waves which enter and leave the interaction volume. The propagator particle only exists within the interaction volume. It must have the wavefunction of a particle in an infinite square well. This constraint leads to the quantization of momentum.

Suppose $x = 0$ is the entrance and $x = \ell$ is the exit then:

$$p_x \ell_x = 2\pi n_x$$

where n_x is an integer. What then is the separation Δp_x of the modes?

$$\begin{aligned} (p_x + \Delta p_x)\ell_x &= 2\pi(n_x + 1) \\ \Rightarrow \Delta p_x &= \frac{2\pi}{\ell_x} \end{aligned}$$

Therefore the number of final states are:

$$\begin{aligned} N_{\text{final states}} &= \frac{dP_x}{2\pi}\frac{dP_y}{2\pi}\frac{dP_z}{2\pi}.\ell_x\ell_y\ell_z \\ &= \frac{V}{(2\pi)^3}d^3p \end{aligned}$$

There are 2E particles in V $\Rightarrow$ the number of final states becomes:

$$N_{\text{final states}} = \frac{V}{(2\pi)^3 2E}d^3p$$

The initial flux (A and B particles) factor $\rho_A\rho_B v_{AB}$ where ρ is the density of particles in the volume, V, with relative velocity v_{AB}

$$\begin{aligned} \text{flux} &= \frac{2E_A}{V}\frac{2E_B}{V}v_{AB} \\ &= \frac{2E_A}{V}\frac{2E_B}{V}\left(\frac{P_A}{E_A} - \frac{P_B}{E_B}\right) \\ \text{in C.M.S:}\;\; P_A &= P_B \\ \Rightarrow \text{flux} &= \frac{4E_AE_B}{V^2}\frac{E_BP_A - E_AP_B}{E_AE_B} \\ &= \frac{4}{V^2}P_A(E_A + E_B) \\ &= \frac{4}{V^2}P_A\sqrt{s} \end{aligned}$$

6.1 Combining all the terms of the cross-section

$$\begin{aligned} d\sigma &= \frac{e^4}{q^4}\frac{(2\pi)^4}{V^4}\left\{(P_A + P_C)_\mu(P_D + P_B)^\mu\right\}^2 \delta(P_C + P_D - P_A - P_B)\frac{Vd^3P_C}{2E_C(2\pi)^3}\frac{Vd^3P_D}{2E_D(2\pi)^3} \\ \text{where: } w_{fi} &= \frac{e^4}{q^4}\frac{(2\pi)^4}{V^4}\left\{(P_A + P_C)_\mu(P_D + P_B)^\mu\right\}^2 \delta(P_C + P_D - P_A - P_B) \\ \text{The final states} &= \frac{Vd^3P_C}{2E_C(2\pi)^3}\frac{Vd^3P_D}{2E_D(2\pi)^3} \\ \text{and flux} &= \frac{V^3}{4P_A\sqrt{S}} \end{aligned}$$

Let's consider the number of particles and simplify per unit volume per particle in the final state, we have:

$$dQ = \frac{V}{2E_C}\frac{d^3P_C}{(2\pi)^3}\frac{V}{2E_D}\frac{d^3P_D}{(2\pi)^3}(2pi)^4\delta(P_C + P_D - P_A - P_B)$$

where dQ is a Lorentz invariant phase space (excluding some unimportant terms) Convert this to centre of mass frame:

$$dQ = (2\pi)^4\delta(\sqrt{s} - (E_C + E_D))\ \delta(P_C + P_D)\frac{d^3P_C}{2E_C(2\pi)^3}\frac{d^3P_D}{2E_D(2\pi)^3}V^2$$

reduce by integrating over d^3P_D

$$\begin{aligned} dQ &= (2\pi)^4\delta(\sqrt{s} - (E_C + E_D))\frac{d^3P_C}{2E_C(2\pi)^3}\frac{1}{2E_D(2\pi)^3}V^2 \\ \int d^3P_C &= 2\pi P_C \sin\theta P_C d\theta dP_C \\ &= P_C^2\, dP_C\, d\Omega \\ \Rightarrow dQ &= \frac{V^2}{4\pi^2}\delta(\sqrt{s} - (E_C + E_D))\frac{1}{4E_CE_D}\ P_C^2\, dP_C\, d\Omega \end{aligned}$$

where θ is the scattering angle (**??**) in centre of mass system:

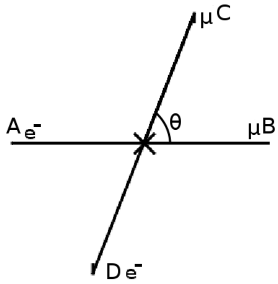

Figure 6.1: schematic of $e^-\mu^-$ scattering

$$\begin{aligned}
\sqrt{s} &= E_C + E_D \\
&= (P_C^2 + m_C^2)^{\frac{1}{2}} + (P_D^2 + m_D^2)^{\frac{1}{2}} \\
\text{in CMS} \;\; P_C &= -P_D \\
\Rightarrow \sqrt{s} &= (P_C^2 + m_C^2)^{\frac{1}{2}} + (P_C^2 + m_D^2)^{\frac{1}{2}} \\
d\sqrt{s} &= \frac{1}{2}\frac{2P_C dP_C}{\sqrt{P_C^2 + m_C^2}} + \frac{1}{2}\frac{2P_C dP_C}{\sqrt{P_C^2 + m_D^2}} \\
&= \frac{P_C(E_D + E_C)}{E_C E_D} dP_C \\
\Rightarrow dQ &= \frac{V^2}{4\pi^2}\delta(\sqrt{s} - (E_C + E_D))\frac{1}{4}d\Omega\frac{P_C}{E_C + E_D}d\sqrt{s}
\end{aligned}$$

Integrate over $\sqrt{s}$, remember $\sqrt{s} = E_C + E_D$

$$\Rightarrow dQ = \frac{V^2}{16\pi^2}d\Omega\frac{P_C}{\sqrt{s}}$$

substitute back into $d\sigma$

$$\begin{aligned}
d\sigma &= \frac{e^4}{q^4}\frac{1}{V^4}\left\{(P_A + P_C)_\mu (P_B + P_D)^\mu\right\}^2 dQ \frac{V^2}{4P_A\sqrt{s}} \\
dQ \rightarrow \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\frac{P_C}{P_A}\frac{e^4}{q^4}\left\{(P_A + P_C)_\mu (P_B + P_D)^\mu\right\}^2
\end{aligned}$$

Work out the cross-section in the massless limit:

$$\begin{aligned}
\text{remember:} \;\; q^2 &= (P_A - P_C)^2 \\
&= (E_A - E_C, \vec{P}_A - \vec{P}_C)^2 \\
\text{as:} \;\; m \rightarrow 0 \Rightarrow |E| \sim |\vec{p}| \\
\Rightarrow q^2 &= E_A^2 + E_C^2 - 2E_A E_C - E_C^2 - E_A^2 + 2E_A E_C \cos\theta \\
&= -2E_A E_C(1 - \cos\theta) \\
\Rightarrow q^4 &= 4E_A^2 E_C^2 (1 - \cos\theta)^2 \\
\left\{(P_A + P_C)_\mu (P_B + P_D)^\mu\right\}^2 &= \left\{P_A.P_B + P_A.P_D + P_C.P_B + P_C.P_D\right\}^2 \\
\text{using:} \;\; P_A &= (P, -\vec{P}) \\
P_B &= (P, -\vec{P}) \\
P_C &= (P, \vec{P}') \\
P_D &= (P, -\vec{P}') \\
\Rightarrow \left\{(P_A + P_C)_\mu (P_B + P_D)^\mu\right\}^2 &= (6P^2 + 2P^2\cos\theta) \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{e^4}{64\pi^2 s}\left(\frac{3 + \cos\theta}{1 - \cos\theta}\right)^2
\end{aligned}$$

6.1.1 Note on decays

What is the number of states and the flux factor for decays?

$$A \rightarrow B + C$$

The number of final states factor is as before:

$$N_{\text{final states}} = \frac{1}{16\pi^2}\frac{P_C V^2}{m_A} d\Omega$$
$$\text{as:} \quad \sqrt{s} = m_a$$

for decaying particles we have $2E_A$ i.e. $2m_A$ in CMS in volume, V

$$\therefore \rho = \frac{2m_A}{V}$$

makes the combined factor:

$$\begin{aligned}
&= \frac{1}{\rho} N_{final} \\
&= \frac{V}{2m_A}\frac{1}{16\pi^2}\frac{P_C V^2}{m_A} d\Omega \\
&= \frac{1}{32\pi^2}\frac{P_C V^3}{m_A^2} d\Omega
\end{aligned}$$

This gives the decay rate:

$$\frac{d\Gamma}{d\Omega} = |T_{fi}|^2 \frac{1}{32\pi^2}\frac{P_C V^3}{m_A^2}$$

Chapter 7

Relativistic spin $\frac{1}{2}$ particles: Dirac equation

7.1 Non-relativistic

We have 2 spin states: up $+\frac{1}{2}$ and down $-\frac{1}{2}$. In spin states we have spin operators given by:

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma}$$

where σ are the Pauli matrices.

Spin algebra is the same as that of orbital angular momentum

$$L^2 = \ell(\ell+1)\hbar^2$$

In spin angular momentum:

$$s^2 = s(s+1)\hbar^2$$

the Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\text{with: } \sigma_x = \sigma_y = \sigma_z = \mathbb{I}_{2\times 2}$$

communication relation:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where ϵ_{ijk} is the Levi-Civita tensor,

$$\epsilon_{ijk} = \begin{cases} +1 & \text{cyclic permutations} \\ -1 & \text{anti-cyclic permutations} \\ 0 & \text{otherwise} \end{cases}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}_{2\times 2}$$

$$\Rightarrow \sigma_i\sigma_j = \delta_{ij}\mathbb{I}_{2\times 2} + i\epsilon_{ijk}\sigma_k$$

consider:

$$\sigma_i A_i \sigma_j B_j = A_i B_j \delta_{ij} + i\epsilon_{ijk}\sigma_k A_i B_j$$

$$\therefore\ (\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A}\times\vec{B})$$

$$\text{if: } \vec{A} = -\vec{B} = \vec{p}$$

$$\Rightarrow (\vec{\sigma}.\vec{p})^2 = \vec{p}^{\,2}$$

if you were to start from $\frac{1}{2}(\vec{\sigma}.\vec{p})^2 + V = E$ rather than $\frac{p^2}{2m} + V = E$ and include the electromagnetic coupling you would get the correct gyro-magnetic ratio, g.

From the Dirac equation, g=2, experimentally this has been found as g = 2.00232.

7.2 The Dirac Equation

To avoid the problems of negative probabilities we interpret the E¡0 solutions of the Klein-Gordon, Dirac proposed, an equation linear in $\frac{\partial}{\partial t}$

$$H\Psi = (\vec{\alpha}.\vec{p} + \beta.m)\Psi$$

where α and β are 4×4 matrices and the solutions Ψ are multi-component objects.

The formation must be consistent with

$$\begin{aligned} E^2\Psi &= (p^2+m^2)\Psi \\ \text{if then: } E\Psi &= (\alpha_i p_i + \beta m)\Psi \\ \text{then: } E^2\Psi &= (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)\Psi \\ &= (\alpha_i p_i + \beta m)^2\Psi \\ &= (\alpha_i\alpha_j p_i p_j + (\alpha_i\beta + \beta\alpha_j)p_i m + \beta^2 m^2)\Psi \\ &= \left\{\left(\frac{\alpha_i\alpha_j + \alpha_j\alpha_i}{2}\right)p_i p_j + (\alpha_i\beta + \beta\alpha_j)p_j m + \beta^2 m^2\right\}\Psi \end{aligned}$$

$$\begin{aligned} \Rightarrow \alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{ij} \\ \alpha_i\beta + \beta\alpha_j &= 0 \\ \beta^2 &= \mathbb{I} \end{aligned}$$

7.2.1 Properties of α and β

- α and β are Hermitian $\alpha = \alpha^\dagger$ $\beta = \beta^\dagger$
- $\alpha^2 = \mathbb{I}$ $\beta^2 = \mathbb{I}$
- α and β are traceless
- $$\alpha_i\beta + \beta\alpha_i = 0$$
  multiply from right by β:
  $$\begin{aligned} 0 &= \alpha_i\beta^2 + \beta\alpha_i\beta \\ \therefore \alpha_i\mathbb{I} &= -\beta\alpha_i\beta \\ \mathrm{Tr}(\alpha_i) &= -\mathrm{Tr}(\beta\alpha_i\beta) \\ \Rightarrow &= -\mathrm{Tr}(\alpha_i\beta^2) \qquad \text{may move elements of a trace cyclically} \\ &= -\mathrm{Tr}(\alpha_i) \\ &= 0 \end{aligned}$$

Using these properties we find common choices for α_i and β are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$
$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

where σ_i and $\mathbb{I}$ are 2×2 matrices

7.3 The covariant form of the Dirac Equation

We have:

$$\begin{aligned} E\Psi &= (\alpha_i p_i + \beta m)\Psi \\ E &\rightarrow i\frac{\partial}{\partial t} \\ p &\rightarrow -i\nabla \\ i\frac{\partial\Psi}{\partial t} &= -i\alpha_i\nabla\Psi + \beta m\Psi \\ i\beta\frac{\partial\Psi}{\partial t} &= -i\beta\alpha_i\nabla\Psi + \beta^2 m\Psi \qquad \beta \text{ from right} \end{aligned}$$

let $\beta = \gamma^0$ and $\beta\alpha_i = \gamma^i$

$$\begin{aligned} \underbrace{i\gamma^0\frac{\partial\Psi}{\partial t} + i\gamma^i\nabla\Psi}_{(i\gamma^\mu\partial_\mu} \quad -m\Psi &= 0 \\ (i\gamma^\mu\partial_\mu \quad -m)\Psi &= 0 \end{aligned}$$

where:

$$\gamma^\mu = (\gamma^0, \gamma^i)$$
$$\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$$

7.3.1 Properties of γ

- γ^0 is Hermitian
- γ^i is anti-Hermitian i.e. $(\gamma^i)^\dagger = -\gamma^i$

$$\begin{aligned} \gamma^i &= \beta\alpha^i \\ (\gamma^i)^\dagger &= (\beta\alpha^i)^\dagger \\ &= (\alpha^i)^\dagger\beta^\dagger \\ &= \alpha^i\beta \\ &= -\beta\alpha^i \\ &= -\gamma^i \end{aligned}$$

- $(\gamma^0)^2 = \mathbb{I}$ since $\beta^2 = \mathbb{I}$
- $(\gamma^i)^2 = -\mathbb{I}$ since:

$$\begin{aligned} (\gamma^i)^2 &= \gamma^i\gamma^i \\ &= \beta\alpha^i\beta\alpha^i \\ &= -\beta\alpha^i\alpha^i\beta \\ &= -\beta\beta \\ &= -\mathbb{I} \end{aligned}$$

7.4 The adjoint Dirac Equation and conserved current

As we are now dealing with a matrix equation we have to consider the Hermitian, rather than the complex, conjugate in order to find currents.

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \tag{1}$$
$$(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\nabla - m)\Psi = 0$$

Take the Hermitian:

$$\begin{aligned}
(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x^i} - m)^\dagger\Psi^\dagger &= 0\\
-i\frac{\partial\Psi^\dagger}{\partial t}(\gamma^0)^\dagger - i\frac{\partial\Psi^\dagger}{\partial x^i}(\gamma^i)^\dagger - m\Psi^\dagger &= 0\\
-i\frac{\partial\Psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial\Psi^\dagger}{\partial x^i}(-\gamma^i) - m\Psi^\dagger &= 0\\
&\leftarrow \times\gamma^0\\
-i\frac{\partial\Psi^\dagger}{\partial t}\gamma^0\gamma^0 + i\frac{\partial\Psi^\dagger}{\partial x^i}\gamma^i\gamma^0 - m\Psi^\dagger\gamma^0 &= 0\\
\text{remember: }\ \gamma^i\gamma^0 &= -\gamma^0\gamma^i\\
\Rightarrow -i\frac{\partial\Psi^\dagger}{\partial t}\gamma^0\gamma^0 - i\frac{\partial\Psi^\dagger}{\partial x^i}\gamma^0\gamma^i - m\Psi^\dagger\gamma^0 &= 0
\end{aligned}$$

now define the adjoint as:

$$\overline{\Psi} = \Psi^\dagger\gamma^0$$

$$\begin{aligned}
\Rightarrow -i\frac{\partial\overline{\Psi}}{\partial t}\gamma^0 - i\frac{\partial\overline{\Psi}}{\partial x^i}\gamma^i - m\overline{\Psi} &= 0\\
\Rightarrow i\partial_\mu\overline{\Psi}\gamma^\mu + m\overline{\Psi} &= 0 \qquad (2)
\end{aligned}$$

derive the continuity equation, $\partial_\mu j^\mu = 0$:

$$\begin{aligned}
\overline{\Psi}i\gamma^\mu\partial_\mu\Psi - \overline{\Psi}m\Psi &= 0 && \overline{\Psi}\times(1)\\
\Rightarrow i\partial_\mu\overline{\Psi}\gamma^\mu\Psi + m\overline{\Psi}\Psi &= 0 && (2)\times\Psi\\
i\overline{\Psi}\gamma^\mu\partial_\mu\Psi + i\partial_\mu\overline{\Psi}\gamma^\mu\Psi &= 0 && \overline{\Psi}\times(1)+(2)\times\Psi\\
\Rightarrow \partial_\mu(\overline{\Psi}\gamma^\mu\Psi) &= 0
\end{aligned}$$

so we associate $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ and satisfies the continuity equation. We identify j^μ with the probability and flux densities, ρ and $\vec{j}$.

The probability density is:

$$\begin{aligned}
\rho &= \overline{\Psi}\gamma^0\Psi\\
&= \Psi^\dagger\Psi\\
&= |\Psi|^2
\end{aligned}$$

i.e. positive and definite as required.

7.5 Free-particle solution to the Dirac Equation

Look for solutions of the form $\Psi = U(p)e^{-ip.x}$ and substitute into the Dirac Equation:

$$\begin{aligned}
(i\gamma^\mu \partial_\mu - m)\Psi &= 0 \\
(i\gamma^\mu \partial_\mu - m)U(p)e^{-ip.x} &= 0 \\
(i\gamma^\mu(-ip_\mu) - m)U(p)e^{-ip.x} &= 0 \\
(\gamma^\mu p_\mu - m)U(p) &= 0 \\
(\not{p} - m)U(p) &= 0 \qquad\qquad \text{Dirac slash notation}
\end{aligned}$$

to get a solution for $U(P)$ we write the above in terms of β and α matrices

$$\begin{aligned}
(\gamma^0 E - \gamma^i \vec{p}_i - m)U(p) &= 0 \\
((\gamma^0)^2 E - \gamma^0\gamma^i \vec{p}_i - \gamma^0 m)U(p) &= 0 \qquad\qquad \gamma^0 \times | \\
(E - \alpha^i p_i - \beta m)U(p) &= 0
\end{aligned}$$

for particles at rest set $p_i = 0$

$$\begin{aligned}
(\mathbb{I}E - \beta m)U(p) &= 0 \\
m\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} U(p) &= \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} EU(p)
\end{aligned}$$

solutions exist if:

$$\begin{aligned}
\begin{vmatrix} (m-E)\mathbb{I} & 0 \\ 0 & -(m+E)\mathbb{I} \end{vmatrix} &= 0 \\
\Rightarrow (m-E)(m-E)(m+E)(m+E) &= 0
\end{aligned}$$

so there are 4 eigenvalues:$E = m, m, -m, -m$ i.e. negative energy solutions remain. We now associate U_1 and U_2 with the $E = m$ solutions while u_3 and u_4 are associated with the $E = -m$ solutions.

For the case $p \neq 0$ find the solution from

$$\begin{aligned}
(\alpha.p + \beta m)U &= EU \\
\left[\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} .p + m\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\right]\begin{pmatrix} U_A \\ U_B \end{pmatrix} &= E\begin{pmatrix} U_A \\ U_B \end{pmatrix} \\
\Rightarrow \sigma.pU_B + mU_A &= EU_A \\
\sigma.pU_A - mU_B &= EU_B \\
\Rightarrow U_A &= \frac{\sigma.p}{E-m}U_B \\
U_B &= \frac{\sigma.p}{E+m}U_A
\end{aligned}$$

For E>0:

$$U_B = \frac{\sigma.p}{E+m} U_A \qquad \text{as at rest } E = m$$

$$U_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow U_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{\sigma.p}{E+m} \\ 0 \end{pmatrix} \quad \text{and} \quad U_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma.p}{E+m} \end{pmatrix}$$

Where N is a renormalisation constant. Now for E<0:

$$\begin{aligned} U_A &= \frac{\sigma.p}{E-m} U_B \\ &= \frac{\sigma.p}{-|E|-m} U_B \\ &= \frac{-\sigma.p}{|E|+m} U_B \end{aligned}$$

$$\Rightarrow U_3 = N \begin{pmatrix} \frac{-\sigma.p}{|E|+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad U_4 = N \begin{pmatrix} 0 \\ \frac{-\sigma.p}{|E|+m} \\ 0 \\ 1 \end{pmatrix}$$

In summary all the above comes from $(\not{p} - m)U = 0$ and we now associate the negative energy solutions, U_3 and U_4 such that they describe positron solutions propagating backwards in time with a propagation factor: e^{ipx} compared to $e^{-}ipx$ for electrons.

$$U^{(3,4)}(-p)e^{-i(-p)x} \equiv V^{(2,1)}(p)e^{ipx}$$

$$V_2 = N \begin{pmatrix} \frac{\sigma.p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_1 = N \begin{pmatrix} 0 \\ \frac{\sigma.p}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

Considering the original equation with an electron of energy, -E, and momentum, -p, we have:

$$\begin{aligned} (-\not{p} - m)U(-p) &= 0 \\ (\not{p} + m)V(p) &= 0 \end{aligned}$$

7.5.1 Orthonormality of the spinors

$$\Psi_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{\sigma.p}{E+m} \\ 0 \end{pmatrix} e^{-ip.x} \quad \text{and} \quad \Psi_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma.p}{E+m} \end{pmatrix} e^{-ip.x}$$

For Orthonormality:

$$\int \Psi_1^\dagger \Psi_2 d^3x = 0$$

We normalise to $2E$ particles in a volume, V:

$$\int \Psi_1^\dagger \Psi_1 d^3x = 2E$$

$$\int N^* N \left\{ 1 + \left(\frac{\sigma.p}{E+m} \right)^2 \right\} d^3x = 2E \qquad (\sigma.p)^\dagger = (\sigma.p)$$

$$\int N^* N \left\{ 1 + \left(\frac{p^{\ 2}}{(E+m)^2} \right)^2 \right\} d^3x = 2E$$

$$\int N^* N \left\{ \frac{(E+m)^2 + (E+m)(E-m)}{(E+m)^2} \right\} d^3x = 2E$$

$$\int N^* N \left\{ \frac{E+m+E-m}{(E+m)} \right\} d^3x = 2E$$

$$\int N^* N \left\{ \frac{2E}{(E+m)} \right\} d^3x = 2E$$

$$\Rightarrow \frac{N^* N}{E+m} V = 1$$

$$\Rightarrow N = \sqrt{\frac{E+m}{V}}$$

7.5.2 Spin

Neither the angular momentum nor the spin angular momentum commune with the Dirac Hamiltonian. We seek an operator that which commutes, whose eigenvalues can be taken to distinguish the states.

Using the Hermitian form of the Dirac Equation:

$$H \begin{pmatrix} U_A \\ U_B \end{pmatrix} = (\alpha.p + \beta m) \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$
$$= \begin{pmatrix} m\mathbb{I} & \sigma.p\mathbb{I} \\ \sigma.p\mathbb{I} & -m\mathbb{I} \end{pmatrix} \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$
$$= E \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$

By inspection H commutes with $\sigma.p\mathbb{I}$ check:

$$
\begin{aligned}
[H, \sigma.p\mathbb{I}] &= 0 \\
\begin{pmatrix} m\mathbb{I} & \sigma.p\mathbb{I} \\ \sigma.p\mathbb{I} & -m\mathbb{I} \end{pmatrix}\begin{pmatrix} \sigma.p\mathbb{I} & 0 \\ 0 & \sigma.p\mathbb{I} \end{pmatrix} - \begin{pmatrix} \sigma.p\mathbb{I} & 0 \\ 0 & \sigma.p\mathbb{I} \end{pmatrix}\begin{pmatrix} m\mathbb{I} & \sigma.p\mathbb{I} \\ \sigma.p\mathbb{I} & -m\mathbb{I} \end{pmatrix} &= 0 \\
\begin{pmatrix} m\sigma.p\mathbb{I} & (\sigma.p)^2\mathbb{I} \\ (\sigma.p)^2\mathbb{I} & -m\sigma.p\mathbb{I} \end{pmatrix} - \begin{pmatrix} m\sigma.p\mathbb{I} & (\sigma.p)^2\mathbb{I} \\ (\sigma.p)^2\mathbb{I} & -m\sigma.p\mathbb{I} \end{pmatrix} &= 0
\end{aligned}
$$

We define the helicity operator as:

$$\frac{1}{2}\vec{\sigma}.\hat{\vec{p}} = \frac{1}{2}\frac{\vec{\sigma}.\vec{p}}{|\vec{p}|}$$

i.e. the helicity is the projection of spin in the direction of motion. Its eigenvalues are:

$$
\begin{aligned}
&+\frac{1}{2} \Rightarrow\rightarrow && \text{positive helicity} \\
&-\frac{1}{2} \Leftarrow\rightarrow && \text{negative helicity}
\end{aligned}
$$

for an arbitrary vector, $\vec{p}$, angle θ from z-axis and ϕ in the x-y plane (measured from the x-axis).

$$
\begin{aligned}
\hat{\vec{p}} &= \sin\theta\cos\phi\,\hat{\vec{i}} + \sin\theta\sin\phi\,\hat{\vec{j}} + \cos\theta\,\hat{\vec{k}} \\
\Rightarrow \vec{\sigma}.\hat{\vec{p}} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta \\
\Rightarrow \frac{1}{2}\vec{\sigma}.\hat{\vec{p}}\begin{pmatrix} U_A \\ U_B \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} U_A \\ U_B \end{pmatrix}
\end{aligned}
$$

so the eigen values equation is:

$$
\begin{aligned}
\frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} U_A \\ U_B \end{pmatrix} &= \lambda\begin{pmatrix} U_A \\ U_B \end{pmatrix} \\
\begin{vmatrix} \cos\theta - 2\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - 2\lambda \end{vmatrix} &= 0 \\
-(\cos\theta - 2\lambda)(\cos\theta + 2\lambda) - \sin^2\theta &= 0 \\
\Rightarrow -1 + 4\lambda^2 &= 0 \\
\Rightarrow \lambda &= \pm\frac{1}{2}
\end{aligned}
$$

7.5.3 γ^5 Matrix

This matrix is introduced to simplify notation:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

it can be shown that:

$$(\gamma^5)^\dagger = \gamma^5$$
$$(\gamma^5)^2 = \mathbb{I}$$

In Dirac-Pauli representation

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

Consider the effects of γ^5 on the Dirac Equation solutions:

$$\begin{aligned}
\text{Let} \quad \chi &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \gamma^5 \begin{pmatrix} U_A \\ U_B \end{pmatrix} &= \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \frac{\sigma.p}{E+m}\chi \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma.p}{E+m}\chi \\ \chi \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma.p}{2|p|}\chi \\ \chi \end{pmatrix} && \text{in the massless limit } E \to |p| \\
&= \begin{pmatrix} \sigma.\hat{p}\chi \\ \chi \end{pmatrix} \\
&= \begin{pmatrix} \sigma.\hat{p}\chi \\ (\sigma.\hat{p})^2\chi \end{pmatrix} && (\sigma.\hat{p})^2 = 1 \text{ as } \hat{p} \text{ is unitary} \\
&= \sigma.\hat{p} \begin{pmatrix} \chi \\ \sigma.\hat{p}\chi \end{pmatrix} \\
&= \sigma.\hat{p} \begin{pmatrix} \chi \\ \frac{\sigma.p}{E+m}\chi \end{pmatrix} \\
&= \sigma.\hat{p} \begin{pmatrix} U_A \\ U_B \end{pmatrix}
\end{aligned}$$

So in the massless limit (i.e. as $E \to \infty$ or as $m \to 0$) γ^5 becomes the helicity operator, we can now define two further operators:

$$P_R = \frac{1}{2}(1 + \gamma^5)$$
$$P_L = \frac{1}{2}(1 - \gamma^5)$$

which are the right and left hand projection operators.

In the general ($m \neq 0$)case we call $\frac{1}{2}(1 + \gamma^5)$ the right-handed chirality operator. If m is small it is essentially a right-handed helicity state but will also contain a small left-handed component.

7.6 Completeness relations

Completeness relationships are extensively used in the evaluation of Feynman diagrams.

$$\sum_{S=1,2} U_S(P)\overline{U}_S(P) = A$$

$$row \times column = scalar$$

$$column \times row = matrix \qquad \text{(outter product)}$$

$$\Rightarrow A = \sum_{s=1,2} NN^* \begin{pmatrix} \chi_s\chi_s^\dagger & -\left(\frac{\sigma.p}{E+m}\right)^\dagger \chi_s\chi_s^\dagger \\ \left(\frac{\sigma.p}{E+m}\right)\chi_s\chi_s^\dagger & \left(\frac{-(E^2-m^2)}{(E+m)^2}\right)\chi_s\chi_s^\dagger \end{pmatrix}$$

$$\begin{aligned} \sum_{s=1,2}\chi_s\chi_s^\dagger &= \begin{pmatrix}1\\0\end{pmatrix}(1,0) + \begin{pmatrix}0\\1\end{pmatrix}(0,1) \\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\ &= \mathbb{I} \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum_{S=1,2} U_S(P)\overline{U}_S(P) &= \begin{pmatrix} \mathbb{I} & \frac{-\sigma.p}{E+m}\mathbb{I} \\ \frac{-\sigma.p}{E+m}\mathbb{I} & \frac{-(E^2-m^2)}{(E+m)^2}\mathbb{I} \end{pmatrix}.(E+m) \qquad \text{NB: } NN^* = (E+m) \\ &= \begin{pmatrix} (E+m)\mathbb{I} & -\sigma.p\mathbb{I} \\ -\sigma.p\mathbb{I} & (m-E)\mathbb{I} \end{pmatrix} \qquad \text{(A)} \end{aligned}$$

Now:

$$\begin{aligned} (\not{P}+m) &= \gamma^\mu P_\mu + m\mathbb{I} \\ &= \gamma^0 E - \gamma^i \vec{p}_i + m\mathbb{I} \\ &= \begin{pmatrix} E\mathbb{I} & 0 \\ 0 & -E\mathbb{I}\end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}_i \\ \vec{\sigma}_i & 0\end{pmatrix}\vec{p}_i + \begin{pmatrix} m\mathbb{I} & 0 \\ 0 & m\mathbb{I}\end{pmatrix} \\ &= \begin{pmatrix} (E+m)\mathbb{I} & -\sigma.p\mathbb{I} \\ \sigma.p\mathbb{I} & (m-E)\mathbb{I} \end{pmatrix} \qquad \text{(B)} \end{aligned}$$

We can see from this that A = B

$$\Rightarrow \sum_{S=1,2} U_S(P)\overline{U}_S(P) = (\not{P}+m)$$

and similarly we can see that:

$$\sum_{S=1,2} V_S(P)\overline{V}_S(P) = (\not{P}-m)$$

Chapter 8

Electron-Muon Scattering

We will now calculate a real cross-section, i.e. with no constructs such as spin-0, but rather something which could be compared to experiment. We will start with $e^-\mu^+$ scattering as this can be simply modified to also calculate electron-quark scattering and e^+e^- annihilation to a $\mu^+\mu^-$ pair.

8.1 An electron in an electromagnetic field

The Dirac Equation $(\vec{\alpha}\vec{p} + \beta m)\Psi = E\Psi$. Now substitute $p^\mu \rightarrow p^\mu + eA^\mu$

$$\begin{aligned} \therefore E &\rightarrow E + eV \\ p^i &\rightarrow p^i + eA^i \\ (\vec{\alpha}_i\vec{p}_i + \beta m + e\{\alpha A - V\mathbb{I}\})\Psi &= E\Psi \\ \text{where: } e\{\alpha A - V\mathbb{I}\} &= \text{Dirac potential, } V_{Dir} \end{aligned}$$

The amplitude for the scattering of an electron from a state Ψ_i to Ψ_f

$$
\begin{aligned}
T_{fi} &= -i \int d^4x \Psi_f^\dagger V_{dir} \Psi_i \\
&= -ie \int d^4x \Psi_f^\dagger \{-V\mathbb{I} + \vec{\alpha}.\vec{A}\} \Psi_i \\
&= -ie \int d^4x \Psi_f^\dagger \{-A_0\mathbb{I} + \vec{\alpha}.\vec{A}\} \Psi_i \\
&= -ie \int d^4x \Psi_f^\dagger \gamma^0\gamma^0 \{-A_0\mathbb{I} + \vec{\alpha}.\vec{A}\} \Psi_i \\
&= -ie \int d^4x \overline{\Psi}_f^\dagger \{-\gamma^0 A_0\mathbb{I} + \gamma^k A_k\} \Psi_i \qquad \gamma^0 = \beta;\ \beta\alpha = \gamma^k \\
&= \cancel{-}ie \int d^4x \overline{\Psi}_f^\dagger \gamma^\mu A_\mu \Psi_i \qquad \gamma^\mu A_\mu = (\gamma^0 A_0; -\gamma^k A_k) \\
&= -i \int j^\mu_{fi} A_\mu d^4x
\end{aligned}
$$

where: $j^\mu_{fi} = -e\overline{U}_f \gamma^\mu U_i e^{i(p_f-p_i).x}$

j^μ_{fi} is the electromagnetic transition current between states i and f. Recall for a spin-less electron that j^μ_{fi} was:

$$
j^\mu_{fi} = e(p_f + p_i)^\mu e^{i(p_f-p_i)} x \qquad \text{spin-less}
$$

j^μ_{fi} is the current at each vertex so in electron-muon scattering we have two vertices, at vertex 1 is the electron and vertex 2 the muon (8.1).This gives the transition amplitude as:

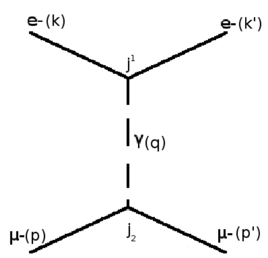

Figure 8.1: Schematic of electron muon scattering

$$
\begin{aligned}
T_{fi} &= -i \int j^1_\mu \frac{-1}{q^2} j^\mu_2 d^4x \\
&= +ie \int d^4x \overline{U}(k')e^{+ik'x}\gamma_\mu U(k)e^{-ikx}\frac{-1}{q^2}\overline{U}(p')e^{+ip'x}\gamma_\mu U(p)e^{-ipx}(-e) \\
&= \frac{ie^2}{q^2}\int d^4x e^{i(k'+p'-k-p).x}\{\overline{u}(k')\gamma_\mu u(k)\}\{\overline{u}(p')\gamma^\mu u(p)\}
\end{aligned}
$$

As before for $|T_{fi}|^2$ one exponential term goes into the phases space factor (number of final states) and the second becomes Vt so we just need@

$$|T_{fi}|^2 = \frac{ie^4}{q^4}\{\overline{u}(k')\gamma_\mu u(k)\}\{\overline{u}(p')\gamma^\mu u(p)\} \times \{\overline{u}(k')\gamma_\nu u(k)\}^\dagger\{\overline{u}(p')\gamma^\nu u(p)\}^\dagger$$

Considering the final term:

$$
\begin{aligned}
\{\ \overline{u}(p')\gamma^\nu u(p)\ \}^\dagger &= \{\ u^\dagger(p')\gamma^0\gamma^\nu u(p)\ \}^\dagger \\
&= u^\dagger(p)(\gamma^\nu)^\dagger(\gamma^0)^\dagger u(p') \\
&= u^\dagger(p)\gamma^0\gamma^\nu u(p') \qquad \text{as: } \begin{cases} (\gamma^\nu)^\dagger & = -\gamma^\mu \\ (\gamma^0)^\dagger & = \gamma^0 \\ \gamma^\nu\gamma^0 & = -\gamma^0\gamma^\nu \\ \Rightarrow (\gamma^\nu)^\dagger(\gamma^0)^\dagger & = \gamma^0\gamma^\nu \end{cases} \\
&= \overline{u}(p)\gamma^\nu u(p')
\end{aligned}
$$

Similarly for the third term:

$$
\begin{aligned}
&\{\overline{u}(k')\gamma_\nu u(k)\}^\dagger = \overline{u}(k)\gamma_\nu u(k') \\
&\qquad \Rightarrow |T_{fi}|^2 = \frac{ie^4}{q^4}\{\ \overline{u}(k')\gamma_\mu u(k)\ \}\ \{\ \overline{u}(p')\gamma^\mu u(p)\ \} \times \{\ \overline{u}(k)\gamma_\nu u(k')\ \}\ \{\ \overline{u}(p)\gamma^\nu u(p')\ \}
\end{aligned}
$$

This is often written as:

$$|T_{fi}|^2 = \frac{e^4}{q^4}\, {}^eL_{\mu\nu}\ {}^\mu L^{\mu\nu}$$

where ${}^eL$ and ${}^\mu L$ are tensors representing the electron and muon respectively.

We now need the $e^-\mu^-$ scattering transition amplitude summed over all the initial states, summed over all the final states and averaged over initial spin states:

$${}^eL_{\mu\nu} = \frac{1}{2}\sum_s\sum_{s'}\ \overline{U}_{s'}(k')\gamma_\mu U_s(k)\ \overline{U}_s(k)\gamma_\nu U_{s'}(k')$$

Where the factor $\frac{1}{2}$ is the spin average, s denotes the initial states and s' denotes final states.

We can re-write this explicitly in terms of matrix elements: α, β, γ and δ.

$${}^eL_{\mu\nu} = \frac{1}{2}\sum_s\sum_{s'}\ \overline{U}_{s'}(k')_\alpha\ \gamma_\mu^{\alpha\beta}\ U_s(k)_\beta\ \overline{U}_s(k)_\gamma\ \gamma_\nu^{\gamma\delta}\ U_{s'}(k')_\delta$$

we can make cyclic changes to the order so group $\overline{U}$ and U terms together, i.e. move the final term $U_{s'}(k')$ next to the first $\overline{U}_{s'}(k')$:

$$^{e}L_{\mu\nu} = \frac{1}{2}\sum_{s}\sum_{s'} \underbrace{U_{s'}(k')_{\delta}\,\overline{U}_{s'}(k')_{\alpha}}_{(\not{k}'+m)_{\delta\alpha}} \;\gamma_{\mu}^{\alpha\beta}\; \underbrace{U_{s}(k)_{\beta}\,\overline{U}_{s}(k)_{\gamma}}_{(\not{k}+m)_{\beta\gamma}} \;\gamma_{\nu}^{\gamma\delta}$$

so $^{e}L_{\mu\nu}$ is reduced to the trace of four 4×4 matrices, similarly for $^{\mu}L_{\mu\nu}$.

This gives us the final results for the L tensors:

$$^{e}L_{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left\{(\not{k}'+m_e)\gamma_{\mu}(\not{k}+m_e)\gamma_{\nu}\right\}$$
$$^{\mu}L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left\{(\not{p}'+m_{\mu})\gamma^{\mu}(\not{p}+m_{\mu})\gamma^{\nu}\right\}$$

Chapter 9

Trace Theorems

1. $\mathrm{Tr}(\mathbb{I}_{4\times4}) = 4$ generally $\mathrm{Tr}(\mathbb{I}_{n\times n}) = n$

2.

$$\begin{aligned}
\mathrm{Tr}(\not{a}\,\not{b}) &= \mathrm{Tr}(\not{b}\,\not{a}) \\
&= \frac{1}{2}\mathrm{Tr}(\not{a}\,\not{b} + \not{b}\,\not{a}) \\
&= \frac{1}{2}\mathrm{Tr}(\gamma^\mu a_\mu \gamma^\nu b_\nu + \gamma^\nu a_\nu \gamma^\mu b_\mu) \\
&= \frac{1}{2}\mathrm{Tr}(\gamma^\mu \gamma^\nu a_\mu b_\nu + \gamma^\nu \gamma^\mu a_\nu b_\mu) \\
\text{We know: } 2g^{\mu\nu}\mathbb{I} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\
\Rightarrow \mathrm{Tr}(\not{a}\,\not{b}) &= \frac{1}{2}a.b.2\mathrm{Tr}(\mathbb{I}) \\
&= 4a.b
\end{aligned}$$

3.

$$\begin{aligned}
\mathrm{Tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}) &= \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma a_\mu b_\nu c_\delta d_\sigma) \\
\text{now: } \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma) &= -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\delta\gamma^\sigma) + \mathrm{Tr}(2g_{\mu\nu}\gamma^\delta\gamma^\sigma) \\
&= \mathrm{Tr}(\gamma^\nu\gamma^\delta\gamma^\mu\gamma^\sigma) - \mathrm{Tr}(2g_{\mu\delta}\gamma^\nu\gamma^\sigma) + \mathrm{Tr}(2g_{\mu\nu}\gamma^\delta\gamma^\sigma) \\
&= -\mathrm{Tr}(\gamma^\nu\gamma^\delta\gamma^\sigma\gamma^\mu) + \mathrm{Tr}(2g_{\mu\sigma}\gamma^\nu\gamma^\delta) - \mathrm{Tr}(2g_{\mu\delta}\gamma^\nu\gamma^\sigma) + \mathrm{Tr}(2g_{\mu\nu}\gamma^\delta\gamma^\sigma) \\
&= -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma) + \mathrm{Tr}(2g_{\mu\sigma}\gamma^\nu\gamma^\delta) - \mathrm{Tr}(2g_{\mu\delta}\gamma^\nu\gamma^\sigma) + \mathrm{Tr}(2g_{\mu\nu}\gamma^\delta\gamma^\sigma) \quad \text{(cyclic permutati} \\
\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma) &= \mathrm{Tr}(g_{\mu\sigma}\gamma^\nu\gamma^\delta) - \mathrm{Tr}(g_{\mu\delta}\gamma^\nu\gamma^\sigma) + \mathrm{Tr}(g_{\mu\nu}\gamma^\delta\gamma^\sigma) \\
&= 4(g_{\mu\sigma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\sigma} + g_{\mu\nu}g_{\delta\sigma}) \\
\Rightarrow \mathrm{Tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}) &= 4\left\{(a.d)(b.c) - (a.c)(b.d) + (a.b)(c.d)\right\}
\end{aligned}$$

9.1 Other Identities

We will use these to calculate the cross-sections for compton scattering (among other things)

1.

$$\begin{aligned}\gamma_\mu \gamma^\nu \gamma^\mu &= -\gamma_\mu \gamma^\mu \gamma^\nu + g_{\nu\mu}\gamma_\mu \\ &= -(\mathbb{I})\gamma^\nu + 2\gamma^\nu \\ &= -4\gamma^\nu + 2\gamma^\nu \\ &= -2\gamma^\nu \\ \text{or: } \gamma_\mu \not{a} \gamma^\mu &= -2\not{a}\end{aligned}$$

2.

$$\begin{aligned}\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\mu &= -\gamma^\nu \gamma_\mu \gamma^\sigma \gamma^\mu + 2g_{\mu\nu}\gamma^\sigma \gamma^\mu \\ &= 2\gamma^\nu \gamma^\sigma + 2\gamma^\sigma \gamma^\nu \\ &= 4g_{\sigma\nu} \\ \text{or: } \gamma_\mu \not{a} \not{b} \gamma^\mu &= 4a.b\end{aligned}$$

3.

$$\begin{aligned}\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu &= \gamma_\mu \not{a} \not{b} \gamma^\nu \gamma^\mu c_\nu \\ &= -\gamma_\mu \not{a} \not{b} \gamma^\mu \gamma^\nu c_\nu + 2\gamma_\mu \not{a} \not{b} g^{\nu\mu} c_\nu \\ &= -4a.b\not{c} + 2\not{c}\not{a}\not{b} \\ &= -4a.b\not{c} + 2\not{c}\gamma^\alpha \gamma^\beta a_\alpha b_\beta \\ &= -4a.b\not{c} + 2\not{c}\gamma^\beta \gamma^\alpha a_\alpha b_\beta + 2\not{c}2g_{\alpha\beta}a_\alpha b_\beta \\ &= -4a.b\not{c} - 2\not{c}\gamma^\beta \gamma^\alpha a_\alpha b_\beta + 4a.b\not{c} \\ &= -2\not{c}\gamma^\beta \gamma^\alpha a_\alpha b_\beta \\ &= -2\not{c}\not{b}\not{a}\end{aligned}$$

9.2 Trace Theorems with γ^5

1.

$$\begin{aligned}\mathrm{Tr}(\gamma^5 \gamma^\mu) &= -\mathrm{Tr}(\gamma^\mu \gamma^5) && \text{(inverse a trace)} \\ &= -\mathrm{Tr}(\gamma^5 \gamma^\mu) && \text{(commutation relation)}\end{aligned}$$

2.

$$
\begin{aligned}
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) &= \mathrm{Tr}(\gamma^5(\gamma^\mu)^2) && \text{(case: } \mu = \nu\text{)}\\
&= \mathrm{Tr}(\gamma^5\mathbb{I})\\
&= \mathrm{Tr}\begin{bmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{bmatrix}\\
&= 0\\
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) &= i\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2) && \text{(case: } \mu = 1 \text{ and } \nu = 2\text{)}\\
&= i\mathrm{Tr}(\gamma^0(\gamma^1)^2\gamma^2\gamma^3\gamma^2) && (\gamma^1\gamma^\mu = -\gamma^\mu\gamma^1)\\
&= i\mathrm{Tr}(-\gamma^0\mathbb{I}(\gamma^2)^2\gamma^3)\\
&= -i\mathrm{Tr}(\gamma^0\gamma^3)\\
\text{And: } \mathrm{Tr}(\gamma^0\gamma^3 + \gamma^3\gamma^0) &= 2\mathrm{Tr}(g_{o3}\mathbb{I})\\
&= 0\\
\Rightarrow \mathrm{Tr}(\gamma^0\gamma^3) &= 0\\
\Rightarrow \mathrm{Tr}(\gamma^5\gamma^1\gamma^2) &= 0\\
\Rightarrow \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) &= 0
\end{aligned}
$$

This final assumption can be made because no matter which substitutions for μ and ν myou make you will always end up with an off-diagonal matrix which will always have a trace of 0.

3.

$$
\begin{aligned}
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\delta) &= \mathrm{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^\nu\gamma^\delta)\\
&= 0
\end{aligned}
$$

As the trace of an odd number of gamma matrices is 0. Consider:

$$
\begin{aligned}
\mathrm{Tr}(\not{a}_1\not{a}_2\ldots\not{a}_n) &= \mathrm{Tr}(\not{a}_1\not{a}_2\ldots\not{a}_n\gamma^5\gamma^5) && \text{as: } (\gamma^5)^2 = \mathbb{I}\\
&= (-1)^n\mathrm{Tr}(\gamma^5\not{a}_1\not{a}_2\ldots\not{a}_n\gamma^5) && \text{(move } \gamma^5 \text{ n places left)}\\
&= (-1)^n\mathrm{Tr}(\gamma^5\gamma^5\not{a}_1\not{a}_2\ldots\not{a}_n) && \text{(cyclic permutations)}\\
&= (-1)^n\mathrm{Tr}(\not{a}_1\not{a}_2\ldots\not{a}_n)\\
\Rightarrow \text{for odd } n \quad \mathrm{Tr}(\not{a}_1\not{a}_2\ldots\not{a}_n) &= -\mathrm{Tr}(\not{a}_1\not{a}_2\ldots\not{a}_n)\\
&= 0
\end{aligned}
$$

4.

$$\begin{aligned}
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma) &= \\
\text{consider:}& \\
\mathrm{Tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) &= \frac{-1}{i}\mathrm{Tr}((\gamma^5)^2) \\
&= 4i
\end{aligned}$$

In general:

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma) = 4\epsilon_{\mu\nu\delta\sigma}$$

$$\text{Where: } \epsilon = \begin{cases} +1 & \text{even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{otherwise} \end{cases}$$

Chapter 10

Return to Transition Amplitude for $e^- \mu^-$ scattering

Recall that:

$$|T_{fi}|^2 = \frac{e^4}{q^4}\left\{\frac{1}{2}\text{Tr}[\ (\not{K'} + m_e)\gamma_\mu(\not{K} + m_e)\gamma_\nu\ ]\frac{1}{2}\text{Tr}[\ (\not{P'} + m_\mu)\gamma^\mu(\not{P} + m_\mu)\gamma^\nu\ ]\right\}$$

The only non-zero traces are those involving even numbers of the γ matricies eg. $\text{Tr}[\not{K'}\gamma_\mu m\gamma_\nu] = 0$. (m = m_e and M = m_μ)

$$\begin{aligned}
|T_{fi}|^2 =& \frac{e^4}{4q^4}\left\{\text{Tr}[\gamma_\delta\gamma_\mu\gamma_\sigma\gamma_\nu(K')^\delta K^\sigma + \gamma_\mu\gamma_\nu m^2]\ \text{Tr}[\gamma^\delta\gamma^\mu\gamma^\sigma\gamma^\nu(P')_\delta P_\sigma + \gamma^\mu\gamma^\nu M^2]\right\} \\
=& \frac{e^4}{4q^4}\left\{4[\ (g_{\delta\nu}g_{\mu\sigma} - g_{\delta\sigma}g_{\mu\nu} + g_{\delta\mu}g_{\sigma\nu})(K')^\delta K^\sigma + g_{\mu\nu}m^2\ ]\right. \\
& \left.\times\ 4[\ (g^{\delta\nu}g^{\mu\sigma} - g^{\delta\sigma}g^{\mu\nu} + g^{\delta\mu}g^{\sigma\nu})(P')_\delta P_\sigma + g^{\mu\nu}M^2\ ]\right\} \\
=& \frac{8e^4}{q^4}\left\{(K'.P')(K.P) + (K'.P)(K.P') - m^2(P'.P) - M^2(K'.K) + 2m^2M^2\right\}
\end{aligned}$$

In the relativistic limit m^2 ande M^2 terms can be neglected.

$$|T_{fi}|^2 = \frac{8e^4}{q^4}\{\ (K'.P')(K.P) + (K'.P)(K.P')\ \}$$

expressing in terms of the Mandelstam variabls:

$$\begin{aligned}
s &= (K+P)^2 \\
&\approx 2K.P \\
&\approx 2K'.P' \\
t &= (K-K')^2 \\
&= q^2 \\
&\approx -K'.K \\
&\approx -P'.P \\
u&(K-P')^2 \\
&\approx -2K.P' \\
&\approx -2K'.P \\
\Rightarrow |T_{fi}|^2 &= \frac{8e^4}{t^2}\left\{\frac{s}{2}\cdot\frac{s}{2}+\frac{-u}{2}\cdot\frac{-u}{2}\right\} \\
&= 2e^4\left(\frac{s^2+u^2}{t^2}\right) \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\cdot 2e^4\left(\frac{s^2+u^2}{t^2}\right)
\end{aligned}$$

10.1 Cross-section for $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}$

This cross section can be easily derived from $e^{-}\mu^{-} \rightarrow e^{-}\mu^{-}$ to do this compare the Feynman diagrams (10.1):

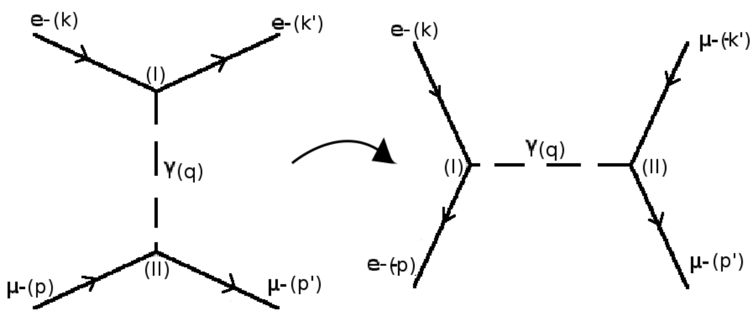

Figure 10.1: Transfortation of s-channel electron muon scattering to t-channel electron anihilation producing two muons

Comparing the verticies we can see that:

$$\begin{aligned} &(I) k \to k' && \text{for } e\mu \to e\mu \\ &k \to -p && \text{for } ee \to \mu\mu \\ \\ &(II) p \to p' && \text{for } e\mu \to e\mu \\ &-k' \to p' && \text{for } ee \to \mu\mu \end{aligned}$$

between the two processes exchange k' for $-p$ and p for $-k'$. This changes $|T_{fi}|^2$:

$$|T_{fi}|^2_{e\mu\to e\mu} = \frac{8e^4}{4(K.K')^2}\{\ (K'.P')(K.P) + (K'.P)(K.P')\ \}$$
$$|T_{fi}|^2_{ee\to\mu\mu} = \frac{8e^4}{4(K.P)^2}\{\ ((-P).P')(K.(-K')) + ((-P).-K')(K.P')\ \}$$

and the Mandelstam variables stay the same

$$\begin{aligned} \Rightarrow |T_{fi}|^2_{ee\to\mu\mu} &= \frac{8e^4}{4(\frac{s}{2})^2}\left\{ \frac{-t}{2}\frac{-t}{2} + \frac{-u}{2}\frac{-u}{2} \right\} \\ &= 2e^4\left\{\frac{t^2+u^2}{s^2}\right\} \\ \Rightarrow \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\cdot 2e^4\left\{\frac{t^2+u^2}{s^2}\right\} \end{aligned}$$

This is the same as for $e^-\mu^- \to e^-\mu^-$ other than the exchange of s and t Mandelstam variabls

10.2 The total $e^-e^+ \to \mu^-\mu^+$ cross section

We want to integrate over Ω and evaluate s, t and u in terms of energy and scattering angle (10.2)

$$\begin{aligned} s &\approx 2k.p \\ &= 4E_{e^-}E_{e^+} \\ t^2 &\approx 4(k.k')^2 \\ &= 4(E_{e^-}E_{\mu^+} - p_{e^-}p_{\mu^+})^2 \\ &= 4E_{e^-}^2E_{\mu^+}^2(1-\cos\theta)^2 \\ u^2 &\approx 4(k.p')^2 \\ &= 4E_{e^-}^2E_{\mu^-}^2(1+\cos\theta)^2 \end{aligned}$$

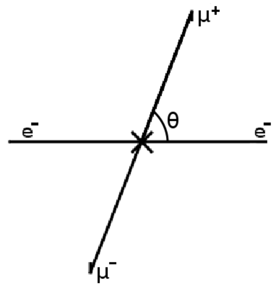

Figure 10.2: Schematic of electron annhilation going to muons

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \cdot 2e^4 \left\{ \frac{4E_{e^-}^2 E_{\mu^+}^2 (1-\cos\theta)^2 + 4E_{e^-}^2 E_{\mu^-}^2 (1+\cos\theta)^2}{16 E_{e^-} E_{e^+}} \right\}$$

because of working in the centre of mass frame at the massless limit:

$$\begin{aligned}
E_{e^-} = E_{e^+} = E_{\mu^-} &= E_{\mu^-} \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s} \cdot e^4 \left\{ \frac{(1-\cos\theta)^2 + (1+\cos\theta)^2}{2} \right\} \\
&= \frac{1}{64\pi^2 s} \cdot e^4 (1+\cos^2\theta) \\
\therefore \sigma &= \int_0^\pi \frac{\alpha^2}{4s}(1+\cos^2\theta) \cdot 2\pi \sin\theta d\theta \qquad \text{(where } \alpha = \frac{e^2}{4\pi}\text{)} \\
&= \frac{2\pi\alpha^2}{4s} \left[-\cos\theta - \frac{1}{3}\cos^3\theta \right]_0^\pi \\
&= \frac{4\pi\alpha^2}{3s} \\
&\approx \frac{87}{s} \; \frac{(\text{nb})}{(\text{GeV}^2)}
\end{aligned}$$

From this we can see that the anhilation process falls off as $\frac{1}{s}$. Had it not been for $q\bar{q}$ resonances, eg. Z^0, high-energy particle physics would be very un-interesting.

Note: using a similar method we can find Bhathe scattering: $e^+e^- \to e^+e^-$ in s and t channels.

10.3 The ratio, R, at e^+e^- colliders

$$R = \frac{\sigma(e + e^- \to hadrons)}{\sigma(e + e^- \to \mu^+\mu^-)}$$

In this ratio muons are looked for rather than electrons as muons are much easier to detect.

At low energies electrons anhilate into systems containing u and d quarks which appear as $q\overline{q}$ pairs, these pairs subsequently hadronise. They can anhilate through the virtual photon to make $q\overline{q}$ resonances egρ. As the energy increase $q\overline{q}$ pairs of s, c, b and eventually t quarks can be produced. What is the contribution to R of one generation of $q\overline{q}$ pairs?

$$\begin{aligned}\frac{\sigma(e+e^- \to q\overline{q}, q'\overline{q}')}{\sigma(e+e^- \to \mu^+\mu^-)} &= \frac{\left[(\frac{2}{3})^2 + (\frac{-1}{3})^2\right] e^2 3}{e^2} \\ &= \frac{15}{9} \\ &= \frac{5}{3}\text{per generation}\end{aligned}$$

where q is a u quark and q' is a d quark, the factor of 3 is due to colour charge while the e's and fractions are the relavent electromagnetic charges.

By the time we have $\sqrt{s} = 2m_b$, R should be $\frac{11}{3} = 3.66$. The actual value at 2m$_b$ is in the region 3.8$\tilde{4}$ but this is to be expected as our calculated value of R is a very simple electromagnetic calculation.

This ratio shows resonces at ρ = (770 MeV), J/Ψ = (3.1 GeV) γ = (10.6 GeV) the b$\overline{\text{b}}$ experiments work at this energy at which we can measure R.

Although the simple EM model is a reasonable description of the data it can be improved. The actual value of R is higher due to gluon radiation in the final state (10.3). These and higher order (O(α_s)) corrections to R produce the

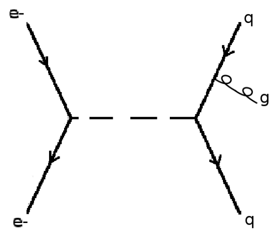

Figure 10.3: radiation of gluons from final state q$\overline{\text{q}}$ pair

difference between experiment and theory. e.g. there is a Z-mass correction. There is an analogous ration for partial decay widths of the Z^0

$$R_Z = \frac{\Gamma(e+e^- \to hadrons)}{\Gamma(e+e^- \to \mu^+\mu^-)}$$

to lowest order R$_Z$ =20.09. However the measured value is R$_{Z(m)}$ $= 20.79 \pm 0.04$ i.e about 3.4% higher but clearly many (experimental) standard deviations away. As this entirely due to higher order QCD corrections it is used as a way of measuring α_s.

10.4 Fermion processes

10.4.1 Møller Scattering

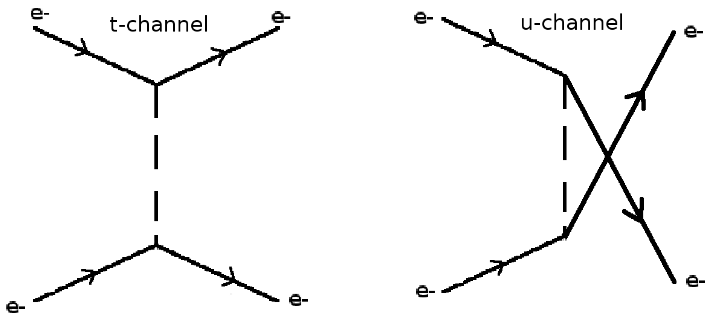

Figure 10.4: Moller scattering of electrons: t and u channel

Because processes like Moller scattering have identical particles it is impossible to tell from which vertex any single final state electron originated.

$$\sigma \sim \frac{s^2+u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2+t^2}{u^2} \qquad \text{(where: } \frac{2s^2}{tu} \text{ is a cross term)}$$

$$|T_{fi}|^2_{ee\to ee} = \left| T_{fi}^{t-channel} + T_{fi}^{u-channel} \right|^2$$

10.4.2 Bhatha Scattering

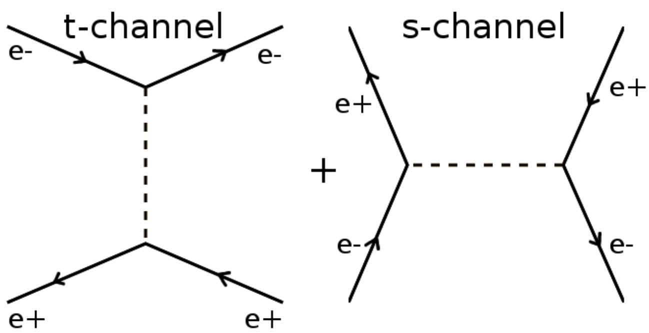

Figure 10.5: Bhatha scattering: s and t channels of $e^+e^- \to e^+e^-$

$$\sigma \sim \frac{s^2 + u^2}{t^2} + \frac{2u^2}{su} + \frac{t^2 + u^2}{s^2}$$

10.4.3 $e^-\mu^- \to e^-\mu^-$

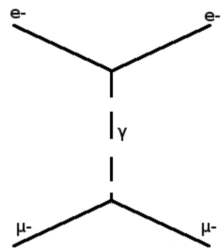

Figure 10.6: t-channel electron-muon scattering

$$\sigma \sim \frac{s^2 + u^2}{t^2}$$

10.4.4 $e^+e^- \to \mu^+\mu^-$

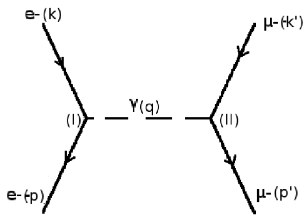

Figure 10.7: s-channel electron anhilition to muons

$$\sigma \sim \frac{t^2 + u^2}{s^2}$$

Chapter 11

Helicity conservation at high energies

We can gain further insight into the cross-section calculations and their angular distributions by looking at the helicity of particles.

The states are $U_L = \frac{1}{2}(1-\gamma^5)U$ and $U_R = \frac{1}{2}(1+\gamma^5)U$ consider

$$\begin{aligned}
\overline{U}_L &= U_L^\dagger \gamma^0 \\
&= \frac{1}{2}U^\dagger(1-\gamma^5)^\dagger\gamma^0 \\
&= \frac{1}{2}U^\dagger\gamma^0(1+\gamma^5) \\
&= \frac{1}{2}\overline{U}(1+\gamma^5)
\end{aligned}$$

At high energies, E≫m, the electromagnetic interaction conseres helicity

Consider the electromagnetic current: $\overline{U}\gamma^\mu U$

$$\begin{aligned}
\overline{U}\gamma^\mu U &= (\overline{U}_L + \overline{U}_R)\gamma^\mu(U_L + U_R) \\
\overline{U}_L\gamma^\mu U_R &= \frac{1}{2}\overline{U}(1+\gamma^5)\gamma^\mu\frac{1}{2}(1+\gamma^5)U \\
&= \frac{1}{4}\overline{U}\gamma^\mu(1-\gamma^5)(1+\gamma^5)U \\
&= \frac{1}{4}\overline{U}\gamma^\mu(1-(\gamma^5)^2)U \\
&= 0 \qquad\qquad ((\gamma^5)^2 = \mathbb{I})
\end{aligned}$$

i.e. there is no contribution from cross products.

Helicity conservation requires that the incoming e$^-$ and e$^+$ have opposite helicities as do the outgoing muons. In CMS $e_R^- e_L^+ \to \mu_L^- \mu_R^+$

The reaction preceeds via a photon of spin 1 so the amplitudes are proportional to the rotation matrices.

$$d^{j}_{\lambda\lambda^{prime}}(\theta) \equiv \langle j\lambda^{prime}|e^{-i\theta j_y}|j\lambda\rangle$$

where j is the spin of the propagator, λ and λ' are the spins of the particles and where y is perpendiculat to the reaction plane and the matricies can be worked out in angular momentum theory.

$$\begin{aligned}
d^{1}_{11}(\theta) &= d^{1}_{-1-1}(\theta) \\
&= \frac{1}{2}(1+\cos\theta) \\
&\sim -\frac{u}{s} \\
d^{1}_{1-1}(\theta) &= d^{1}_{-11}(\theta) \\
&= \frac{1}{2}(1-\cos\theta) \\
&\sim -\frac{t}{s}
\end{aligned}$$

square and add the above gets:

$$\frac{d\sigma}{d\Omega} \sim \frac{u^2+t^2}{s^2}$$

2 contributions depend on whether the spins ‘swap’.

Chapter 12

Massless, spin-1 particle: The photon

Maxwell's equations and definition of the classical potentials:

1. $\vec{\nabla}.\vec{E} = \rho$

2. $\vec{\nabla}.\vec{B} = 0$

3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

4. $\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$ $\vec{D}$ and $\vec{H}$ are used for situations where there is polarisation, the further constants that they use are removed due to natural units

12.1 Potentials

$$
\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \\
&= -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \\
\Rightarrow 0 &= \vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t})
\end{aligned}
$$

This has as a solution:

$$\begin{aligned}
\vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\nabla\phi \\
\rho &= -\nabla^2\phi - \frac{\partial}{\partial t}\vec{\nabla}.\vec{A} && \text{(using Maxwell 1)} \\
-\rho &= \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\vec{\nabla}.\vec{A} \\
\vec{\nabla} \times \vec{H} &= \vec{j} + \frac{\partial \vec{D}}{\partial t} && \text{(using Maxwell 4)} \\
\vec{\nabla} \times \vec{H} &= \vec{j} + \frac{\partial \vec{E}}{\partial t} && \text{(0 polarisation)} \\
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{j} + \frac{\partial \vec{E}}{\partial t} \\
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{j} + (-\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\frac{\partial\phi}{\partial t}) \\
\nabla(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} &= \vec{j} + (-\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\frac{\partial\phi}{\partial t}) \\
-\vec{j} &= \nabla^2\vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla(\vec{\nabla}.\vec{A}) - \nabla\frac{\partial\phi}{\partial t} \\
\Rightarrow j^\mu &= \Box^2 \vec{A}^\mu - \partial^\mu\partial_\nu A^\nu
\end{aligned}$$

Where:

$$\begin{aligned}
\Box^2 &= (\frac{\partial^2}{\partial t^2}, -\nabla^2) \\
A^\mu &= (\phi, \vec{A}) \\
j^\mu &= (\rho, \vec{j})
\end{aligned}$$

The electromagnetic field is described by the 4-potential A^μ which satisfies the above equations.

For a free electromagnetic field $j^\mu = 0$

Chapter 13

What are the polarisation states of a free photon?

Since we have a 4-potential, we expect to also have 4 polarisation states. How do these reduce to the well known 2 polarisation states of a free photon? the four states could be:

$$\begin{aligned}
\text{(time like)} &= \quad |1,0,0,0\rangle \\
\text{(space like)} &= \begin{cases} |0,1,0,0\rangle & \text{(x-polarisation)} \\ |0,0,1,0\rangle & \text{(y-polarisation)} \\ |0,0,0,1\rangle & \text{(z-polarisation)} \end{cases}
\end{aligned}$$

For virtual photons all of the above polarisation states exist, conversely for real photons only the transverse ($\vec{x}$ and $\vec{y}$) states exist. Firstly, apply the Lorenz (not Lorentz) condition:

$$\begin{aligned}
\partial_\mu A^\mu &= 0 \\
\text{gives: } \Box^2 A^\mu &= j^\mu
\end{aligned}$$

This makes the time like components depend on the space like components. For a free photon $j^\mu = 0$, so:

$$\Box^2 A^\mu = 0$$

And the solutions are plane waves $A^\mu = \epsilon_i^\mu e^{-iq.x}$ where ϵ_i are the polarisation states. Apply the Lorenz condition:

$$\begin{aligned}
\partial_\mu A^\mu &= \partial_\mu \epsilon_i^\mu e^{-iq_\mu x^\mu} \\
\Rightarrow -iq_\mu \epsilon_i^\mu e^{-iq_\mu x^\mu} &= 0
\end{aligned}$$

so the two Lorenz conditions reduce to:

$$q_\mu \epsilon_i^\mu = 0$$

We now apply a gauge transformation. Recall that $\vec{B}$ and $\vec{E}$ in classical electromagnetism come from the field tensor, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. $F^{\mu\nu}$ is unchanged and thus $\vec{E}$ and $\vec{B}$ are unchanged under the gauge transform:

$$A'^\mu \rightarrow A^\mu + \partial^\mu \Lambda$$

where Γ is a scalar field which has to satisfy the Lorenz condition, let $\Lambda = i\alpha e^{-iq^\mu x_\mu}$.

Is $\partial_\mu \partial^\mu \Lambda = 0$? If so then it will satisfy the Lorenz condition:

$$\begin{aligned}\partial_\mu \partial^\mu \Lambda &= (-iq_\mu)(-iq^\mu) i\alpha e^{-iq^\mu x_\mu} \\ &= -i\alpha q^2 e^{-iq^\mu x_\mu}\end{aligned}$$

As charge of a real photon $= 0 \Rightarrow q^2 = 0$ and therefore $\partial_\mu \partial^\mu \Lambda = 0$.

Substitute Λ and A^μ into the gauge transform

$$A^\mu \rightarrow \epsilon^\mu e^{-iq.x} + (-iq^\mu) i\alpha e^{-iq.x}$$

so the gauge transform simplifies to

$$\epsilon'^\mu = \epsilon^\mu + \alpha q^\mu$$

so two polarisation vectors $(\epsilon_\mu, \epsilon'_\mu)$ which differ by a multiple of q_μ, describe the same photon. This means we can set the time-like components to be 0. This means that the Lorenz condition reduces to $\vec{\epsilon}.\vec{q} = 0$. Therefore only two independent polarisation vectors exist and they must be perpendicular to each other.

So the states are:

$$\begin{aligned}\vec{\epsilon}_1 &= (1, 0, 0) \\ \vec{\epsilon}_s &= (0, 1, 0)\end{aligned}$$

There are also the circular polarisation vectors:

$$\begin{aligned}\vec{\epsilon}_R &= -\frac{1}{\sqrt{2}}(\epsilon_1 + i\epsilon_2) \\ \vec{\epsilon}_L &= -\frac{1}{\sqrt{2}}(\epsilon_1 - i\epsilon_2)\end{aligned}$$

13.1 Virtual photons and the photon propagator

For virtual photons by imposing the Lorenz condition we have:

$$\begin{aligned}\Box^2 A^\mu &= j^\mu \\ &= g^{\mu\nu} j_\nu\end{aligned}$$

by inspection we have seen that:

$$A^\mu = \frac{-g^{\mu\nu} j_\nu}{q^2}$$

we now show that this solution can be derived from the propagator approach:

$$A^\mu(x') = \int G(x', x) j^\mu(x) d^4x \tag{1}$$

Where we also know that:

$$\begin{aligned}
\Box^2 A^\mu(x') &= j^\mu(x') \\
&= \int d^4x \; \delta^4(x' - x) j^\mu(x) \\
&= \int d^4x \; \Box^2 G(x', x) j^\mu(x) \qquad\qquad \text{(substitute 1)}
\end{aligned}$$

comparing the last two equations:

$$\Box^2 G(x', x) = \delta^4(x' - x)$$

Translating into 4-momentum space via a Fourier transform:

$$\frac{1}{(2\pi)^4} \int d^4q \; (-iq)^2 G(q) e^{-iq(x'-x)} = \frac{1}{(2\pi)^4} \int d^4q \; e^{-iq(x'-x)}$$

For this to be true we can see that $G(q) = \frac{1}{q^2}$ so:

$$\begin{aligned}
A^\mu(x) &= \frac{-j^\mu(x)}{q^2} \\
&= \frac{-g^{\mu\nu}}{q^2} j_\nu(x)
\end{aligned}$$

When we did the propagator approach theory we found that the propagator in 4-momentum space is the inverse of the equation describing the free propagation of virtual particles. So for the Klein-Gordon equation we obtain the propagator by inverting it then multiplying by 'i':

$$i(\Box^2 + m^2)\phi = -iV\phi$$

so the Klein-Gordon propagator is:

$$\begin{aligned}
\frac{1}{i(\Box^2 + m^2)} &= \frac{-i}{(\Box^2 + m^2)} \\
\Box^2 &= \frac{i\partial_\mu i\partial^\mu}{i^2} \\
&= \frac{p_\mu p^\mu}{-1} \\
&= -p^2 \\
\therefore \text{ the propagator is } &= \frac{i}{P^2 - m^2}
\end{aligned}$$

Consider the Dirac Equation:

$$(\vec{\alpha}.\vec{p} + \beta m + e(\vec{\alpha}.\vec{A} - A^0\mathbb{I}))\Psi = E\Psi$$

convert into the covariant form by multiplying through by β

$$\begin{aligned}(\beta E - \beta\vec{\alpha}.\vec{p} - \beta^2 m)\Psi &= e(\beta\,\vec{\alpha}.\vec{A} - \beta A^0)\Psi \\ (\gamma^0 E - \gamma^k p^k - \mathbb{I}m)\Psi &= -e(\gamma^0 A^0 - \gamma^k A^k)\Psi \\ -i(\not{p} - m)\Psi &= ie\not{A}\Psi\end{aligned}$$

so the propagator is

$$\begin{aligned}\frac{1}{i(\not{p} - m)} &= \\ \Rightarrow \frac{i(\not{p} + m)}{(\not{p} - m)(\not{p} + m)} &= \frac{i\sum_s U\overline{U}}{p^2 - m^2}\end{aligned}$$

Where $\sum_s U\overline{U}$ is a sum over spin states of $U\overline{U}$

This is the general form of a virtual propagator where the sum is over all spin states of the electron or polarisation states of the photon.

13.2 The significance of longitudinal and time-like photons

Consider a typical process involving photon exchange i.e. the photon propagator is sandwiched between two currents

$$j^A_\mu(x)\left(\frac{-g^{\mu\nu}}{q^2}\right)j^B_\nu(x) = j^A_\mu(x)\left(\frac{1}{q^2}\right)j^{\mu,B}(x)$$

writing this out in full:

$$\frac{1}{q^2}(j^A_1(x)j^{1,B}(x) + j^A_2(x)j^{2,B}(x) + j^A_3(x)j^{3,B}(x) - j^A_0(x)j^{0,B}(x))$$

we know that the electromagnetic current is conserved:

$$\begin{aligned}\partial_\mu j^\mu &= 0 \\ \Rightarrow q_\mu j^\mu &= 0\end{aligned}$$

Since q is along the t or z axis then $q_1 j^1 = q_2 j^2 = 0$ so then apply the condition $q_\mu j^\mu = 0$ to the time-like and longitudinal components

$$\begin{aligned}q_\mu j^\mu &= q_0 j^0 - q_3 j^3 \\ &= 0\end{aligned}$$

substitute this back into the amplitude:

$$\begin{aligned}\frac{1}{q_3^2 q^2} q_0^2\, j_0^A j_0^B - \frac{1}{q^2}\, j_0^A j_0^B &= \frac{1}{q^2}\, j_0^A(x)\, j_0^B(x) \left(\frac{q_0^2 - q_3^2}{q_3^2}\right) \\ &= \frac{J_0^A(x) J_o^B(x)}{q_3^2} \qquad\qquad (\text{as } q^2 = q_0^2 - q_3^2)\end{aligned}$$

which is Coloumb's law in 3-momentum space.

The completion relation for real photons

$$\begin{pmatrix}1\\0\end{pmatrix}(1\;0) + \begin{pmatrix}0\\1\end{pmatrix}(0\;1)$$

but for virtual photons we have $g_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \lceil 1 & 0 \rceil & 0 \\ 0 & \lfloor 0 & 1 \rfloor & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where the highlighted 2×2 sub-matrix accounts for the real photons

So we can use the generalisation of virtual photons $g_{\mu\nu}$ for the completion relation for real photons

Chapter 14

Compton Scattering

There are 2 leading order Feynman Diagrams for $\gamma e \to \gamma e$ This is closely related

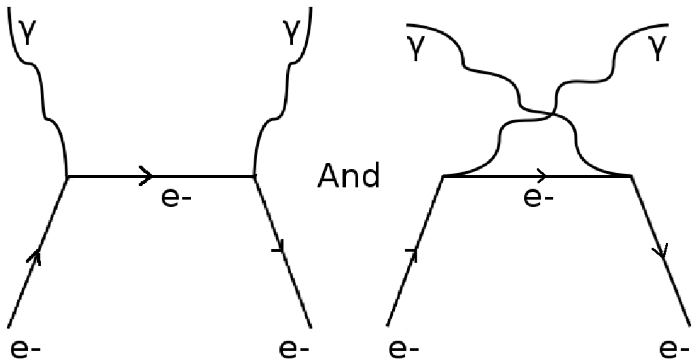

Figure 14.1: Feynman diagram of Compton scattering

to γ emission from e^+e^- annihilation Calculating the cross section for Compton scattering is also useful for deriving the cross-section or the QCD Compton process: $\gamma q \to qg$ (14.3)

Consider the Mandelstam variables:

$$
\begin{aligned}
s &= (k+p)^2 \\
t &= (k-k')^2 \\
u &= (k-p')^2
\end{aligned}
$$

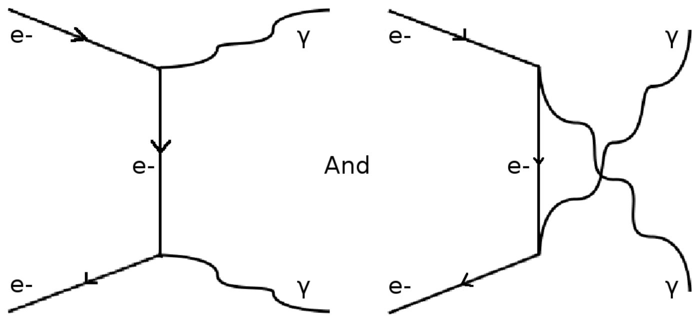

Figure 14.2: t and u channel emission of photons from electrons

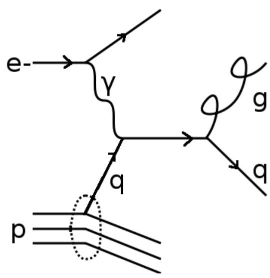

Figure 14.3: QCD Compton scattering

We are evaluating a double scattering:

$$
\begin{aligned}
&= - i \int d^4x_1 \int d^4x_2 \; \phi^*(2)V(2)G_0(2;1)V(1)\phi(1) \\
&= - i \int d^4x_1 \int d^4x_2 \; (-e)\overline{U} \; e^{ip'x_2} \; e^{ik'x_2} \; \epsilon^*_\nu \; \gamma^\nu \\
&\quad \times \frac{1}{(2\pi)^4} e^{-i(p+k)(x_2-x_1)} \, \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \\
&\quad \times (-e)\epsilon_\mu \; e^{-ikx_1} \; \gamma_\mu \; u \; e^{-ipx_1}
\end{aligned}
$$

In the above, $\int d^4x_1 e^{-i(p+k)(x_2-x_1)}$ gives $\delta^4(x_2 - x_1)$ so we can put both x_1 and x_2 to x in the above. The the exponential in terms of the 4-momentum, combined with the other from $T^\dagger_{fi}$ gives $V^\dagger$ and the number of states from factor

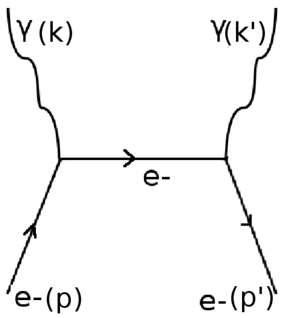

Figure 14.4: Schematic of Compton scattering

as before so T_{fi} simplifies to:

$$T_{fi} = -i\overline{U}(p')(-e)\epsilon^*_\nu \gamma^\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2}(-e)\epsilon_\mu \gamma^\mu U(p)$$

using the massless limit and tidying up:

$$\begin{aligned} T_{fi} &= \frac{-ie^2}{s}\, \overline{u}(p')\, \epsilon^*_\nu \gamma^\nu\, (\not{p} + \not{k})\, \epsilon_\mu \gamma^\mu\, U(p) \\ \Rightarrow |T_{fi}|^2 &= \frac{e^4}{s^2}\left[\epsilon^*_{\nu'}\epsilon_{\mu'}\, \epsilon^*_\mu \epsilon_\nu \;\times \overline{u}(p)\gamma^{\mu'}\, (\not{p} + \not{k})\, \gamma^{\nu'} u(p') \times \overline{u}(p')\gamma^\nu\, (\not{p} + \not{k})\, \gamma^\mu u(p)\right] \end{aligned}$$

This must be summed over the initial and final spin states and averaged over the initial spin states ($=\frac{1}{4}$). For the sum over the photon polarisation states:

$$
\begin{aligned}
\sum \epsilon^*_\mu \epsilon_{\mu'} &= -g_{\mu\mu'} \\
\Rightarrow |T_{fi}|^2 &= \frac{e^4}{4s^2} g_{\mu\mu'} g_{\nu\nu'} \mathrm{Tr}\ \left[(\not{p}' + m)\gamma^{\mu'}(\not{p} + \not{k})\gamma^{\nu'}(\not{p} + m)\gamma^{\nu}(\not{p} + \not{k})\gamma^{\mu}\right] \\
&= \frac{e^4}{4s^2} \mathrm{Tr}\ \left[\gamma^{\mu}\not{p}'\gamma_{\mu}(\not{p} + \not{k})\gamma_{\nu}\not{p}\gamma^{\nu}(\not{p} + \not{k})\right] && (m \to 0) \\
&\text{where:} \\
\gamma^{\mu}\not{p}'\gamma_{\mu} &= -2\not{p}' \\
\gamma_{\nu}\not{p}\gamma^{\nu} &= -2\not{p} \\
\Rightarrow |T_{fi}|^2 &= \frac{e^4}{4s^2} \mathrm{Tr}\ \left[(-2\not{p}')(\not{p} + \not{k})(-2\not{p})(\not{p} + \not{k})\right] \\
&= \frac{e^4}{s^2} \mathrm{Tr}\ \left[\not{p}'\not{k}\not{p}\not{k}\right] && (p.p = m_e \approx 0) \\
&= \frac{4e^4}{s^2} \left[(p'.k)(p.k) - (p'.p)(k.k) + (p'.k)(k.p)\right] && (\text{using trace theorem}) \\
&= \frac{4e^4}{s^2} \left[(p'.k)(p.k) + (p'.k)(k.p)\right] && (k.k = m_\gamma = 0) \\
&= 2e^4 \frac{-u}{s}
\end{aligned}
$$

to calculate the second (u-channel) diagram the same steps yield:

$$|T_{fi}|^2_u = 2e^4 \frac{-s}{u}$$

But we still need $T_{fi(u)}$ for the total cross section as we need to calculate the interference term of $T_{fi(u)}$ and $T_{fi(s)}$:

$$T_{fi(u)} = \frac{-ie^2}{u} \overline{U}(p')\epsilon_\mu \gamma^\mu (\not{p} - \not{k})\epsilon_\nu \gamma^\nu u(p)$$

we are calculating:

$$
\begin{aligned}
|T_{fi(u)} + T_{fi(s)}|^2 &= |T_{fi(u)}|^2 + |T_{fi(s)}|^2 + T^{\dagger}_{fi(u)} T_{fi(s)} + T^{\dagger}_{fi(s)} T_{fi(u)} \\
T^{\dagger}_{fi(u)} T_{fi(s)} &= \frac{e^2}{u} \epsilon^*_{\nu'} \epsilon^*_{\mu'} \, \overline{U}(p) \gamma^{\nu'} \, (\not{p} - \not{k}') \, \gamma^{\mu'} U(p') \\
&\quad \times \ \frac{e^2}{s} \epsilon^*_{\nu} \epsilon^*_{\mu} \, \overline{U}(p') \gamma^{\nu} \, (\not{p} + \not{k}) \, \gamma^{\mu} U(p) \\
&= \frac{e^4}{us} \cdot \frac{1}{4} g_{\nu\nu'} g_{\mu\mu'} \sum_{spin} \sum \overline{U}(p) \gamma^{\nu'} \, (\not{p} - \not{k}) \gamma^{\mu'} U(p') \\
&\quad \times \ \overline{U}(p') \gamma^{\nu} \, (\not{p} + \not{k}) \gamma^{\mu} U(p) \\
&= \frac{e^4}{4us} \mathrm{Tr} \left[\not{p} \gamma_{\nu} (\not{p} - \not{k}') \gamma_{\mu} \not{p}' \gamma^{\nu} (\not{p} + \not{k}) \gamma^{\mu} \right] \\
&= \frac{e^4}{4us} \mathrm{Tr} \left[-2 (\not{p} - \not{k}') \gamma_{\nu} \not{p} \not{p}' \gamma^{\nu} (\not{p} + \not{k}) \right] && (\gamma_{\mu} \not{a} \gamma^{\nu} \not{b} \gamma^{\mu} = -2 \not{b} \gamma^{\nu} \not{a}) \\
&= -2 \frac{e^4}{4us} \mathrm{Tr} \left[(\not{p} - \not{k}') (4p.p') (\not{p} + \not{k}) \right] && (\gamma_{\nu} \not{a} \not{b} \gamma^{\nu} = 4a.b) \\
&= -2 \frac{e^4}{4us} (4p.p') \mathrm{Tr} \left[(\not{p} - \not{k}') (\not{p} + \not{k}) \right] \\
&= -2 \frac{e^4}{4us} (4p.p') 4 \left[(p.p) + (p.k) - (p.k') - (k.k') \right] && (\mathrm{Tr}[\not{a} \not{b}] = 4a.b) \\
&= -2 \frac{e^4}{4us} 4 \left(\frac{-t}{2} \right) 4 \left[0 + \frac{s}{2} - \frac{-u}{2} - \frac{-t}{2}) \right] && \text{(Substitute in Mandelstam variables)} \\
&= 2e^4 \frac{t}{us} \left[s + u + t \right]
\end{aligned}
$$

for real photons $s + u + t = 0$ while for a virtual photon $s + u + t \neq 0$

For real photons $s + u + t = 0 = 2(m_e^2 + m_\gamma^2) \approx 0$, thus the interchange terms for real photons scattering off nearly massless electrons do not contribute to the cross-section however if the incoming photon is virtual $s + u + t = Q^2$ for a photon of mass $k^2 = Q^2 = -q^2$.

Therefore for a real photon:

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s} \cdot 2e^4 \left\{ -\frac{u}{s} - \frac{s}{u} \right\} \\
&= \frac{\alpha^2}{2s} \left\{ -\frac{u}{s} - \frac{s}{u} \right\}
\end{aligned}
$$

and for a virtual photon:

$$
\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \cdot 2e^4 \left\{ -\frac{u}{s} - \frac{s}{u} + \frac{2t}{su} Q^2 \right\}
$$

The cross-section and $|T_{fi}|^2$ for $e^+ e^- \to \gamma\gamma$ are the same as the above except u,s and t are rotated according to the diagrams (14.2)

14.1 Compton scattering in the limit $m_e \longrightarrow 0$ and $s \longrightarrow \infty$

$$\begin{aligned}
\left.\frac{d\sigma}{d\Omega}\right|_{s\to\infty} &= \frac{1}{64\pi^2 s}\cdot 2e^4\left\{-\frac{s}{u}\right\} \\
&= \frac{1}{64\pi^2}\cdot 2e^4\left\{-\frac{1}{u}\right\} \\
u &\sim -2p.k' \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2}\cdot\frac{e^4}{p.k'}
\end{aligned}$$

$p.k'$ can be evaluated in the centre of mass frame using $E_e = (p_e^2 + m_e^2)^{\frac{1}{2}}$. First the 4-vectors are dotted then a Taylor expansion is used:

$$p.k' = p_e E_\gamma (1 + \cos\theta + \frac{m_e^2}{2p_e^2} + \ldots)$$

where higher orders are neglected and treating all energies as equal

$$\begin{aligned}
\Rightarrow p_e^2 &= p_e E_\gamma \\
&= \frac{s}{4} \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\cdot\frac{e^4}{\frac{s}{4}(1+\cos\theta+\frac{2m_e^2}{s})} \\
d\sigma &= \frac{\alpha^2}{2}\cdot\frac{2\pi\ d(\cos\theta)}{(1+\cos\theta+\frac{2m_e^2}{s})} \\
\therefore \sigma &= \frac{2\pi\alpha^2}{s}\int\frac{d\ell}{\ell} \qquad\qquad (\ell = (1+\cos\theta+\frac{2m_e^2}{s})) \\
&= \frac{2\pi\alpha^2}{s}\ln\left[1+\cos\theta+\frac{2m_e^2}{s}\right]_\pi^0 \\
&= \frac{2\pi\alpha^2}{s}\left[\ln\left(2+\frac{2m_e^2}{s}\right)-\ln\left(\frac{2m_e^2}{s}\right)\right] \\
&= \frac{2\pi\alpha^2}{s}\ln\left[\frac{2s+2m_e^2}{s}\cdot\frac{s}{2m_e^2}\right] \\
\sigma &\approx \frac{2\pi\alpha^2}{s}\cdot\ln\left[\frac{s}{m_e^2}\right]
\end{aligned}$$

Chapter 15

Electro-Weak

15.1 Massive spin-1 particles

For a massless photon the potential satisfies:

$$\Box^2 A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu$$

for a free particle, $j^\mu = 0$, and so

$$\begin{aligned}
\Box^2 A^\mu - \partial^\mu \partial_\nu A^\nu &= 0 \\
(\frac{\partial^2}{\partial t^2} - \nabla^2) A^\mu - \partial^\mu \partial_\nu A^\nu &= 0 \\
(-E^2 + p^2) A^\mu - \partial^\mu \partial_\nu A^\nu &= 0
\end{aligned}$$

For massive particles, $E^2 = p^2 + m^2$ or $-E^2 + p^2 + m^2 = 0$ therefore a massive particle satisfies:

$$\begin{aligned}
\Box^2 A^\mu + m^2 A^\mu - \partial^\mu \partial_\nu A^\nu &= 0 && \text{(when real)} \\
&= j^\mu && \text{(when virtual)}
\end{aligned}$$

This is the Proca equation, now differentiate with respect to ∂_μ

$$\begin{aligned}
\partial_\mu \partial^\mu \partial_\nu A^\nu + m^2 \partial_\mu A^\mu - \partial_\mu \partial^\mu \partial_\nu A^\nu &= \partial_\mu j^\mu && \text{(in free field)} \\
\Rightarrow m^2 \partial_\mu A^\mu &= 0 \\
\text{but } m^2 &\neq 0
\end{aligned}$$

Therefore the free Proca field satisfies the Lorenz condition. The polarisation vectors for free massive vector bosons are:

$$A^\mu = \epsilon^\mu e^{-i p_\mu x^\mu}$$

The field satisfies Lorenz condition as

$$\begin{aligned}
\partial_\mu A^\mu &= 0 \\
p_\mu \epsilon^\mu &= 0
\end{aligned}$$

in the rest frame $p_\mu = (M, 0, 0, 0)$ and the polarisation states are:

$$\begin{aligned}\epsilon_1 &= (0, 1, 0, 0)\\ \epsilon_2 &= (0, 0, 1, 0)\\ \epsilon_3 &= (0, 0, 0, 1)\end{aligned}$$

What happens to the polarisation states when M is boosted along z? ϵ_1 and ϵ_2 remain unchanged and the particle is now described by the 4-vector: $(E, 0, 0, -p_z)$ we determine ϵ_3 from the Lorentz condition:

$$\underbrace{\frac{1}{M}(p_z, 0, 0, E)}_{\epsilon_3}(E, 0, 0, -p_z) = 0$$

The completeness relation for massive vector bosons:

$$\begin{aligned}\sum_i \epsilon_i \epsilon_i^\dagger &= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}(0,1,0,0) + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}(0,0,1,0) + \frac{1}{M}\begin{pmatrix}p_z\\0\\0\\E\end{pmatrix}(p_z, 0, 0, E)\\ &= \begin{pmatrix}\frac{p_z^2}{M^2} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{E^2}{M^2}\end{pmatrix}\\ &= -g_{\mu\nu} + \frac{p_\nu p^\nu}{M^2}\end{aligned}$$

Checking the 00 term:

$$\begin{aligned}-g_{00} + \frac{p_0 p^0}{M^2} &= -1 + \frac{E^2}{M^2}\\ &= \frac{E^2 - M^2}{M^2}\\ &= \frac{p_z^2}{M^2}\end{aligned}$$

and checking 33:

$$\begin{aligned}-g_{33} + \frac{p_3 p^3}{M^2} &= 1 + \frac{p_z^2}{M^2}\\ &= \frac{p_z^2 + M^2}{M^2}\\ &= \frac{E^2}{M^2}\end{aligned}$$

Can do similarly and write down polarisation vectors for virtual photons. Imposing the Lorentz condition removes their time-like polarisation states.

For the virtual photon we have:

$$\begin{aligned} q^\mu &= (v, 0, 0, p_z) \\ q^2 &= v^2 - p_z^2 \\ \Rightarrow p_z^2 &= \sqrt{v^2 + Q^2} \\ \therefore q^\mu &= (v, 0, 0, \sqrt{v^2 + Q^2}) \end{aligned}$$

This gives the polarisation states of a virtual photon as:

$$\begin{aligned} \epsilon_1 &= (0, 1, 0, 0) \\ \epsilon_2 &= (0, 0, 1, 0) \\ \epsilon_3 &= \frac{1}{Q^2}(\sqrt{v^2 + Q^2}, 0, 0, v) \end{aligned}$$

15.2 Massive virtual vector boson propagator

$$(\Box^2 + m^2)A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu$$

but we already know that:

$$\begin{aligned} m^2 \partial_\nu A^\nu &= \partial_\mu j^\mu \\ \Rightarrow (\Box^2 + m^2)A^\mu - \partial^\mu \partial_\nu j^\nu \left(\frac{1}{m^2}\right) &= j^\mu \\ (\Box^2 + m^2)A^\mu &= \left(\frac{1}{m^2}\right) \partial^\mu \partial_\nu j^\nu + g_{\mu\nu} j_\nu \\ &= -\left(\frac{1}{m^2}\right) \partial^\mu \partial^\nu j_\nu + g^{\mu\nu} j_\nu \\ &= \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2}\right) j_\nu \end{aligned}$$

so the propagator is

$$\frac{i\left(g^{\mu\nu} - \dfrac{\partial^\mu \partial^\nu}{m^2}\right)}{-q^2 + m^2}$$

Where the 'i' is included as convention. This is then the propagator for the exchange of a massive spin-1 particle. If the particle is made in an annihilation process it can decay and so the expression is modified.

The quantum state of a decaying particle in the rest frame must be of the form

$$\Psi \sim e^{-iMt} e^{-\Gamma \frac{t}{2}}$$

Such that $\Psi\Psi^* = e^{-iMt}e^{iMt}e^{-\Gamma t}$

This suggests that for a decaying particle we should replace '$-iM$' with '$-iM - \frac{\Gamma}{2}$' in the propagator

$$\Rightarrow = \frac{i\left(g^{\mu\nu} - \dfrac{\partial^\mu\partial^\nu}{m^2}\right)}{-q^2 + \left(-iM - \dfrac{\Gamma}{2}\right)^2}$$

$$\approx \frac{i\left(g^{\mu\nu} - \dfrac{\partial^\mu\partial^\nu}{m^2}\right)}{-q^2 - M^2 + iM\Gamma} \qquad (M > \Gamma)$$

Chapter 16

Charged current weak interactions (CC)

Originally Fermi theory and his coupling constant, G_F, from 1934 described weak interactions. This developed from a point-like theory to being mediated via a vector boson with coupling g_w.

Consider the leptonic process:

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$$
$$\text{or } \nu_\mu + e^- \to \mu^- + \nu_e$$

Fermi would have considered these reactions to be point-like between two vector currents (16.1) This theory was modeled on the electromagnetic interaction but

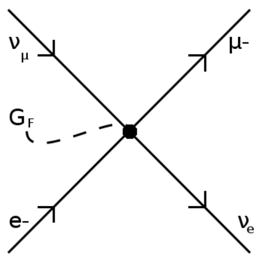

Figure 16.1: Point-like Fermi weak interaction

without the exchanged particle because the range of the weak force was known to be small. Fermi suggested the following:

$$T_{fi} = G_F^2 \bar{U}_{\mu^-} \gamma^\mu U_{\nu_\mu} \cdot \bar{U}_{\nu_e} \gamma_\mu U_{e^-}$$

G_F can be determined by experiment eg by measuring the decay width of a muon or through some nuclear β decays.

Parity violation, discovered in 1956, meant that the weak current had to be modified from vector to vector-axial current, this changes the transition to:

$$T_{fi} = \frac{G_F^2}{2} \bar{U}_{\mu^-} \gamma^\mu (1-\gamma^5) U_{\nu_\mu} \cdot \bar{U}_{\nu_e} \gamma^\mu (1-\gamma^5) U_{e^-}$$

Presently the known form of the charged current weak interaction is seen in So:

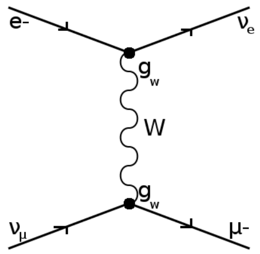

Figure 16.2: electron weak scattering $e^-\nu_e \to e^-\nu_e$

$$\begin{aligned} T_{fi} =& \bar{U}_{\mu^-} \frac{\gamma^\mu}{2}(1-\gamma^5) U_{\nu_\mu} \left(\frac{g_w}{\sqrt{2}}\right) \\ &\times \frac{i\left(-g^{\mu\nu} + \dfrac{\partial^\mu \partial^\nu}{m^2}\right)}{q^2 - M^2} \\ &\times \left(\frac{g_w}{\sqrt{2}}\right) \bar{U}_{\nu_e} \frac{\gamma^\mu}{2}(1-\gamma^5) u_{e^-} \end{aligned}$$

Comparing the old and the new descriptions for transmission (T_{fi}) with $q \ll M$:

$$\begin{aligned} \Rightarrow \frac{G_F^2}{2} &= \left(\frac{1}{2} \cdot \frac{g_w}{\sqrt{2}} \cdot \frac{1}{M^2} \cdot \frac{g_w}{\sqrt{2}} \cdot \frac{1}{2}\right)^2 \\ &= \frac{g_w^4}{64M^4} \\ \Rightarrow \frac{G_F}{\sqrt{2}} &= \frac{g_w^4}{8M^2} \end{aligned}$$

Now look at the chiral doublet :

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \frac{1}{2}(1-\gamma^5) \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$$

and the right handed singlet:

$$e_R^- = \frac{1}{2}(1+\gamma^5)e^-$$

We are writing these down as it appears that in $SU(2) \otimes U(1)$, electroweak unification, the fundamental entities are left handed chiral doublets and right handed chiral singlets in weak isospin space. Now show that the CC weak interaction is of the form:

$$\bar{U}_L \gamma^\mu U_L$$

starting from:

$$\bar{U}_e \frac{1}{2}\gamma^\mu(1-\gamma^5)U_{\nu_e}$$

now

$$\begin{aligned}
\left(\frac{1-\gamma^5}{2}\right)^2 &= \frac{1}{4}(1-2\gamma^5+(\gamma^5)^2) \\
&= \frac{1}{4}\cdot 2(1-\gamma^5) && ((\gamma^5)^2 = \mathbb{I}) \\
&= \frac{1}{2}(1-\gamma^5) \\
\therefore \bar{U}_e \frac{1}{2}\gamma^\mu(1-\gamma^5)U_{\nu_e} &= \bar{U}_e \gamma^\mu \left(\frac{(1-\gamma^5)}{2}\right)^2 U_{\nu_e} \\
&= \bar{U}_e \frac{1}{2}(1+\gamma^5)\gamma^\mu \frac{1}{2}(1-\gamma^5)U_{\nu_e} && (\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5)
\end{aligned}$$

recall:

$$\begin{aligned}
\bar{U}_{eL} &\equiv \bar{U}_e \left(\frac{1+\gamma^5}{2}\right) \\
U_{\nu_e L} &\equiv \left(\frac{1-\gamma^5}{2}\right) U_{\nu_e} \\
\therefore \bar{U}_e \frac{1}{2}\gamma^\mu(1-\gamma^5)U_{\nu_e} &= \bar{U}_{eL}\gamma^\nu U_{\nu_e L}
\end{aligned}$$

16.1 leptonic CC process

- Muon decay has been exhaustively studied theoretically and experimentally since the late 1940s
- We are going to calculate the charged current weak contribution to neutrino-electron scattering. With small modifications this will apply to neutrino-quark scattering. The calculation will be done for $Q^2 \ll M_W^2$ i.e W-propagator effects can be neglected. In fact ν experiments have not been of high enough energy to see the W-propagator effects.

- However W-propagator effects have been seen at the HERA through the process $e^- q \rightarrow \nu q'$ (16.3)

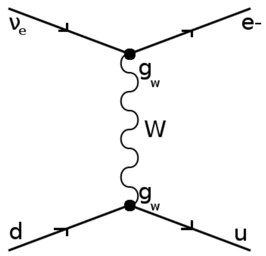

Figure 16.3: Weak quark processes

There are also neutral current processes $\nu_e + e^- \xrightarrow{Z^0} \nu_e + e^-$ (16.4)

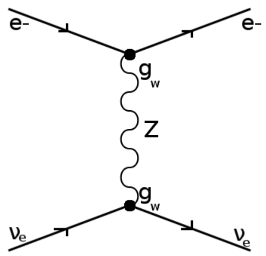

Figure 16.4: Neutral current electron-neutrino scattering

In the point-like limit ($Q^2 \ll M^2$ 16.5) This gives a transmission of:

$$T_{fi} = \frac{G_F}{\sqrt{2}} \bar{U}(P') \gamma^\mu (1 - \gamma^5) U(K) \cdot \bar{U}(K') \gamma_\mu (1 - \gamma^5) U(P) \frac{G_F}{\sqrt{2}}$$

As usual, we need $T_{fi}.T_{fi}^\dagger$ summed over all spin states and averages over initial states.

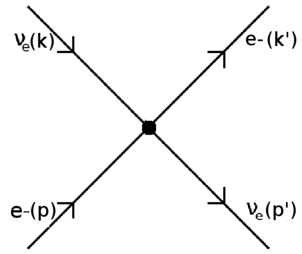

Figure 16.5: point-like schematic of electron-neutrino scattering

For the Hermitian conjugate consider one term:

$$
\begin{aligned}
&= \left[\bar{U}(K')\gamma_\mu(1-\gamma^5)U(P)\right]^\dagger \\
&= U^\dagger(P)(1-\gamma^5)^\dagger\gamma_\mu^\dagger\gamma_0^\dagger U(K')
\end{aligned}
$$

recall:

$$
\begin{aligned}
(\gamma^0)^\dagger &= \gamma^0 \\
(\gamma^5)^\dagger &= \gamma^5 \\
(\gamma^k)^\dagger &= -\gamma^k
\end{aligned}
$$

for $\gamma_\mu = \gamma_0$

$$
\begin{aligned}
&= U^\dagger(P)\underset{-1}{\underleftarrow{\overset{-1}{\overleftarrow{(1-\gamma^5)^\dagger}}\gamma_0}}\gamma_0 U(K') \\
&= \bar{U}(P)\underbrace{\gamma_0}_{\gamma_\mu}(1-\gamma^5)U(K')
\end{aligned}
$$

for $\gamma_\mu = \gamma_i$

$$
\begin{aligned}
&= U^\dagger(P)\underset{-1\quad -1}{\underleftarrow{\overset{-1}{\overleftarrow{(1-\gamma^5)^\dagger}}(-\gamma_i)\gamma_0}}U(K') \\
&= \bar{U}(P)\underbrace{\gamma_i}_{\gamma_\mu}(1-\gamma^5)U(K')
\end{aligned}
$$

therefore the Hermitian has the same order of $\gamma_\mu(1-\gamma^5)$

$$
\begin{aligned}
\Rightarrow T_{fi}^\dagger T_{fi} =& \frac{G_F^2}{2}\left[\bar{U}(K')\gamma^\nu(1-\gamma^5)U(P)\right]\left[\bar{U}(P)\gamma_\nu(1-\gamma^5)U(K')\right] \\
&\cdot\left[\bar{U}(P')\gamma^\mu(1-\gamma^5)U(K)\right]\left[\bar{U}(K')\gamma_\mu(1-\gamma^5)U(P)\right] \\
|T_{fi}|^2 =& \frac{G_F^2}{2}\frac{1}{2}\mathrm{Tr}\left[\gamma^\nu(1-\gamma^5)\not{P'}\gamma^\mu(1-\gamma^5)\not{K}\right]\cdot\mathrm{Tr}\left[\gamma_\nu(1-\gamma^5)\not{K'}\gamma_\mu(1-\gamma^5)\not{P}\right]
\end{aligned}
$$

in the above equation terms have been collected together and completeness relationships formed. The massless limit has also been used, otherwise the '$\not{P}$' and '$\not{K}$' terms would include a mass as well.

$$|T_{fi}|^2 = \frac{G_F^2}{4} \cdot 256(P'.K')(P.K)$$

Where we have used the trace theorems.

$$\begin{aligned}
\text{Recall: } s &= (K+P)^2 \\
&\sim 2K.P \\
&= 2P'.K' \\
\Rightarrow |T_{fi}|^2 &= 64\ G_F^2 \cdot \frac{s}{2} \cdot \frac{s}{2} \\
&= 16\ G_F^2\ s^2 \\
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s} \cdot 16\ G_F^2\ s^2 \\
&= \frac{G_F^2\ s}{4\pi^2} \\
\Rightarrow \sigma(\nu_e + e^- \to \nu_e + e^-) &= \frac{G_F^2\ s}{\pi} \qquad \text{(Leading order)}
\end{aligned}$$

Chapter 17

Approaches to O(n), U(n) and SU(n)

17.1 Orthogonal transformation

These are transformations that preserve the normalisation of a vector in n-dimensional spaces. The requirement on these matrices is that:

$$OO^{-1} = \mathbb{I} = O^{-1}O = OO^T$$

i.e. the inverse matrix is the transpose matrix, these are called orthogonal matrices.

For a transformation that to preserve normalisation we need to show that

$$O^{-1} = O^T$$

the normalisation condition is:

$$\begin{aligned}
x'_i x'_i &= x_i x_i \\
&= a_{ij} x_j \; a_{ik} x_k \qquad\qquad (a_{ij} \text{ and } a_{ik} \text{ are matrices}) \\
\Rightarrow \delta_{jk} &= a_{ij} \; a_{ik} \qquad\qquad (1) \\
x_k &= a'_{kl} x'_l \\
x'_j &= a_{jk} x_k \\
&= a_{jk} a'_{kl} x'_l \\
\text{or } \delta_{jl} x'_l &= a_{jk} a'_{kl} x'_l \\
\Rightarrow \delta_{jl} &= a_{jk} a'_{kl} \qquad\qquad (2)
\end{aligned}$$

Consider:

$$\begin{aligned}
\underbrace{(a_{jm}a_{jk})}_{(1)}\,a'_{kl} &= a_{jm}\,\underbrace{(a_{jk}a'_{kl})}_{(2)} \\
\delta_{mk}a'_{kl} &= a_{jm}\delta_{jl} \\
\Rightarrow a'_{ml} &= a_{ml} \\
&= a^T_{ml} \\
\text{i.e. } O^{-1} &= O^T
\end{aligned}$$

17.2 Independent elements

How many independent elements are there in an $n \times n$ orthogonal matrix?

In an $n \times n$ matrix there are n^2 elements. From the diagonal we get n equations equal to 1 (as $OO^T = \mathbb{I}$). From the off-diagonal we have $n^2 - n$ equations that yield 0. In fact we only have half of these as the transposed off-diagonals must cancel. Therefore the number of independent equations are:

$$\frac{1}{2}n(n-1)$$

eg for $n = 3$ we get 3 independent equations:

$$\begin{pmatrix} a & b & c \\ b & a & d \\ c & d & a \end{pmatrix}$$

where the free equations are b, c and d; a is constrained as it must equal 1.

Examples of 3D orthogonal matrices are:

$$\begin{aligned}
R_z(\gamma) &= \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
R_x(\gamma) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix}
\end{aligned}$$

17.3 The SU(n) group of transformations

These transformations preserve the normalisation of quantum states, subject to the condition $|U(n)| = \mathbb{I}$.

$$\begin{aligned}
\text{if } \Psi' &= U\Psi \\
\Rightarrow (\Psi')^\dagger &= \Psi^{dagger}U^\dagger \\
(\Psi')^\dagger\Psi' &= \Psi^{dagger}U^\dagger U\Psi \\
\therefore U^\dagger U &= U^{-1}U \\
&= \mathbb{I}
\end{aligned}$$

And U is a unitary matrix.

How many independent parameters are there in a SU(n) transformation?

There are $2n^2$ elements in principle as each element is complex. If we multiply $U^T U = \mathbb{I}$ then there are n equations yielding 1 from the diagonals. From the off-diagonals there are $\frac{2n}{2}(n-1)$ equations. The number of free parameters is therefore:

$$2n^2 - n - n(n-1) = n^2$$

but since we require SU(n) transformations to satisfy the condition $|U(n)| = \mathbb{I}$ we constrain one more equation giving us

$$2n^2 - n - n(n-1) - 1 = n^2 - 1$$

parameters.

From this we see that for SU(2) we need 3 parameters. A common choice of transformation matrix is

$$\begin{pmatrix} \cos\theta e^{i\alpha} & \sin\theta e^{i\gamma} \\ -\sin\theta e^{i(\beta-\gamma)} & \cos\theta e^{i(\beta-\alpha)} \end{pmatrix}$$

In SU(3) you use 8 parameters.

17.3.1 What are the generators of SU(n)?

For each parameter we can write:

$$U = e^{i\theta H}$$

is H Hermitian?

$$\begin{aligned} \text{we know: } U^\dagger U &= \mathbb{I} \\ e^{-i\theta H^\dagger} \cdot e^{i\theta H} &= \mathbb{I} \\ \Rightarrow i\theta(H - H^\dagger) &= 0 \\ \Rightarrow H^\dagger &= H \end{aligned}$$

For SU(n) groups the generators H (or G) are also traceless

$$\begin{aligned} |e^{iH}| &= \begin{vmatrix} e^{i\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{i\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\lambda_n} \end{vmatrix} \\ &= e^{i(\lambda_1+\lambda_2+\cdots\lambda_n)} \\ &= e^{i\mathrm{Tr}[H]} \end{aligned}$$

$$\begin{aligned} \text{if } e^{i\mathrm{Tr}[H]} &= 1 \\ \Rightarrow \mathrm{Tr}[H] &= 0 \end{aligned}$$

H is traceless as long as it is diagonalisable, there are non-diagonal H that are traceless but these are non-trivial.

In summary: for SU(n) there are $n^2 - 1$ parameters and $n^2 - 1$ generators which are traceless Hermitian matrices. For SU(2) the generators are the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ where:}$$
$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}$$

we can regard the Pauli matrices as:

- giving eigenvalues
- Combinations of them giving raising and lowering operators

In SU(3) the fundamental entity is made up of 3 objects: colour charge (R,G,B) there are 8 generators which are represented by 3×3 matrices denoted λ_i, these are the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

λ_1 λ_2 λ_3 correspond to the Pauli matrices, therefore SU(2) is a subgroup of SU(3).

The Gell-Mann matrices are all traceless and normalised thus:

$$\mathrm{Tr}[\lambda_i, \lambda_j]N^2 = 2\delta_{ij}$$

Where N is a normalisation constant. They commute according to:

$$[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}] = \frac{i}{2}f_{ijk}\lambda_k$$

where f_{ijk} is an $8 \times 8 \times 8$ object which has 512 elements. The none vanishing values are permutations of:

$$
\begin{aligned}
f_{123} &= 1 \\
f_{458} = f_{678} &= \frac{\sqrt{3}}{2} \\
f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} &= \frac{1}{2} \\
f_{other} &= 0
\end{aligned}
$$

Chapter 18

Charged Current interactions involving quarks

We now extend the calculations of weak cross-sections and decays to interactions involving quarks as well as leptons.

We have already considered (16.2) we want to extend this to processes like (16.3) where 'u' and 'd' represent quarks of those respective types (ie u could be a u, c or t).

Going back to when only u,d and s quarks were important it became apparent that the weak eigen-states were not the same as the mass eigen-states. In 1963 Cabibbo proposed the following to address this:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

For this theory to make sense the d and s masses have to be different. θ_c is the Cabibbo angle ($\sim 13°$) and its origin is still a mystery (there is some insight from the Higgs mechanism).

There must be a weak current which couples quarks eg a 'u' to a 's' quark. Instead of introducing a new coupling we assume that charged current couples to "rotated" quark states.

We know that the form the charged current weak interaction for 2 generations is of the form:

$$(\bar{u}\ \bar{c})\gamma_\mu(1-\gamma^5) \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

The Cabibbo theory would have been valid if there had only been 4 quarks.

However as there is another generation the theory needs to be extended:

$$(\bar{u}\ \bar{c}\ \bar{t}\ )\gamma_\mu(1-\gamma^5)V\begin{pmatrix}d\\s\\b\end{pmatrix}$$

where V is the Cabibbo-Kobayashi-Maskaura (CKM) matrix which is unitary and generally written as:

$$V=\begin{pmatrix}V_{ud} & V_{us} & V_{ub}\\V_{cd} & V_{cs} & V_{cb}\\V_{td} & V_{ts} & V_{tb}\end{pmatrix}$$

where $V_{ud}\sim V_{cs}\sim V_{tb}\sim 1$, the other terms are small ($\lesssim 0.2$)

For this matrix there are $n^2=9$ elements but since a unitary matrix is orthogonal there are only $\frac{1}{2}n(n-1)=3$ independent elements. The remaining 6 parameters (normally given as angles) can be absorbed into arbitrary phase rotations of the quark fields leaving us with only 1 independent phase.

The standard parameterisation is then:

$$V=\begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}\ e^{-i\delta}\\-s_{12}c_{23}-c_{12}s_{23}s_{13}\ e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}\ e^{i\delta} & s_{23}c_{13}\\s_{12}s_{23}-c_{12}c_{23}s_{13}\ e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}\ e^{i\delta} & c_{23}c_{13}\end{pmatrix}$$

where:

$$c_{ab}=\cos\theta_{ab}$$
$$s_{ab}=\sin\theta_{ab}$$

The phase $e^{i\delta}$ is what causes CP violation. Note: even if δ is maximal it can't, alone, explain the matter/anti-matter asymmetry that we observe in the universe.

In the early 1970s (before the charm quark was discovered), recall that the form of the charged current interaction was:

$$(\bar{u}\ \bar{c})\begin{pmatrix}\cos\theta_c & \sin\theta_c\\-\sin\theta_c & \cos\theta_c\end{pmatrix}\begin{pmatrix}d\\s\end{pmatrix}$$

Glashow, Illiopoulos and Maiani (GIM) proposed the existence of the charm quark by using measurements of the weak decay of $K^0\to\mu^+\mu^-$ (18.1) they also predicted the mass of the charm quark. The amplitude of $K^0\to\mu^+\mu^-$ is $\sim\cos\theta_c\sin\theta_c$.

The predicted rate:

$$R=\frac{K^0\to\mu^+\mu^-}{K^0\to anything}$$

was measured to be higher than what was observed. This was solved by GIM where they predicted an additional diagram (18.2). This additional diagram has

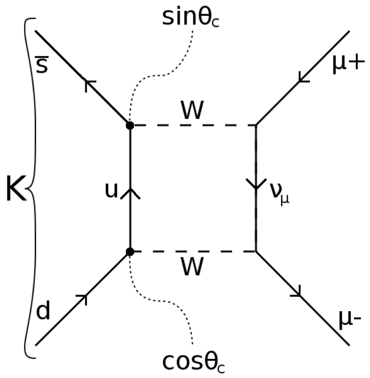

Figure 18.1: Weak decay of $K^0 \rightarrow \mu^+\mu^-$

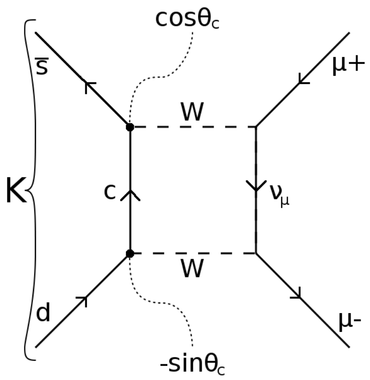

Figure 18.2: Kaon decay through c-quark exchange

an amplitude $\sim \sin\theta_c \cos\theta_c$, it is this term that causes the lower than predicted rate.

If the u and c masses had been the same then these two diagrams would cancel. Hence using the measured rate it was possible to predict the mass of the c-quark. The charm quark was discovered directly soon afterwards.

These Cabibbo terms cause certain processes to be suppressed, an example of this preferential decay is that of $D^+(c\bar{d})$. The preferred decay mode is $D^+(c\bar{d}) \rightarrow \bar{K}^0(s\bar{d}) + \pi^+(u\bar{d})$ even though the decay $D^+(c\bar{d}) \rightarrow \pi^0(d\bar{d}) + K^+(u\bar{s})$ is also allowed. The latter decay is 'doubly' suppressed by a factor $\sin^2\theta_c$ compared to the former which has a factor $\cos^2\theta_C$.

Note decay rates $\sim \cos^4\theta_c$ or $\sim \sin^4\theta_c$ as squared and contribution from each vertex.

18.1 νq scattering

We have previously calculated the weak current and cross-section for $\nu_e e^-$ scattering, we can use these results to easily find the cross-section for νq scattering. We obviously don't have a ν and quark beams so the quarks are generally constituents of hadrons which would need to be considered as deep inelastic scattering, despite this limitation we can still form a weak current for point-like scattering.

Considering 16.3 we construct the current just as we did for leptons, making the appropriate substitions to account for u and d quarks:

$$\begin{aligned} j^\mu_{lepton} &= \bar{U}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)U_e \\ \rightarrow j^\mu_{quark} &= \bar{U}_u \gamma^\mu \frac{1}{2}(1-\gamma^5)U_d \end{aligned}$$

Therefore the same procedure as before leads to:

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\nu_e d \rightarrow e^- u) &= \frac{G_F^2 s}{4\pi^2} \\ &= \left(\frac{d\sigma}{d\Omega}(\bar{\nu}_e \bar{d} \rightarrow e^+ \bar{u})\right) \\ \frac{d\sigma}{d\Omega}(\bar{\nu}_e u \rightarrow e^+ d) &= \frac{G_F^2 s}{16\pi^2}(1+\cos\theta)^2 \\ &= \left(\frac{d\sigma}{d\Omega}(\nu_e \bar{u} \rightarrow e^- \bar{d})\right) \end{aligned}$$

where θ is the centre of mass scattering angle. The form of the cross-section is often written as:

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\bar{\nu}_e u \rightarrow e^+ d) &= \frac{G_F^2 s}{16\pi^2}(1-y)^2 \\ \text{where: } (1-y)^2 &\sim \frac{1}{2}(1+\cos\theta)^2 \end{aligned}$$

and y is one of the variables associated with deep inelastic scattering.

18.2 Charged Pion and Kaon decay

Consider the decays of the charged pions and kaons (18.3):

$$\pi^\pm / K^\pm \rightarrow e^\pm / \mu^\pm + \overset{(-)}{\nu}_e / \overset{(-)}{\nu}_{\mu -}$$

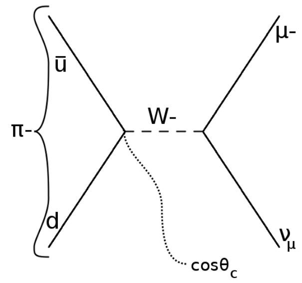

Figure 18.3: Pion decay to two muons

For $\pi \to \mu^- + \bar{\nu}_\mu$ the four momenta are:

$$\begin{aligned}\pi(q) &\to \mu^-(p) + \bar{\nu}_\mu(k)\\ q &= p + k\\ \therefore T_{fi} &= \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c q_\mu \bar{U}(P)\gamma^\mu(1-\gamma^5)V(K)\end{aligned}$$

We may have thought that at the quark-W vertex we could write the form as $\bar{U}_d\gamma^\mu \frac{1}{2}(1-\gamma^5)V_{\bar{u}}$ however the quarks are bound states and not 'free'. The meson's 4-momenta q_μ is the only 4-vector available. f_π is a function of q^2, but $q^2 = m_\pi^2$ so f_π is constant. The pion decay constant characterises the strong interaction probability of the $d\bar{u}$ process and should, in principle, be calculable in QCD

Simplify T_{fi} further

$$T_{fi} = \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c \left\{ \bar{U}(P) \underbrace{[P_\mu\gamma^\mu + K_\mu\gamma^\mu]}_{q=p+k}(1-\gamma^5)V(K) \right\}$$

recall that:

$$\begin{aligned}\not{p}\bar{U} &= m_\mu \bar{U}\\ (\not{K} + m_\nu)V &\sim \not{K}V = 0\\ \Rightarrow T_{fi} &= \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c m_\mu \left\{\bar{U}(P)(1-\gamma^5)v(K)\right\}\\ \therefore |T_{fi}|^2 &= \frac{G_F^2}{2} f_\pi^2 \cos^2\theta_c m_\mu^2 \mathrm{Tr}\left\{(\not{p} + m_\mu)(1-\gamma^5)\not{k}(1+\gamma^5)\right\}\\ &= 4G_F^2 f_\pi^2 \cos^2\theta_c m_\mu^2 (P.K)\end{aligned}$$

In the pion rest frame $\vec{k} = -\vec{p}$, so

$$\begin{aligned}P.K &= EE' + k^2\\ &= EE' + E'^2\end{aligned}$$

The decay rate is given by:

$$\begin{aligned} d\Gamma &= \frac{1}{2m_\pi} \cdot |T_{fi}|^2 \cdot \frac{d^3P}{(2\pi)^3\, 2E} \cdot \frac{d^3K}{(2\pi)^3\, 2E'} \cdot (2\pi)^4\, \delta(q-p-k) \\ &= \frac{1}{2m_\pi} \cdot \frac{G_F^2 f_\pi^2 \cos^2\theta_c m_\mu^2}{(2\pi)^2} \int \frac{d^3P\, d^3K}{EE'} \delta(m_\pi - E - E')\delta(\vec{k}+\vec{p})E'(E+E') \end{aligned}$$

The integration over d^3P is removed by the delta, there is no angular dependence, $d\Omega \to 4\pi$ leaving an integration over E'

$$\Rightarrow \Gamma = \frac{G_F^2 f_\pi^2 \cos^2\theta_c m_\mu^2}{2m_\pi(2\pi)^2}.4\pi \int dE' \cdot E'^2 \left(1 + \frac{E'}{E}\right) \delta(m_\pi - E - E')$$

Recall:

$$\begin{aligned} \delta[f(E')] &= \frac{\delta(E'-E_0')}{\left.\frac{\partial f}{\partial E'}\right|_{E'=E_0'}} \\ \text{and } E &= (m_\mu^2 + E'^2)^{\frac{1}{2}} \\ \text{where } E_0' &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \end{aligned}$$

The result of the integration in Γ is E_0^2

$$\Rightarrow \Gamma = \frac{G_F^2 f_\pi^2 \cos^2\theta_c m_\mu^2}{8\pi} \cdot m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)$$

for the decay to an electron: $m_\mu \to m_e$ and for kaon decay: $\cos^2\theta \to \sin^2\theta$ and $m_\pi \to m_K$

The ratio of muon to electron decays is:

$$\begin{aligned} R &= \frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} \\ &= \left(\frac{m_e}{m_\mu}\right)^2 \cdot \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \\ &= 1.28 \times 10^{-4} \end{aligned}$$

The decay to muons is strongly favoured over the decay to electrons, this is understood by considering the spin and direction of travel of the particles. As the pion is spin-0 and assumed to be at rest then the electron and neutrino must have opposite spins as well as travel in opposite directions. This would give them both chirality 1 (both would have spin and direction aligned) as the neutrino can only be right handed the electron must have chirality -1, which in its boosted (ie massless) state is suppressed, this suppression is less pronounced for the muon due to its higher mass.

18.3 The width of the W-boson

This is an "exact" first order calculation where we assume the masses of the products are 0. The possible channels are:

$$
\begin{aligned}
W^+ &\to e^+\nu_e \\
&\to \mu^+\nu_\mu \qquad\qquad (18.4)\\
&\to \tau^+\nu_\tau \\
&\to u + \bar{d}\,type \\
&\to c + \bar{d}\,type \\
(&\to t + \bar{d}\,type) \qquad\qquad \text{(highly suppressed)}
\end{aligned}
$$

The massless assumption is valid for these channels (except the t) as initial mass ∼1 GeV initial mass ∼80 GeV

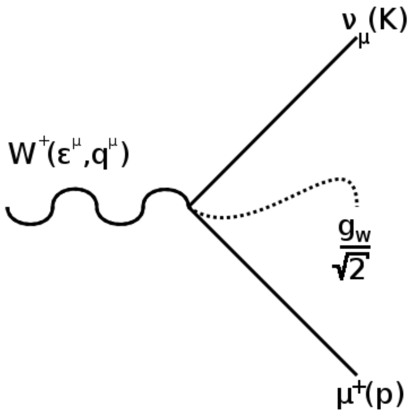

Figure 18.4: decay of the W boson to a muon and a neutrino

$$
\begin{aligned}
T_{fi} &= \frac{g_w}{\sqrt{2}}\epsilon^\mu \bar{U}(K)\gamma_\mu \frac{1}{2}(1-\gamma^5)V(P) \\
\Rightarrow |T_{fi}|^2 &= \frac{g_w^2}{2}\cdot\frac{1}{3}\cdot\sum \epsilon^\mu\epsilon^{\nu *} \\
&\quad \times \left\{\bar{V}(P)\gamma^\nu\frac{1}{2}(1-\gamma^5)U(K)\right\} \\
&\qquad \times \left\{\bar{U}(K)\gamma^\mu\frac{1}{2}(1-\gamma^5)V(P)\right\}
\end{aligned}
$$

The factor of $\frac{1}{3}$ is the average of the W polarisations.

$$|T_{fi}|^2 = \frac{1}{3}\frac{g_w^2}{2}\left[-g_{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right]\cdot \mathrm{Tr}[(\not{P} - m_\mu)\gamma^\nu \frac{1}{2}\underbrace{(1-\gamma^5)}_{\text{move to end}}\not{K}\gamma^\mu \frac{1}{2}(1-\gamma^5)]$$

$$= \frac{g_w^2}{24}\left[-g_{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right]\cdot \mathrm{Tr}[(\not{P} - m_\mu)\gamma^\nu \not{K}\gamma^\mu 2(1-\gamma^5)^2]$$

$$= \frac{g_w^2}{12}\left[-g_{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right]\cdot \mathrm{Tr}[\not{P}\gamma^\nu \not{K}\gamma^\mu (1-\gamma^5)]$$

remove m_μ term as $\mathrm{Tr}[\gamma^\alpha\gamma^\beta\gamma^\gamma] = 0$ now using trace theorem 2 and that the γ^5 term is 0:

$$|T_{fi}|^2 = \frac{g_w^2}{12}\left[-g_{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right]\cdot 4\cdot [P_\mu K_\nu + P_\nu K_\mu - g_{\mu\nu}(P.K)]$$

$$= \frac{g_w^2}{3}\left[- g_{\mu\nu}(P_\mu K_\nu + P_\nu K_\mu) - g_{\mu\nu}(-g_{\mu\nu}(P.K)) + \frac{q^\mu q^\nu}{M^2}(P_\mu K_\nu + P_\nu K_\mu) + \frac{q^\mu q^\nu}{M^2}(-g_{\mu\nu}(P.K))\right]$$

$$= \frac{g_w^2}{3}\left[-2(P.K) + 4(P.K) + \frac{2}{M^2}(q.P)(q.K) - \frac{1}{M^2}q^2(P.K)\right] \quad \text{recall: } (q^2 = M^2)$$

$$= \frac{g_w^2}{3}\left[(P.K) + \frac{2}{M^2}(q.P)(q.K)\right]$$

In the W rest frame:

$$q.K = \frac{M^2}{2}\left(1 - \frac{m_\mu^2}{M^2}\right)$$

$$q.P = \frac{M^2}{2}\left(1 + \frac{m_\mu^2}{M^2}\right)$$

$$\Rightarrow |T_{fi}|^2 = \frac{g_w^2}{3}\left[\frac{2}{M^2}\frac{M^2}{2}\left(1 + \frac{m_\mu^2}{M^2}\right)\frac{M^2}{2}\left(1 - \frac{m_\mu^2}{M^2}\right)\right]$$

$$= \frac{g_w^2}{3}\frac{M^2}{2}\left(1 + \frac{m_\mu^2}{M^2}\right)\left(1 - \frac{m_\mu^2}{M^2}\right)$$

$$\Gamma = \frac{g_w^2}{48\pi}M\left(1 + \frac{m_\mu^2}{M^2}\right)\left(1 - \frac{m_\mu^2}{M^2}\right)$$

$$= 0.227 \text{ GeV}$$

The total W width is

$$\Gamma_{total} = N_l\Gamma_{l\nu} + N_c N_q \Gamma_{l\nu}$$

$$= (3 + 3\times 2)\Gamma_{l\nu} \qquad \text{(only 2 generations of quark used)}$$

$$= 2.05 \text{ GeV}$$

The value (PDG) is measured as $= 2.141 \pm 0.041$ GeV

Chapter 19

Neutral Current

19.1 Weinberg-Salam model

This will allow us to find the neutral current coupling of the Z^0 to fundamental particles. After covering QCD we will return to the Weinberg-Salam model starting from the gauge symmetry and then the breaking of this symmetry by the Higgs mechanism.

A certain structure was perceived in the charged current weak process and similarities were seen between the weak and electromagnetic interactions. The charged current weak interaction operates between left-handed leptons and quark doublets.

$$\chi_L = \frac{1}{2}(1-\gamma^5)\begin{pmatrix}\nu_e \\ e-\end{pmatrix}$$

When a W^+ is absorbed the matrix element is:

$$\begin{aligned}
&= \bar{U}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)U_e \\
&= \bar{U}_\nu \gamma^\mu \frac{1}{4}(1-\gamma^5)^2 U_e \\
&= \bar{U}_\nu \frac{1}{2}(1+\gamma^5)\gamma^\mu \frac{1}{2}(1-\gamma^5)U_e
\end{aligned}$$

Using the 2 component weak isospinors we can write@

$$\chi_L \gamma^\mu \tau^+ \chi_L$$

where τ^+ is the raising operator in weak isospin space.

We also know the form of the electromagnetic current

$$-e\bar{U}\gamma^\mu U$$

or

$$j^\mu_{EM} = -e(\bar{U}_L\gamma^\mu U_L + \bar{U}_R\gamma^\mu U_R)$$

The charged current interaction is:

$$\chi_L \gamma^\mu \tau^+ \chi_L = (\bar{\nu}_e, \bar{e}^-)_L \gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$$

if the symmetry were exact we would expect the following interaction:

$$\begin{aligned} &= (\bar{\nu}_e, \bar{e}^-)_L \gamma^\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \\ &= \frac{1}{2}\bar{\nu}_e \gamma^\mu \nu_e - \frac{1}{2}\bar{e}_L^- \gamma^\mu e_L^- \end{aligned}$$

But this neutral current does not exist. This theory was used before the discovery of neutral current which is not compatible with the above form.

There is another entity which is the right-handed electron. Since this object is not involved in charged current weak interactions it must be a singlet in weak isospin space.

So we must invent a new quantity which differentiates between the states in the electron left-handed doublet and the electron right-handed singlet, i.e. the members of the doublet must have the same value of their of their quantum number.

The solution to this is weak hypercharge:

$$Y = 2Q - 2T_3$$

	T_3	Q	Y
ν_e	$\frac{1}{2}$	0	-1
e_L^-	$-\frac{1}{2}$	-1	-1
e_R^-	0	-1	-2
u_L	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
d_L	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
u_R	0	$\frac{2}{3}$	$\frac{4}{3}$
d_R	0	$-\frac{1}{3}$	$-\frac{2}{3}$

For the electromagnetic interaction we can multiply the wavefunction by phase factors i.e.

$$\Psi' = e^{iq\chi}\Psi$$

if χ is a number such a transformation is called a global phase or gauge transformation. However if χ is $\chi(x,t)$ then we have a local gauge transformation and the electromagnetic field, A^μ, has to be introduced so that the gauge invariance of the Lagrangian is preserved.

If we require the new theory to be invariant with respect to the local transformation of the form

$$e^{iY\chi(x,t)}$$

then a massless B^μ field has to be introduced.

The weak left handed isospinor is transformed by an (S)unitary transformation involving 3 parameters.

Alternatively, the SU(2) spinor is in an O(3) space which requires 3 axis, these axis can be defined locally under gauge transformation and 3 weak massless fields must be introduced if the system is to be invariant with respect to these gauge transformations. The interactions will be of the form:

$$g_w \bar{\chi}_L \gamma^\mu \frac{\tau_i}{2} \chi_L W_\mu^2 + \frac{g'}{2} j_\mu^Y B^\mu$$

or:

$$g_w \bar{\chi}_L \gamma^\mu \frac{\tau_i}{2} \chi_L W_\mu^2 + \frac{g'}{2} (j_\mu^{EM} - j_\mu^3) B^\mu$$

When the symmetries breaks the massless W_μ^1 and W_μ^2 fields combine to give the massive W^+ and W^- bosons. The W_μ^3 and B^μ fields mix to give a massless photon and the massive Z^0 i.e.

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}$$

where θ_w is the Weinberg, or weak, mixing angle.

Now express, in terms of A_μ and Z_μ:

$$\begin{aligned} &= g_w j_\mu^3 W_3^\mu + g'(j_{EM}^\mu - j_3^\mu) B_\mu \\ &= (g_w j_\mu^3 \sin\theta_w + g' j_\mu^{EM} \cos\theta_w - g' j_\mu^3 \cos\theta_w) A_\mu \\ &\quad + (g_w j_\mu^3 \cos\theta_w - g' j_\mu^{EM} \sin\theta_w + g' j_\mu^3 \sin\theta_w) Z_\mu \end{aligned}$$

given that:

$$\begin{aligned} B_\mu &= A_\mu \cos\theta_w - Z_\mu \sin\theta_w \\ W_\mu^3 &= A_\mu \sin\theta_w + Z_\mu \cos\theta_w \end{aligned}$$

The A^μ term only couples to the electromagnetic interaction so the first and third terms within the bracket cancel:

$$\begin{aligned} g_w j_\mu^3 \sin\theta_w &= -g' j_\mu^3 \cos\theta_w \\ &= 0 \\ \Rightarrow g' &= g_w \tan\theta_w \end{aligned}$$

The second term in the A^μ bracket must equal the electron charge:

$$\begin{aligned} g_w \tan\theta_w \cos\theta_w &= Q = e \\ g_w \sin\theta_w &= e \\ \Rightarrow g' \cos\theta_w &= e \end{aligned}$$

the 2 couplings: g_w and g' can be replaced by e and θ_w, where is determined by experiment .

From the Z_μ terms we can get the coupling of the Z^0 to leptons and quarks.

$$\begin{aligned}
&= (g_w j^3_\mu \cos\theta_w - g' j^{EM}_\mu \sin\theta_w + g' j^3_\mu \sin\theta_w) Z_\mu \\
&= (g_w j^3_\mu \cos\theta_w - g_w \tan\theta_w j^{EM}_\mu \sin\theta_w + g_w \tan\theta_w j^3_\mu \sin\theta_w) Z_\mu \\
&= \frac{g_w}{\cos\theta_w}(j^3_\mu \cos^2\theta_w - j^{EM}_\mu \sin^2\theta_w + j^3_\mu \sin^2\theta_w) Z_\mu \\
&= \frac{g_w}{\cos\theta_w}(j^3_\mu - j^{EM}_\mu \sin^2\theta_w) Z_\mu \\
\Rightarrow j^{NC}_\mu &= j^3_\mu - j^{EM}_\mu \sin^2\theta_w
\end{aligned}$$

Using the forms of the current:

$$j^{NC}_\mu = \bar{\Psi}_f \gamma^\mu \left(\frac{1}{2}(1-\gamma^5)T^3 - \sin^2\theta_w Q\right)\Psi_f$$

Where Ψ_f is the wavefunction of a fermion.

$$j^{NC}_\mu = \bar{\Psi}_f \gamma^\mu \frac{1}{2}\left(C^f_V - C^f_A \gamma^5\right)\Psi_f$$

Where the vector (C_V) and vector-axial (C_A) couplings are determined in the standard model, given θ_w and equal to:

$$\begin{aligned}
C^f_V &= T^3_f - 2\sin^2\theta_w Q_f \\
C^f_A &= T^3_f
\end{aligned}$$

Where T^3_f and Q_f are the third component of weak isospin and the electromagnetic charge of the fermion, f, respectively.

	C_v	C_A
$(\nu_e, \nu_\mu, \nu_\tau)$	$\frac{1}{2}$	$\frac{1}{2}$
(e^-, μ^-, τ^-)	$\frac{1}{2} + 2\sin^2\theta_w$	$-\frac{1}{2}$
(u, c, t)	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$
(d, s, b)	$-\frac{1}{2} - \frac{2}{3}\sin^2\theta_w$	$-\frac{1}{2}$

19.2 Width of the Z^0

For $Z^0 \to f\bar{f}$ we get a transmission of:

$$T_{fi} = \frac{g_w}{\cos\theta_w}\epsilon_\mu \bar{U}(P')\gamma^\mu \frac{1}{2}(C^f_V - C^f_A\gamma^5)V(P)$$

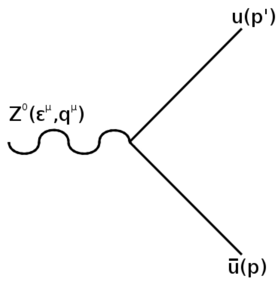

Figure 19.1: decay of the Z^0 to a $u\bar{u}$ pair

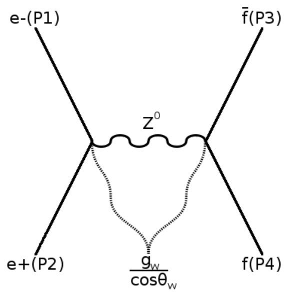

Figure 19.2: fermion production via electron annihilation mediated by a Z^0

19.3 Cross-section for $e^+e^- \xrightarrow{Z^0} f\bar{f}$

$$
\begin{aligned}
|T_{fi}|^2 = \frac{g_w}{\cos^2\theta_w} & \left[\bar{V}(P_2)\gamma^\mu \frac{1}{2}(C_V^e - C_A^e\gamma^5)U(P_1)\right] \\
& \cdot \left(\frac{-g_{\mu\nu} + \frac{q^\mu q^\nu}{M^2}}{s - M^2 - iM\Gamma}\right) \\
& \qquad \cdot \left[\bar{U}(P_4)\gamma_\nu \frac{1}{2}(C_V^f - C_A^f\gamma^5)V(P_3)\right]
\end{aligned}
$$

The '$iM\Gamma$' term is included as the Z^0 can decay. For unpolarised electrons undergoing $e^-e^+ \to \mu^-\mu^+$:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left[A_o(1+\cos^2\phi) + A_1\cos\phi\right]$$

Where A_0 and A_1 are functions of C_V and C_A, Z^0 introduces an asymmetry, the A_1 term.

For lowest order QED calculations, we take $A_0 = 1$ and $A_1 = 0$, so a symmetric distribution is predicted and the weak interaction introduces a forward-backwards asymmetry. Measurements at the e^+e^- PETRA collider confirmed these interference effects of the virtual Z^0 and γ contributions. At $\sqrt{s} = 34$ GeV the cross-section versus $\cos\theta$ was measured and compared with QED.

The cross-section at the Z^0 peak for $e^+e^- \rightarrow f\bar{f}$ is ~30 mb. However the resonance curve is modified by radiative processes (19.3) These corrections can

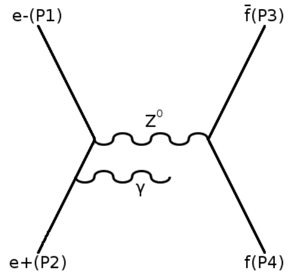

Figure 19.3: Radiative correction to $e^+e^- \xrightarrow{Z^0} f\bar{f}$

be calculated; they lead to a shift in the peak and a reduction in the peak cross-section.

Chapter 20

The Strong Force (QCD)

20.1 Deep inelastic scattering

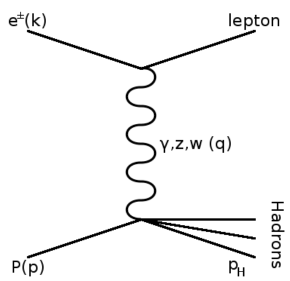

Figure 20.1: Deep inelastic scattering of an electron from a proton

For such a process the following dimensionless variables are defined:

$$
\begin{aligned}
x &= \frac{-q^2}{2p.q} \\
&= \frac{Q^2}{2p.q} \\
y &= \frac{q.p}{k.p}
\end{aligned}
$$

at the proton vertex:

$$
\begin{aligned}
q + p &= p_H \\
q^2 + p^2 + 2p.q &= p_H^2 \\
q^2 + 2p.q + m_p^2 &= m_H^2
\end{aligned}
$$

so m_H is a function of the invariant q^2 and $2p.q$ and x is the ratio of the invariants characterising the hadronic vertex.

Historically, the ignorance of the hadron vertex was (is) described by structure functions.

We know that probes: $\gamma, W^\pm, Z^0$ interact with quarks and anti-quarks in the proton/neutron. The q and $\bar{q}$ interact with gluons which carry around 50 % of the nucleon momentum. The nucleon is a QCD system of $q\bar{q}$ and gluons all interacting together.

We therefore calculate deep inelastic scattering cross-section by calculating the scattering of leptonic probes by q and $\bar{q}$ constituents and modify to account for qg interactions.

20.2 The quark-parton model

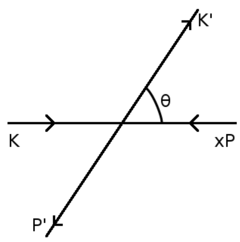

Figure 20.2: Schematic of deep inelastic scattering under the quark-parton model

At the hadronic vertex the incoming quark is assumed to carry a fraction, x, of the whole proton's 4-momenta, P; i.e the individual quark has 4-momenta xP.

$$\begin{aligned} q + xP &= p' \\ q^2 + 2xP.q + x^2P^2 &= (P')^2 \\ \Rightarrow x &= \frac{-q^2}{2q.P} \qquad\qquad (q \gg xp \text{ and } q^2 \gg m_H^2) \end{aligned}$$

What is $\hat{s}$, the centre of mass energy, in the quark-parton system?

$$\begin{aligned} s &= (K+P)^2 \\ &\simeq 2\vec{K}.\vec{P} \\ \Rightarrow \hat{s} &= (K+xP)^2 \\ &\simeq 2x\vec{K}.\vec{P} \\ \Rightarrow \hat{s} &= xs \end{aligned}$$

y in the quark parton model.

$$\begin{aligned} y &= \frac{q.P}{K.P} \\ &\rightarrow \frac{xq.P}{xK.P} \\ &= \frac{(K-K').P}{K.P} \\ &= 1 - \frac{K'.P}{K.P} \qquad\qquad (= 1 - \frac{u}{s}) \\ &= 1 - \frac{E'E_q - E'E_q\cos\theta}{E'E_q - E'E_q(-1)} \\ &= 1 - \frac{1+\cos\theta}{2} \\ &= \frac{1}{2}(1-\cos\theta) \end{aligned}$$

Therefore $y = 1$ signifies a very deeply inelastic collision and occurs when $\cos\theta = -1$

Now we want an expression for $e^{\pm}$ scattering off a nucleon via photon exchange. Recall $e\mu$ scattering:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \cdot 2e^4 \left(\frac{s^2+u^2}{t^2}\right)$$

for quarks we replace e^4 with $e^4 q_i^2$ where $q_i^2 = +\frac{2}{3}, -\frac{1}{3}$ etc

As:

$$
\begin{aligned}
y &= \frac{1}{2}(1-\cos\theta) \\
dy &= -\frac{d(\cos\theta)}{2} \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{d\sigma}{2\pi d(\cos\theta)} \\
&= \frac{1}{64\pi^2\hat{s}}\cdot 2e^4q_i^2\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right) \\
\frac{d\sigma}{2\pi 2dy} &= \frac{1}{64\pi^2 Q^4\hat{s}}\cdot 2e^4q_i^2\hat{s}^2\left(1+\frac{\hat{u}^2}{\hat{s}^2}\right) && (t^2 = Q^4) \\
\frac{d\sigma}{dy} &= \frac{1}{8\pi}\cdot\frac{e^4q_i^2}{Q^4}\cdot\hat{s}(1+(1-y)^2) && (\frac{u}{s} = 1-y) \\
\therefore \frac{d\sigma}{dy} &= \frac{2\pi\alpha^2}{Q^4}q_i^2\hat{s}(1+(1-y)^2)
\end{aligned}
$$

In the nucleon the fraction of quarks having x between x and $x+\delta x$ is $Q_i(x,Q^2)dx$ where Q_i is the parton distribution function (PDF) for the quark concerned. PDFs cannot be calculated in QCD, but their evolution with x and Q^2 can be.

The aim of deep inelastic scattering experiments is to measure $Q_i(x,Q^2)$ and compare their evolution with theory: Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations.

$$
\begin{aligned}
\left.\frac{d\sigma}{dy}\right|_{x\to x+\delta x} &= \frac{2\pi\alpha^2}{Q^4}q_i^2(1+(1-y)^2)\cdot xs\; Q(x)dx \\
\frac{d\sigma}{dxdy} &= \frac{2\pi\alpha^2}{Q^4}q_i^2(1+(1-y)^2)\cdot s\left(xQ_i(x)+x\bar{Q}_i(x)\right)
\end{aligned}
$$

The photon interacts equally with q a $\bar{q}$. In the absence of QCD effects the Q_i are functions of x, this is known as "scaling". However QCD effects cause Q_i to become a function of x and Q^2 (hence "scaling violation"). There is also a gluon distribution in the proton. For a ν beam we worked out the proton cross-sections so:

$$
\frac{d\sigma}{dxdy} \propto \left(xQ_d(x)+x\bar{Q}_u(x)\;(1-y)^2\right)
$$

as this is a charged current w exchange. Here particular quarks are 'picked out' and one can measure a specific density (either u or d type). To encompass quark densities they are written in terms of a structure function:

$$
\frac{F_2}{x} \sim \sum_i \left(Q_i(x,Q^2)+\bar{Q}_i(x,Q^2)\right)
$$

At fixed Q^2, experiments determine F_2. The experiments are $e^\pm + p, \nu + n$ etc. All experimental points are fed into a fit to determine $u(x), d(x), s(x)$ etc

20.3 Drell-Yan process

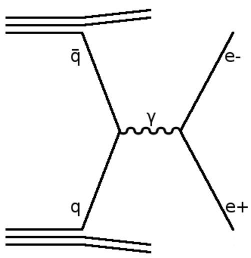

Figure 20.3: Drell-Yan interaction between two quarks resulting in two electrons

In the centre of mass frame the kinematics for the Drell-Yan process (20.3) are:

$$\begin{aligned}
x_1 p_1 &= x_1(p, 0, 0, p) \\
x_2 p_2 &= x_2(p, 0, 0, -p) \\
q &= ([x_1 + x_2]p, 0, 0, [x_1 - x_2]p) \\
q^2 &= \left([x_1 + x_2]^2 - [x_1 - x_2]^2\right) p^2 \\
&= 4x_1 x_2 p^2 \\
&= x_1 x_2 s \\
&= \hat{s}
\end{aligned}$$

since we can approximate Drell-Yan processes to $e^- e^+ \to \mu^- \mu^+$:

$$\begin{aligned}
\sigma(e^- e^+ \to \mu^- \mu^+) &= \frac{4\pi\alpha^2}{3s} \\
\Rightarrow \frac{d^2\sigma}{dx_1 dx_2} &= \frac{1}{3} \cdot \frac{4\pi\alpha^2}{3\hat{s}} q_i^2 \left(Q_i(x_1)\, \bar{Q}_i(x_2) + \bar{Q}_i(x_1)\, Q_i(x_2) \right)
\end{aligned}$$

Many processes are Drell-Yan like: eg. $u + \bar{d} \to e^+ + \nu_e$.

As well as the parton distributions their gluon equivalents must also be considered as these are important for predicting certain processes: eg. Higgs production at the LHC (20.4). Colour interactions are assumed to be a very similar to electromagnetic interactions so the rules for QED are used with the substitution: $\alpha \to \alpha_s$ For example in quark scattering, $q_1 q_2 \to q_1 q_2$:

$$|T_{fi}|^2 \sim \frac{4}{9} \left(\frac{s^2 + u^2}{t^2} \right) \qquad \left(\frac{4}{9} \text{ is a colour factor}\right)$$

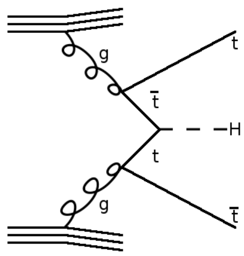

Figure 20.4: Higgs production through a Drell-Yan type process

or for $q\bar{q} \to q\bar{q}$

$$
\begin{aligned}
|T_{fi}|^2 &\sim \frac{4}{9}\left(\frac{s^2+u^2}{t^2}+\frac{t^2+u^2}{s^2}\right) \\
&\sim \frac{8}{27}\frac{u^2}{st}
\end{aligned}
$$

20.4 Evolution of the structure functions (DGLAP)

20.4.1 QCD Compton scattering

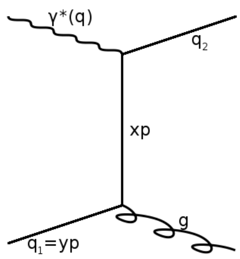

Figure 20.5: QCD Compton-Scattering $\gamma^* p \to qg$

Kinematics in $\gamma^* p$ (photon/proton) frame (20.5)

$$\begin{aligned}
z &= \frac{Q^2}{2q_1.q} \\
&= \frac{x}{y} \\
&= \frac{Q^2}{2yp.q} \\
\hat{s} &= (yp+q)^2 \\
&= y^2p^2 + q^2 + 2yp.q \\
\therefore \hat{s} &= 0 - Q^2 + \frac{Q^2}{z} \\
&= \frac{Q^2(1-z)}{z}
\end{aligned}$$

Kinematics in the $\gamma^* q$ (photon/quark) frame (20.6):

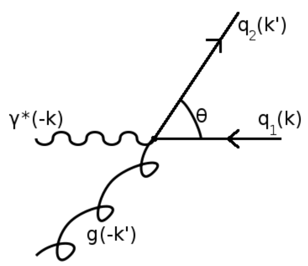

Figure 20.6: $\gamma * q$ frame QCD Compton scattering

$$\begin{aligned}
\hat{s} &= (g+q_2)^2 \\
&\simeq 2g.q_2 \\
&= 2k'.k' - 2|k'||k'|\cos(180°) \\
&= 4(k')^2 \\
\hat{t} &= (g-q1)^2 \\
&= -2k\,k'(1-\cos\theta) \\
\hat{u} &= (q_1-q_2)^2 \\
&= -2k\,k'(1+\cos\theta) \\
\text{also:} \quad -\hat{u}-\hat{t} &= 4k\,k' \\
\text{Recall: } \hat{s}+\hat{u}+\hat{t} &= -Q^2 \\
\therefore -\hat{u}-\hat{t} &= \hat{s}+Q^2 \\
&= 4k\,k'
\end{aligned}$$

The QED Compton process is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left\{-\frac{u}{s} - \frac{s}{u} + \frac{2tQ^2}{su}\right\}$$

Now convert this to QCD:

$$\frac{d\sigma}{d\Omega}\bigg| QCD = C_f \frac{\alpha\alpha_s}{2\hat{s}} \cdot e_i^2 \left\{-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}\right\}$$

Where the $\alpha\alpha_s$ term is due to the one QED and one QCD vertex and C_f is the colour factor, $C_f = \frac{4}{3}$. There are 8 possible colour configurations; then averaging over the number of quark colours gives $\frac{8}{3}$ hence $\frac{4}{3}$ with a factor $\frac{1}{2}$ from α_s.

A more interesting quantity is the transverse momentum of the outgoing quark, $p_\perp = k' \sin\theta$. Now:

$$\begin{aligned}
\hat{s}\hat{t}\hat{u} &= 4(k')^2(-2k.k')(1-\cos\theta)(-2k.k')(1+\cos\theta) \\
&= 16(k')^2(k.k')^2 \sin^2\theta \\
&= 16\, k^2(k')^2 p_\perp^2 \\
&= (\hat{s}+Q^2)^2 p_\perp^2 && (\hat{s}+Q^2 = 4k.k')
\end{aligned}$$

For small scattering angles

$$\begin{aligned}
dp_\perp^2 &= (k')^2 d(\sin^2\theta) \\
&= 2(k')^2 \sin\theta\cos\theta \; d\theta \\
&= \frac{\hat{s}}{2}\sin\theta \; d\theta \\
d\Omega &= 2\pi \sin\theta \; d\theta \\
\Rightarrow dp_\perp^2 &= \frac{\hat{s}}{2}\frac{d\Omega}{2\pi} \\
\Rightarrow d\Omega &= \frac{4\pi}{\hat{s}} dp_\perp^2
\end{aligned}$$

in the small angle limit $-\hat{t} \ll \hat{s}$

$$\begin{aligned}
\Rightarrow p_\perp^2 &= \frac{\hat{s}\hat{t}(-\hat{s} - Q^2)}{(\hat{s} + Q^2)^2} \\
&= \frac{\hat{s}(-\hat{t})}{\hat{s} + Q^2} \qquad (1) \\
\Rightarrow \frac{d\sigma}{dp_\perp^2} &= \frac{4}{3} \cdot \frac{4\pi}{\hat{s}} \cdot \frac{\alpha\alpha_s e_i^2}{2\hat{s}} \cdot \left\{ -\frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}} \right\} \\
&= \frac{8\pi}{3} \cdot \frac{\alpha\alpha_s e_i^2}{\hat{s}^2} \cdot \left(\frac{-1}{\hat{t}} \right) \cdot \left\{ \hat{s} + 2\frac{(\hat{s} + Q^2)Q^2}{\hat{s}} \right\} \\
\text{Using: 1 and } z &= \frac{Q^2}{\hat{s} + Q^2} = \frac{x}{y} \\
\Rightarrow \frac{d\sigma}{dp_\perp^2} &= \frac{4\pi^2\alpha}{\hat{s}} \cdot \frac{2\alpha_s}{3\pi} \cdot \frac{e_i^2}{\hat{s}} \cdot \frac{\hat{s}}{p_\perp^2(\hat{s} + Q^2)} \cdot \left\{ \hat{s} + 2\frac{(\hat{s} + Q^2)Q^2}{\hat{s}} \right\} \\
&= \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \cdot \frac{e_i^2}{p_\perp^2} \cdot \left\{ \frac{\hat{s}}{(\hat{s} + Q^2)} + 2\frac{Q^2}{\hat{s}} \right\}
\end{aligned}$$

Where $\sigma_0 = \frac{4\pi^2\alpha}{s}$ is the $\gamma^* p$ total cross-section

$$\begin{aligned}
\frac{d\sigma}{dp_\perp^2} &= \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \cdot \frac{e_i^2}{p_\perp^2} \cdot \left\{ \frac{\hat{s}^2 + 2Q^2(\hat{s} + Q^2)}{(\hat{s} + Q^2)\hat{s}} \right\} \\
&= \sigma_0 \cdot \frac{\alpha_s}{2\pi} \cdot \frac{e_i^2}{p_\perp^2} \cdot P_{qq}(z)
\end{aligned}$$

Where:

$$P_{qq}(z) = \frac{4}{3} \cdot \left(\frac{1 + z^2}{1 - z} \right)$$

is the probability of a quark emitting a gluon and so becoming a quark with momentum reduced by a factor, z.

$$\sigma(\gamma q \to qg) = \int_{\mu^2}^{\frac{\hat{S}}{4}} dp_\perp^2 \frac{d\sigma}{dp_\perp^2}$$

where μ^2 is a cut of so that the integral is not divergent

$$\begin{aligned}
\sigma(\gamma q \to qg) &= e_i^2 \sigma_o \int_{\mu^2}^{\frac{\hat{S}}{4}} \frac{dp_\perp^2}{p_\perp^2} \cdot \frac{\alpha_s}{2\pi} \cdot P_{qq}(z) \\
&= e_i^2 \sigma_o \frac{\alpha_s}{2\pi} P_{qq}(z) \log\left(\frac{Q^2}{\mu^2} \right)
\end{aligned}$$

$$\frac{F_2}{x} = \sum_q e_q^2 \int_x^1 \frac{dy}{y} Q(y) \left[\delta(1 - \frac{x}{y}) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{y}\right) \log\left(\frac{Q^2}{\mu^2}\right) \right]$$

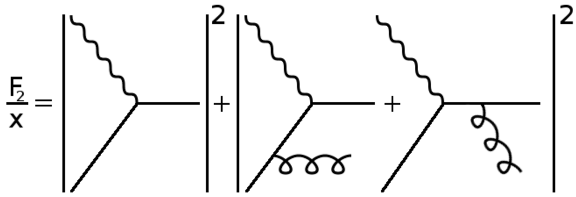

Figure 20.7: Leading order and next to leading order gluon vertices

20.5 Boson-Gluon fusion

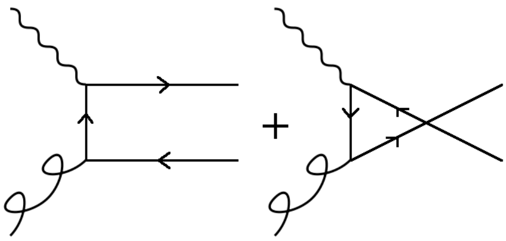

Figure 20.8: Boson-gluon fusion, u and t channel

$$\left.\frac{d\sigma}{d\Omega}\right|_{BGF} = \frac{1}{4}\frac{e_i^2\alpha\alpha_s}{s}\left[\frac{u}{t}+\frac{t}{u}-2\frac{sQ^2}{tu}\right]$$

$$\left.\frac{F_2(x,Q^2)}{x}\right|_{BGF} = \sum_q e_q^2\int_x^1\frac{dy}{y}G(y)\frac{\alpha_s}{2\pi}P_{qg}\left(\frac{x}{y}\right)\log\left(\frac{Q^2}{\mu^2}\right)$$

Where $G(y)$ is the gluon density function in the proton and:

$$P_{qg} = \frac{1}{2}\left(z^2+(1-z)^2\right)$$

is the probability that a gluon splits into a $q\bar{q}$ pair so:

$$\frac{dQ_i(x,Q^2)}{d(\log Q^2)} = \frac{\alpha_s}{2\pi}\int_x^1\frac{dy}{y}\left\{Q_i(y,Q^2)P_{qq}\left(\frac{x}{y}\right)+G(y,Q^2)P_{qg}\left(\frac{x}{y}\right)\right\}$$

$$\frac{dG(x,Q^2)}{d(\log Q^2)} = \frac{\alpha_s}{2\pi}\int_x^1\frac{dy}{y}\left\{Q_i(y,Q^2)P_{gq}\left(\frac{x}{y}\right)+G(y,Q^2)P_{gg}\left(\frac{x}{y}\right)\right\}$$

These are the DGLAP equations at leading order $i\pi\alpha_s$

Chapter 21

Local gauge invariance and determination of the form of interactions

21.1 Local gauge invariance of electromagnetic field

We require the Dirac equation to be covariant if the wavefunction undergoes local phase transformation.

$$\Psi \to \Psi' = e^{iq\alpha(x)}\Psi$$

The Dirac equation is:

$$(i\gamma_\mu\partial^\mu - qA^\mu - m)\Psi = 0$$

Under the local gauge transformation we have:

$$e^{iq\alpha(x)}(i\gamma_\mu\partial^\mu + i\gamma_\mu\partial^\mu(iq\alpha) - q\gamma_\mu A'^\mu - m)\Psi = 0$$

But $A'^\mu = A^\mu - \partial^\mu\alpha$

$$\begin{aligned}
\Rightarrow 0 &= e^{iq\alpha(x)}(i\gamma_\mu\partial^\mu - q\gamma_\mu\partial^\mu\alpha - q\gamma_\mu(A^\mu - \partial^\mu\alpha) - m)\Psi \\
&= e^{iq\alpha(x)}(i\gamma_\mu\partial^\mu - q\gamma_\mu\partial^\mu\alpha + q\gamma_\mu\partial^\mu\alpha - q\gamma_\mu A^\mu - m)\Psi \\
&= e^{iq\alpha(x)}(i\gamma_\mu\partial^\mu - q\gamma_\mu A^\mu - m)\Psi
\end{aligned}$$

If we require the Dirac equation to be gauge invariant (covariant) then the massless electromagnetic field must be introduced where A^μ is invariant under gauge transform

$$A'^\mu = A^\mu - \partial^\mu\alpha$$

This involves replacing $i\partial^\mu$ in the equation with: $i\partial^\mu - qA^\mu$ or

$$D^\mu = \partial^\mu + iqA^\mu$$

We need to demonstrate that D^μ transforms in the same way as Ψ:

$$\begin{aligned}
(D^\mu\Psi)' &= D'^\mu\Psi' \\
&= (\partial^\mu + iqA^\mu)' e^{iq\alpha(x)}\Psi \\
&= e^{iq\alpha(x)}(\partial^\mu + iq\partial^\mu\alpha + iqA'^\mu)\Psi \\
&= e^{iq\alpha(x)}(\partial^\mu + iq\partial^\mu\alpha + iqA^\mu - iq\partial^\mu\alpha)\Psi \\
&= e^{iq\alpha(x)}(\partial^\mu + iqA^\mu)\Psi \\
&= e^{iq\alpha(x)}D^\mu\Psi
\end{aligned}$$

Now look at the Lagrangian for the free Dirac equation, a Lagrangian can be constructed such that a simple operation yields the Dirac equation.

$$\mathcal{L} = i\bar{\Psi}\gamma_\mu\partial^\mu\Psi - m\bar{\Psi}\Psi$$

for the Dirac equation to remain covariant we have to introduce a massless field changing ∂_μ to $\partial_\mu + iqA_\mu$ so the Dirac Lagrangian becomes:

$$\begin{aligned}
\mathcal{L} &= i\bar{\Psi}\gamma_\mu D^\mu\Psi - m\bar{\Psi}\Psi \\
&= \bar{\Psi}(-i\gamma_\mu D^\mu - m)\Psi - q\bar{\Psi}\gamma_\mu\Psi A^\mu
\end{aligned}$$

Hence, by demanding local gauge invariance we are forced to introduce a vector field, A^μ, called the gauge field which couples to the Dirac particle (charge q) in exactly the same way as the photon field. The new interaction may be written as $j_\mu A^\mu$ where $j_\mu = -e\bar{\Psi}\gamma_\mu\Psi$ is the current density.

If we consider this new field as a physics photon field we must add a term corresponding to its kinetic energy. As the kinematic term must be invariant it can only involve the gauge invariant field tensor.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

so the Lagrangian of WED is:

$$\mathcal{L}_{QED} = \bar{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi + e\bar{\Psi}\gamma_\mu A^\mu\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The addition of a mass term is prohibited by gauge invariance. This can be seen by substituting $A'^\mu = A^\mu + \partial^\mu\alpha$ into the Proca equation, if gauge invariance is required then it changes, only as $m = 0$ does it become invariant. This shows that the photon must be massless. The photon couples only to charge and therefore cannot self couple.

21.2 Local gauge invariance and the electro-weak interaction

We have already seen that the electro-weak interaction acts between left-handed weak isospin doublets, χ_L, and right-handed weak isospin singlets, Ψ_R, We have also identified hyper charge as a charge like object in the weak isospin space. The left and right handed components transform as:

$$\begin{aligned}\chi_L \to \chi_L' &= \exp\left\{ i g_w \alpha(x)\frac{\tau}{2} + i g' \beta(x)\frac{Y}{2}\right\}\ \chi_L \\ \Psi_R \to \Psi_R' &= \exp\left\{ i g' \beta(x)\frac{Y}{2}\right\}\ \Psi_R\end{aligned}$$

For a Dirac-like equation to remain covariant under such a transformation we must introduce 4 massless fields: W_μ^1, W_μ^2, W_μ^3 and B^μ in the following way:

$$\begin{aligned}\mathcal{L} = \bar{\chi}_L\gamma^\mu &\left(i\partial_\mu - \frac{g_w}{2}\tau.W_\mu - \frac{g'}{2}Y.B_\mu\right)\chi_L \\ &+ \bar{\Psi}_R\gamma^\mu\left(i\partial_\mu - \frac{g'}{2}Y.B_\mu\right)\Psi_R \\ &\quad - \frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\end{aligned}$$

Where the final two terms are the kinetic energy and self coupling of the W_μ and B^μ fields. Note there are no mass terms above. Adding simple mass terms is not possible without breaking gauge invariance. To add mass (as is experimentally required) in a gauge invariant manner we use the Higgs mechanism.

Mass terms are of the form:

$$\bar{\Psi}m\Psi = (\bar{\Psi}_L + \bar{\Psi}_R)m(\Psi_L + \Psi_R)$$

such terms would link a left-handed electron in the SU(2) weak doublet to a right-handed electron which is a singlet. If SU(2)$\otimes$U(1) symmetry were exact this could not happen. Also, for the fields we have introduced to preserve gauge invariance they have to be massless, since terms like $mB_\mu B^\mu$ do not preserve gauge invariance.

Consider how W_μ changes under a gauge transform.

$$\chi_L' = \exp\left\{ i\frac{g_w}{2}\alpha_i(x)\tau\right\}\chi_L \qquad (g' \text{ term acts with } B^\mu)$$

Then:

$$(D^\mu\chi_L)' = \exp\left\{ i\frac{g_w}{2}\alpha_i(x)\tau\right\} D^\mu\chi_L$$

Make an infinitesimal change to $W_i'^\mu = W_i^\mu + \delta W_i^\mu$

$$(D^\mu \chi_L)' = \left\{\partial^\mu + ig_w \frac{\tau_i}{2} W_i^\mu + ig_w \frac{\tau_i}{2}\delta W_i^\mu\right\} \chi_L' \qquad \text{LHS}$$

Where: $D^\mu = \partial^\mu + ig_w \frac{\tau_i}{2} W_i^\mu$

$$\begin{aligned}
(D^\mu \chi_L)' &= \left\{\partial^\mu + ig_w \frac{\tau_i}{2} W_i^\mu + ig_w \frac{\tau_i}{2}\delta W_i^\mu\right\} \cdot \left(1 + ig_w \alpha_i(x)\frac{\tau_i}{2}\right) \chi_L \quad \text{(first order expansion)} \\
&= \Big\{\partial^\mu + ig_w \partial^\mu \alpha_i \frac{\tau_i}{2} + ig_w \alpha_i \frac{\tau_i}{2}\partial^\mu \\
&\quad + ig_w \frac{\tau_i}{2} W_i^\mu + ig_w ig_w \frac{\tau_i}{2} W_i^\mu \alpha_i \frac{\tau_i}{2} \\
&\quad + ig_w \frac{\tau_i}{2}\delta W_i^\mu + \dots\Big\}\chi_L
\end{aligned}$$

Now consider the RHS

$$\begin{aligned}
&= \left(1 + ig_w \alpha_i \frac{\tau_i}{2})(\partial^\mu + ig_w \frac{\tau_i}{2} W_i^\mu\right) \chi_L \\
&= \left(\partial^\mu + ig_w \frac{\tau_i}{2} W_i^\mu + ig_w \alpha_i \frac{\tau_i}{2}\partial^\mu + ig_w \alpha_i \frac{\tau_i}{2} ig_w \frac{\tau_i}{2} W_i^\mu\right) \chi_L
\end{aligned}$$

equating LHS = RHS and making obvious cancellations:

$$\begin{aligned}
ig_w \frac{\tau_i}{2}\delta W_i^\mu &= -ig_w \partial^\mu \alpha_i \frac{\tau_j}{2} \\
&\quad - ig_w ig_w \frac{\tau_i}{2}\frac{\tau_j}{2} W_i^\mu \alpha_j \\
&\quad + ig_w ig_w \frac{\tau_j}{2}\frac{\tau_i}{2} W_i^\mu \alpha_j \\
\frac{\tau_i}{2}\delta W_i^\mu &= -\frac{\tau_i}{2}\partial^\mu \alpha_i \; - \; ig_w \left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] \alpha_j W_i^\mu \\
&= -\frac{\tau_i}{2}\partial^\mu \alpha_i \; - \; ig_w i\epsilon_{ijk} \frac{\tau_k}{2} \alpha_j W_i^\mu \\
&= -\frac{\tau_i}{2}\partial^\mu \alpha_i \; - \; g_w \frac{\tau_i}{2} \vec{\alpha} \times \vec{W} \\
\Rightarrow \delta W_i^\mu &= -\partial^\mu \alpha_i - g_w \vec{\alpha} \times \vec{W} \\
\Rightarrow W_i^\mu \rightarrow W_i'^\mu &= W_i^\mu - \partial^\mu \alpha_i - g_w \vec{\alpha} \times \vec{W}
\end{aligned}$$

Now consider the self interaction terms in $\mathcal{L}$

$$-\frac{1}{4} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

where

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

However, the $W_{\nu\mu}$ term could not be of the same form and retain gauge invariance. This is because of the extra term in $W_i'^\mu$, this makes

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - g_w \vec{W}_\mu \times \vec{W}_\nu$$

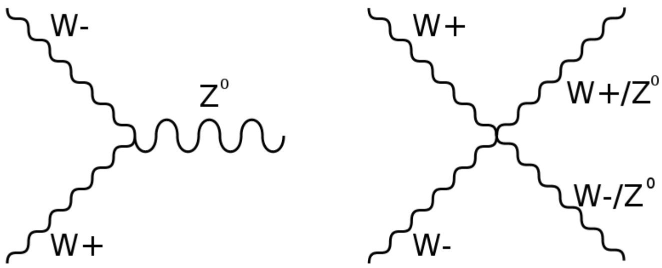

Figure 21.1: Examples of trilinear and quadrilinear self coupling among the weak bosons

The $-\frac{1}{4}W_{\mu\nu}W^{\mu\nu}$ term now yields the trilinear and quadrilinear coupling of the weak bosons (21.1)

$$\begin{aligned}
-\frac{1}{4}W_{\mu\nu}W^{\mu\nu} &= -\frac{1}{4}\left\{\partial_\mu W_\nu - \partial_\nu W_\mu - g_w\vec{W}_\mu \times \vec{W}_\nu\right\}\left\{\partial^\mu W^\nu - \partial^\nu W^\mu - g_w\vec{W}^\mu \times \vec{W}^\nu\right\} \\
&= -\frac{1}{4}\Big\{ \left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\left(\partial^\mu W^\nu - \partial^\nu W^\mu\right) \\
&\quad - g_w\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\vec{W}^\mu \times \vec{W}^\nu - g_w\left(\partial^\mu W^\nu - \partial^\nu W^\mu\right)\vec{W}_\mu \times \vec{W}_\nu \qquad \text{(trilinear coupling)} \\
&\quad + g_w^2(\vec{W}_\mu \times \vec{W}_\nu)(\vec{W}^\mu \times \vec{W}^\nu)\Big\} \qquad \text{(quadrilinear coupling)}
\end{aligned}$$

21.2.1 Trilinear coupling

From the above we see that the second and third terms are the same and that both account for trilinear coupling:

$$-\frac{1}{4}g_w \cdot 4(\partial_\mu W_\nu)\vec{W}^\mu \times \vec{W}^\nu$$

as:

$$\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\vec{W}^\mu \times \vec{W}^\nu = 2(\partial_\mu W_\nu)\vec{W}^\mu \times \vec{W}^\nu$$

so at the vertex (21.2)

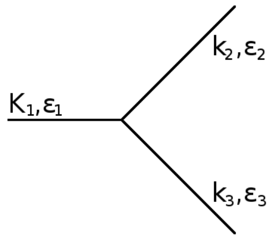

Figure 21.2: Simple schematic of trilinear vertex

$$\begin{aligned}
W^1_\mu &= \epsilon^1_\mu e^{-ik^1_\mu x^\mu} \\
\Rightarrow \partial^1_\mu W^1_\mu &= -ik^1_\mu \epsilon^1_\mu e^{-ik^1_\mu x^\mu} \\
\Rightarrow -g_w(\partial_\mu W_\mu)\, \vec{W}^\mu \times \vec{W}^\nu &= g_w \begin{vmatrix} ik^1_\nu \epsilon^1_\mu & ik^2_\nu \epsilon^2_\mu & ik^3_\nu \epsilon^3_\mu \\ \epsilon^\nu_1 & \epsilon^\nu_2 & \epsilon^\nu_3 \\ \epsilon^\mu_1 & \epsilon^\mu_2 & \epsilon^\mu_3 \end{vmatrix} \\
&= ig_w \left\{ k^1_\nu \epsilon^1_\mu (\epsilon^\nu_2 \epsilon^\mu_3 - \epsilon^\mu_2 \epsilon^\nu_3) - k^2_\nu \epsilon^2_\mu (\epsilon^\nu_1 \epsilon^\mu_3 - \epsilon^\mu_1 \epsilon^\nu_3) + k^3_\nu \epsilon^3_\mu (\epsilon^\nu_1 \epsilon^\mu_2 - \epsilon^\mu_1 \epsilon^\nu_2) \right\} \\
&= ig_w \left\{ (\epsilon_1.\epsilon_2)(k_2 - k_1).\epsilon_3 + (\epsilon_2.\epsilon_3)(k_3 - k_2).\epsilon_1 + (\epsilon_3.\epsilon_1)(k_1 - k_3).\epsilon_2 \right\}
\end{aligned}$$

Quadrilinear coupling will be products of the 4 polarisation vectors eg. $(\epsilon_1.\epsilon_2)(\epsilon_3.\epsilon_4)$

21.3 Local gauge invariance and QCD

As with QED we can infer the structure of QCD from local gauge invariance. Here we replace U(1) with SU(3) group of phase transformations of colour fields. The quark fields transform as:

$$q(x) \rightarrow q'(x) = Uq(x)$$
$$U = \exp\left\{ i\alpha_a(x) \frac{\lambda_a}{2} \right\}$$

Where λ_a are the Gell-Mann matrices (3×3 linear independent and traceless) and α_a are the group parameters. The SU(3) generators do not commute hence the theory is non-abelian

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2} \right] = if_{ijk} \frac{\lambda_k}{2}$$

Following the same formalism we impose SU(3) invariance on the free Lagrangian and consider an infinitesimal change.

$$\begin{aligned} q(x) &\to \left(1 + i\alpha_a(x)\frac{\lambda_a}{2}\right) q(x) \\ \partial_\mu q(x) &\to \left(1 + i\alpha_a(x)\frac{\lambda_a}{2}\right) \left(\partial_\mu q(x) + i\frac{\lambda_a}{2}\ q(x)\ \partial_\mu \alpha_a(x)\right) \end{aligned}$$

we then introduce 8 gauge fields, G^a_μ each of which transforms as:

$$G^a_\mu \to G^a_\mu - \frac{1}{g_s}\partial_\mu \alpha_a(x)$$

and form a covariant derivative:

$$D_\mu = \partial_\mu + ig_s\frac{\lambda_a}{2}G^a_\mu$$

so the Lagrangian becomes:

$$\mathcal{L} = \bar{q}(i\gamma^\mu\partial_\mu - m)q - g_s\bar{q}\gamma^\mu\frac{\lambda_a}{2}q\ G^a_\mu$$

However this is not a gauge invariant Lagrangian, to regain invariance we transform the second term:

$$\begin{aligned} \bar{q}\gamma^\mu\frac{\lambda_a}{2}q &\to \bar{q}\gamma^\mu\frac{\lambda_a}{2}q + i\alpha_a\bar{q}\gamma^\mu\left(\frac{\lambda_a}{2}\cdot\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\cdot\frac{\lambda_a}{2}\right)q \\ &\to \bar{q}\gamma^\mu\frac{\lambda_a}{2}q - f_{abc}\alpha_b\left(\bar{q}\gamma^\mu\frac{\lambda_c}{2}q\right) \end{aligned}$$

Gauge invariance will be achieved if the field transforms as:

$$G^a_\mu \to G^a_\mu - \frac{1}{g_s}\partial_\mu\alpha_a(x) - f_{abc}\alpha_b G^C_\mu$$

Again, we add the kinetic energy term and the QCD Lagrangian becomes:

$$\mathcal{L} = \bar{q}(i\gamma^\mu\partial_\mu - m)q - g_s\bar{q}\gamma^\mu\frac{\lambda_a}{2}q\ G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$$

Alternatively:

$$\mathcal{L} = \text{“}q\bar{q}\text{”} + \text{“}G^2\text{”} + g_s\text{“}q\bar{q}G\text{”} + g_s\text{“}G^3\text{”} + g_s\ \text{“}G^4\text{”}$$

Where we have the quark then gluon propagator terms, the quark/anti-quark/gluon vertex term then tri and quadrilinear gluon coupling terms.

Chapter 22

Evolution of coupling "constants"

The self-coupling of gluons leads to a different behaviour of QCD coupling, α_s, with Q^2 when compared to QED coupling.

22.1 QED coupling

we normally include loops, i.e. higher order terms, into a process we had previously calculated

For example: $e\nu$ scattering has a correction in which there is a virtual electron-positron loop on the photon line. This diagram is calculable but there is no restriction on the 4-momenta of the loop particles and so there are an infinite number of possible loops.

The electron charge, say, becomes dependant on the scale of the process and has to be renormalised in order to remove the infinities caused by the loops.

The relationship between the measured e^2 and the "bare" e_o^2 has to be specified at a particular Q^2. As the coupling is directly related to the charge this then has a Q^2 dependence. The "running coupling constant", α, is a function of Q^2 given by:

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log\left(\frac{Q^2}{\mu^2}\right)}$$

where μ^2 is the renormalisation scale.

22.2 QCD coupling

For QCD the couplings derived previously (gluon self coupling etc) also contribute giving:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 - \frac{\alpha_s(\mu^2)}{12\pi}(33 - 2n_f)\log\left(\frac{Q^2}{\mu^2}\right)}$$

Chapter 23

Spontaneous symmetry breaking

We want to be able to add mass to our theory of the standard model. The W and Z bosons are not massless yet we cannot simply add mass 'by hand' to the relevant equations; simply adding terms to the respective Lagrangians breaks gauge invariance making the theory invalid. To cope with this mass can instead be generated through spontaneous symmetry breaking. We will consider three models:

1. Spontaneous symmetry breaking for a scalar field, ϕ, for which: $\phi \rightarrow -\phi$ is a symmetry

2. A complex scalar field which has a global gauge symmetry (Goldstone boson)

3. A complex scalar field which has a local gauge symmetry. The electromagnetic-like field introduced to allow invariance with respect to local gauge transformation cures the Goldstone boson problem and the A^{μ} field acquires mass (Higgs model, 1964, next chapter)

23.1 Scalar field

First consider the Lagrangian for a scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)^2 - (\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4)$$

where $\lambda > 0$. This Lagrangian is symmetric under the transformation $\phi \rightarrow -\phi$. If $\mu^2 > 0$, there is a minimum at $\phi = 0$, if $\mu^2 < 0$ then the potential has two minima (23.1). The ϕ^4 is a self interacting term.

$$
\begin{aligned}
\frac{dV}{d\phi} &= 0 \\
&= \mu^2\phi + \lambda\phi^3 \\
\Rightarrow \phi_{\text{min}} &= \sqrt{-\frac{\mu^2}{\lambda}} \\
&\equiv \pm v
\end{aligned}
$$

where V is the vacuum expectation of the field.

$$
\begin{aligned}
V &= \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \\
&= \lambda\left(\frac{\mu^2\phi^2}{2\lambda} + \frac{\phi^4}{4}\right) \\
&= -\frac{\lambda}{4}v^4
\end{aligned}
$$

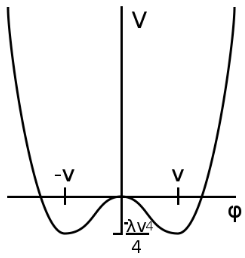

Figure 23.1: Potential field V

A perturbative expansion about the minima $\phi = \pm v$

$$\Rightarrow \phi(x) = v + \eta(x)$$

where $\eta(X)$ represents quantum fluctuations about this minima.

Substituting this into the Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu\eta(x))^2 - (\frac{1}{2}\mu^2(v+\eta(x))^2 + \frac{\lambda}{4}(v+\eta(x))^4) \qquad (v \text{ is a const } \partial_\mu v = \\
&= \frac{1}{2}(\partial_\mu\eta)^2 - (\frac{1}{2}\mu^2(v^2 + 2v\eta + \eta^2) + \frac{\lambda}{4}(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4))
\end{aligned}
$$

but $\mu^2 = -v^2\lambda$

$$
= \frac{1}{2}(\partial_\mu\eta)^2 - \left\{-\frac{1}{4}v^4\lambda + v^2\lambda\eta^2 + v\lambda\eta^3 + \frac{1}{4}\lambda\eta^4\right\}
$$

So now the field η has a mass term of the correct sign. Identifying with the Klein-Gordon equation mass term, $-\frac{1}{2}m^2\phi^2$, this means that

$$\begin{aligned} -\lambda v^2 &= -\frac{1}{2}m_\eta^2 \\ \Rightarrow m_\eta &= \sqrt{2\lambda v^2} \\ &= \sqrt{-2\mu^2} \end{aligned}$$

23.2 Complex field

For a complex field,

$$\begin{aligned} (\phi_1^2 + \phi_2^2) &= -\frac{\mu^2}{\lambda} \\ &= v^2 \end{aligned}$$

This is invariant under

$$\phi' = e^{i\alpha}\phi$$

ie under a global gauge transformation.

Again for $\lambda > 0, \mu^2 < 0$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)$$

This is now a circle of minima and we break the symmetry by choosing: $\phi_1 = v$ and $\phi_2 = 0$ then expand out about the minimum using:

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}(v + \eta(x) + i\rho(x)) \\ \phi(x) &\rightarrow \mathcal{L} \\ \mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^*(\partial_\mu\phi) - \frac{1}{2}\mu^2\phi^*\phi - \frac{\lambda}{4}\phi^*\phi \\ &= \frac{1}{2}(\partial_\mu\eta)^2 + \frac{1}{2}(\partial_\mu\rho)^2 + \mu^2\eta^2 \cdots \end{aligned}$$

The third term looks like a mass term of the form $-\frac{1}{2}m_\eta^2\eta^2$ with $m_\eta = \sqrt{-2\mu^2}$ and there is a kinetic energy term for the ρ field but no mass term

Chapter 24

The Higgs model

Again we have the complex field:

$$\frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$$

and we want the Lagrangian to be invariant with respect to a local gauge transform. To do this we must introduce the electromagnetic field A^μ and replace ∂_μ by $\partial_\mu - ieA_\mu$

$$\Rightarrow \mathcal{L} = (\partial_\mu + ieA_\mu)\phi^*(\partial_\mu - ieA_\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

consider $\mu^2 < 0$

$$\begin{aligned}\phi &= \frac{1}{\sqrt{2}}(v + eta(x) + i\rho(x)) \\ &\simeq \frac{1}{\sqrt{2}}(v + eta(x))e^{i\rho(x)\frac{1}{v}}\end{aligned}$$

Note that the exponential represents the local gauge transformation.

$$\begin{aligned}\Rightarrow \phi \to \phi' &= \frac{1}{\sqrt{2}}(v + eta(x))e^{i\rho(x)\frac{1}{v}} \\ A_\mu \to A'_\mu &= A_\mu + \frac{1}{e}\partial_\mu\frac{\rho(x)}{v}\end{aligned}$$

Breaking the symmetry by choosing $\rho(x) = 0$ i.e. (using 'h' for Higgs rather than η)

$$\phi = \frac{1}{\sqrt{2}}(v + h(x))$$

$$\begin{aligned}\Rightarrow \mathcal{L} &= (\partial_\mu + ieA_\mu)\frac{v + h(x)}{\sqrt{2}}(\partial_\mu - ieA_\mu)\frac{v + h(x)}{\sqrt{2}} - \mu^2\left[\frac{v + h(x)}{\sqrt{2}}\right]^2 - \lambda\left[\frac{v + h(x)}{\sqrt{2}}\right]^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}(\partial_\mu h)^2 - \lambda v^2h^2 + \frac{1}{2}e^2v^2A_\mu A^\mu + \cdots - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

So the Lagrangian describes 2 interacting massive particles: a vector gauge boson A^μ and a massive scalar, h, the Higgs particle

24.1 The Weinberg formulation of the Higgs Mechanism for SU(2) ⊗U(1)

Weinberg chose a weak isospin doublet of complex scalar fields with hypercharge +1

$$\begin{array}{cccc} & T_3 & Q & Y \\ \phi = \begin{pmatrix} \phi^\dagger(x) \\ \phi^0(x) \end{pmatrix} & \begin{array}{c} +\frac{1}{2} \\ -\frac{1}{2} \end{array} & \begin{array}{c} 1 \\ 0 \end{array} & \begin{array}{c} +1 \\ +1 \end{array} \end{array}$$

$$\phi^\dagger(x) = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

$$\phi^0(x) = \frac{\phi_3 + i\phi_4}{\sqrt{2}}$$

The above scalar field was required to be invariant under local gauge transformation in the weak hypercharge space of the electroweak interaction. The doublet transforms as:

$$\begin{pmatrix} \phi^\dagger(x) \\ \phi^0(x) \end{pmatrix} \to \begin{pmatrix} \phi^\dagger(x) \\ \phi^0(x) \end{pmatrix}' = \exp\left\{ i g_w \alpha(x) \frac{\tau}{2} + i g' \beta(x) \frac{Y}{2} \right\} \begin{pmatrix} \phi^\dagger(x) \\ \phi^0(x) \end{pmatrix}$$

$$\Rightarrow \mathcal{L} = \left| \left(\partial_\mu - g_w \alpha(x) \frac{\tau}{2} - g' \beta(x) \frac{Y}{2} \right) \phi \right|^2 - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

Weinberg broke the symmetry by setting:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \phi^\dagger(x) \\ \phi^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

From experience Boson masses have come from the gauge terms in the D_μ operation

$$= \left| \left[\frac{g'}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B_\mu + \frac{g_w}{2} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_\mu^1 + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} W_\mu^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_\mu^3 \right\} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{1}{8} \left| \begin{pmatrix} g_w W_\mu^3 + g' B_\mu & g_w \left(W_\mu^1 + i W_\mu^2 \right) \\ g_w \left(W_\mu^1 - i W_\mu^2 \right) & -g_w W_\mu^3 + g' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \left\{ v^2 g_w^2 \left(W_\mu^1 - i W_\mu^2 \right) \left(W_\mu^1 + i W_\mu^2 \right) - v^2 (g' B_\mu - g_w W_\mu^3)(g' B^\mu - g_w W_3^\mu) \right\}$$

the first term can be re-arranged as follows:

$$= \frac{1}{8} \sqrt{2}\sqrt{2}\, v^2 g_w^2 \left(\frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}} \right) \left(\frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}} \right)$$

$$= \frac{1}{4}\, v^2 g_w^2 W^+ W^-$$

Comparing this with the mass term expected for a charged boson

$$M_W^2 W^+W^- = \frac{1}{4}\, v^2 g_w^2 W^+W^-$$
$$\Rightarrow M_W = \frac{g_w v}{2}$$

Using the second term we get:

$$\begin{aligned}
&= \frac{1}{8} v^2 (g_w^2 + (g')^2) \left(\frac{g' B_\mu - g_w W_\mu^3}{\sqrt{g_w^2 + (g')^2}} \right) \left(\frac{g' B^\mu - g_w W_3^\mu}{\sqrt{g_w^2 + (g')^2}} \right) \\
&= \frac{1}{8} v^2 (g_w^2 + (g')^2) Z_\mu Z^\mu + m_A A_\mu A^\mu \\
Z^\mu &= \frac{g' B^\mu - g_w W_3^\mu}{\sqrt{g_w^2 + (g')^2}} \\
A^\mu &= \frac{g' B^\mu + g_w W_3^\mu}{\sqrt{g_w^2 + (g')^2}} \\
\Rightarrow m_A &= 0 \\
\Rightarrow m_Z &= \frac{1}{2} v (g_w^2 + (g')^2)^{\frac{1}{2}}
\end{aligned}$$

also

$$g' = \tan\theta_w$$
$$\text{as } M_W = \frac{v g_w}{2}$$
$$\Rightarrow M_Z = \frac{M_W}{\cos\theta_W}$$

Relative strength of the charged and neutral current interactions.

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_w}$$

is predicted to be 1, it is measured to be close to one but is modified by loop effects.

24.2 Coupling of Higgs to the W and Z

We insert:

$$\phi = \begin{pmatrix} 0 \\ v + h \end{pmatrix}$$

into the Lagrangian and look at the D_μ term

$$
\begin{aligned}
&= \frac{1}{8}\left|\begin{pmatrix} g_w W_\mu^3 + g' B_\mu & g_w\left(W_\mu^1 + iW_\mu^2\right) \\ g_w\left(W_\mu^1 - iW_\mu^2\right) & -g_w W_\mu^3 + g' B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v+h \end{pmatrix}\right|^2 \\
&= \left\{(v+h)^2 g_w^2\left(W_\mu^1 - iW_\mu^2\right)\left(W_\mu^1 + iW_\mu^2\right) - (v+h)^2(g' B_\mu - g_w W_\mu^3)(g' B^\mu - g_w W_3^\mu)\right\}
\end{aligned}
$$

the WH coupling is

$$
\frac{1}{8} g_w^2 \left\{(v^2 + 2vh + h^2)\sqrt{2}W^+\sqrt{2}W^-\right\}
$$

so here we can see the trilinear, WWH, and quadrilinear, WWHH, coupling.

WWH coupling:

$$
\begin{aligned}
&= \frac{1}{8} g_w^2 .2vh.\sqrt{2}W^+\sqrt{2}W^- \\
&= \frac{1}{2} g_w^2 vhW^+W^- \\
&= g_w^2 M_W W^+W^-
\end{aligned}
$$

for WWHH coupling:

$$
\begin{aligned}
&= \frac{1}{8} g_w^2 .h^2.\sqrt{2}W^+\sqrt{2}W^- \\
&= \frac{1}{4} g_w^2 h^2 W^+W^-
\end{aligned}
$$

for Z^0 consider the remaining terms of the Lagrangian

24.3 Fermion masses

In the Weinberg formulation, the scalar field who's symmetry is broken has hypercharge +1 and allows a possible mechanism for giving masses to fermions. The scalar field, ϕ, can be used to generate masses by connecting singlet and doublet states.

$$
\begin{aligned}
\mathcal{L} &= G_e\left[\frac{1}{\sqrt{2}}(\bar{\nu}_e, \bar{e})_L \begin{pmatrix} 0 \\ v+h \end{pmatrix} e_R + \frac{1}{\sqrt{2}}\bar{e}_R(0, v+h)\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L\right] \\
&= \frac{G_e}{\sqrt{2}} v(\bar{e}_L e_R + \bar{e}_R e_L) - \frac{G_e}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L)h
\end{aligned}
$$

Choose G_e such that

$$
m_e = \frac{G_e}{\sqrt{2}}
$$

and hence generate the electron mass:

$$
\mathcal{L} = -m_e \bar{e}e - \frac{m_e}{v}\bar{e}eh
$$

G_e is arbitrary and so m_e is not predicted by this method.

d-quark mass can be generated in the same way and for the u-quark we break the symmetry by choosing:

$$\phi = \begin{pmatrix} v + h \\ 0 \end{pmatrix}$$

We choose this because u has opposite hypercharge. Then a right-handed singlet u quark can be connected to the left-handed doublet state u quark. Again this gives a Dirac-like mass term.

Generalise this for quark masses, we start with the weak eigenstates.

The mass is generated by $\bar{d}'_L M_d d'_R$ where d represents any d-type quark and M_d is a 3×3 matrix, similarly for u-type quarks $\bar{u}'_L M_u u'_R$.

Now consider the relationship between the weak and massive eigenstates

$$\begin{aligned} u_L &= U^u_L u'_L & d_L &= U^d_L d'_L \\ u_R &= U^u_R u'_R & d_R &= U^d_R d'_R \end{aligned}$$

Where $U^{u/d}_{L/R}$ are unitary 3×3 matrices

$$\Rightarrow \bar{u}'_L M_u u'_R = \bar{u}_L \underbrace{U^u_L M_u U^{u\dagger}_R}_{M} u_R$$

Where M is a diagonal matrix of the quark masses. Recall the nature of the charged current for quarks, first in terms of the weak eigenstates:

$$(\bar{u}', \bar{c}', \bar{t}')\frac{1}{2}\gamma^\mu(1 - \gamma^5) \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}$$

and now in terms of the mass eigenstates:

$$(\bar{u}, \bar{c}, \bar{t})U^u_L \frac{1}{2}\gamma^\mu(1 - \gamma^5)U^{d\dagger}_L \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

In the weak interaction we have $V_{CKM} = U^u_L U^{d\dagger}_L$ and the CKM matrix is related to these transformations which give quarks their mass.

24.4 The cosmological problem

We recall that the Higgs 'well-depth' is

$$\begin{aligned}
V_0 &= \frac{-\lambda v^4}{4} \\
v &= \frac{2M_w}{g_w} \\
g_w^2 &= \frac{G_F 8 M_W^2}{\sqrt{2}} \\
\Rightarrow v &= \frac{2}{\sqrt{\frac{8G_F}{\sqrt{2}}}} \\
&= 246 \text{ GeV} \\
\Rightarrow V_0 &\simeq -\lambda \times 1 \times 10^9 \text{ GeV}^4 \\
&\simeq -\lambda \times 1 \times 10^9 \times 1.3 \times 10^{41} \\
&\simeq -\lambda \times 1.3 \times 10^{50} \text{ GeV cm}^{-3}
\end{aligned}$$

The visible density of the universe is ~ 1 proton per cubic meter, i.e. 10^{-6} GeV cm^{-3}, this needs to be $\times 20$ to account for dark energy and dark matter. Therefore the density in the Higgs field is $\lambda \times 10^{55}$ larger than that in the universe.

Chapter 25

Grand Unification

The electroweak, SU(2)$\otimes$U(1) is in impressive agreement with experimental data. However the theoretical unification is not complete the SU(2) group has coupling strength, g, and the U(1) has coupling strength, g', the relation of the two not being predicted by the theory. The ratio: $\frac{g'}{g} = \tan\theta_w$, is determined by experiment. Only if there is a larger set of gauge transformations which embed g and g' can it be said that electromagnetism and the weak interaction are truly unified. We can also consider the strong force and create a Grand Unification Theory (GUT):

$$G \text{ or } SU(5) \supset SU(3) \otimes SU(2) \otimes U(1)$$

Where we have one coupling constant i.e. that the coupling constant of each of the forces at some scale are equal. Knowing the values at low energies and their variation with energy the scale that this is likely to happen at is

$$M_{GUT} \sim 10^{15} \text{ GeV}$$

The simplest form of the SU(5) group has been eliminated through analysis of various decays. It could still be valid as part of another scheme such as SO(10). In SU(5) the down quarks and the anti-leptons doublet are combined into a quintuplet:

$$\begin{pmatrix} d_R \\ d_G \\ d_B \\ e^+ \\ \bar{\nu}_e \end{pmatrix}$$

To allow local gauge transformations on the quintuplet fields have to be introduced. Combined with their corresponding 5×5 generators. The SU(5) has 24 independent generators which are combined with 24 fields.

$$\left(\begin{array}{ccc|cc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & SU(2) & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & b & \cdot & SU(2) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \begin{pmatrix} d_R \\ d_G \\ d_B \\ e^+ \\ \bar{\nu}_e \end{pmatrix}$$

regions a and b can change leptons into quarks and vise versa. There are 12 new gauge bosons, normally represented by X and Y, which allow for lepton-quark transitions, $Q_X = \pm\frac{4}{3}$ and $Q_Y = \pm\frac{1}{3}$

The charge generator, Q, must be a linear combination of the diagonal generators in SU(5)

$$\begin{aligned} Q &= T_3 + \frac{Y}{2} \\ &= T_3 + cT_0 \end{aligned}$$

where T_3 (isospin) and T_0 are the diagonal generators of SU(5) belonging to SU(2) and U(1) subgroups. The coefficient c relates the operators Y and T_0, it has a value of $\sqrt{\frac{5}{3}}$. The diagonal generator is then:

$$\begin{aligned} M &= \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{2}{3} & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\frac{2}{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & -\frac{2}{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \mathrm{Tr} M^2 \qquad = 2\delta_{ij} \\ \Rightarrow 2 &= N^2\left(\frac{4}{9} + \frac{4}{9} + \frac{4}{9} + 1 + 1\right) \\ \Rightarrow N &= \sqrt{\frac{3}{5}} \end{aligned}$$

The unified coupling constant is g_5, where $g_5 = g_W$ and $g_5 = g_s$ but $g' = \sqrt{\frac{3}{5}} g_5 = \sqrt{\frac{3}{5}} g_w$ but

$$\begin{aligned} g' \cos\theta_W &= g_w \sin\theta_W \\ \Rightarrow \tan\theta_W &= \sqrt{\frac{3}{5}} \end{aligned}$$

at unification energy.

Devolving back to low energies shows that the theory $(\tan\theta_W)_{GUTscale} = \sqrt{\frac{3}{5}}$ does not quite match the data. The evolution of the coupling to low energies can be predicted:

$$\frac{1}{\alpha_3(Q^2)} = \frac{1}{\alpha_3(M^2)} + \frac{1}{4\pi}(11 - \frac{2}{3}n_g) \ln \left(\frac{Q^2}{M^2} \right)$$
$$\frac{1}{\alpha_2(Q^2)} = \frac{1}{\alpha_2(M^2)} + \frac{1}{4\pi}(\frac{22}{3} - \frac{4}{3}n_g) \ln \left(\frac{Q^2}{M^2} \right)$$
$$\frac{1}{\alpha_1(Q^2)} = \frac{1}{\alpha_1(M^2)} + \frac{1}{4\pi}(-\frac{4}{3}n_g) \ln \left(\frac{Q^2}{M^2} \right)$$

Where n_g is the number of generations. In fact the coupling constants do not

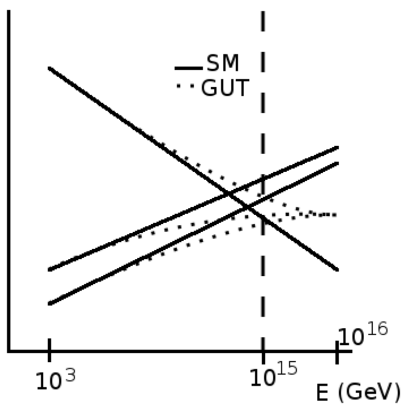

Figure 25.1: evolution of coupling constants under SM and GU theories

meet at the GUT scale (25.1, also had they done so at 10^{15} GeV this would have meant that protons decay with a lifetime that is measurable. This could be evidence for SUSY i.e. each boson has a fermion partner and vise versa, such particles would slow down evolution of the coupling constants.

Chapter 26

Neutrino masses and mixing

As discussed previously a discrepancy was observed between measured and expected numbers of electron neutrinos arriving at the Earth from the Sun. The required new phenomenology can be made to be consistent with SU(2)⊗U(1) broken gauge symmetry of the standard model. We take it that (anti) neutrinos are distinct Dirac fermions.

That neutrinos have mass was a major discovery: the only physics beyond the standard model. Violation of lepton flavour has consequences for its conservation in other circumstances.

The most general Lorentz invariant neutrino mass term that can be introduced to the Lagrangian density is:

$$\mathcal{L}^{\nu}_{mass}(x) = -\sum_{\alpha,\beta} \nu^{\dagger}_{\alpha L}(x) m_{\alpha\beta} \nu_{\beta R} + Hermitian\ constants$$

Where $m_{\alpha\beta}$ is an arbitrary 3×3 matrix, α and β run over the 3 neutrino types (e, μ, τ) and $\nu_{\alpha L}(x)$ and $\nu_{\beta R}(x)$ are the left and right handed spinors.

We can write

$$m_{\alpha\beta} = \sum_{i} U^{L}_{\alpha i} m_i U^{R}_{\beta i}$$

where m_i are 3 real positive masses and $U^{L}_{\alpha i}$ and $U^{R}_{\beta i}$ are unitary matrices.

We now define the fields:

$$\nu_{iL}(x) = \sum_{\alpha} U^{L}_{\alpha i} \nu_{\alpha L}(x)$$
$$\nu_{iR}(x) = \sum_{\alpha} U^{R}_{\alpha i} \nu_{\alpha R}(x)$$

where i indicates mass eigenstates and α the flavour eigenstates The mass terms takes the standard Dirac form:

$$\mathcal{L}^{\nu}_{mass}(x) = \sum_i m_i(\nu^{\dagger}_{iL}\nu_{iR} + \nu^{\dagger}_{iR}\nu_{iL})$$

The transformations return the Dirac form of the dynamical terms

$$\mathcal{L}^{\nu}_{dyn}(x) = \sum_i i(\nu^{\dagger}_{iL}\hat{\sigma}^{\mu}\partial_{\mu}\nu_{iL} + \nu^{\dagger}_{iR}\hat{\sigma}^{\mu}\partial_{\mu}\nu_{iR})$$

and $\mathcal{L}^{\nu}_{dyn}(x) + \mathcal{L}^{\nu}_{mass}(x)$ is the Lagrangian density of free neutrinos of masses m_1, m_2, m_3.

Since U^L and U^R are unitary:

$$\begin{aligned} \nu_{\alpha L}(x) &= \sum_i U^{L*}_{\alpha i}\nu_{iL}(x) \\ \nu_{\alpha R}(x) &= \sum_i U^{R*}_{\alpha i}\nu_{iR}(x) \end{aligned}$$

since neutrinos (e, μ, τ) each have a combination of the neutrino masses we get the phenomena of neutrino mixing.

This modifies eg $\tau(\pi^- \to e^-\bar{\nu}_e)$ which becomes:

$$\frac{1}{\tau(\pi^- \to e^-\bar{\nu}_e)} = \frac{\alpha_\pi^2}{4\pi}\left(1 - \frac{v_e}{\ell}\right) p_e^2 E_e |V_{ei}^2|^2 \qquad (i = 1, 2, 3)$$

As the mass differences are small the different decay modes have not been seen. As a total decay rate is measured and $\sum_i V^L_{ei} = 1$ we recover the original expression.

We have:

$$\begin{aligned} i\hat{\sigma}^{\mu}\partial_{\mu}\nu_{\alpha L} - m_{\alpha\beta}\nu_{betaR} &= 0 \\ i\hat{\sigma}^{\mu}\partial_{\mu}\nu_{\alpha R} - m^*_{\alpha\beta}\nu_{betaL} &= 0 \end{aligned}$$

Zero mass neutrinos would have plane wave solutions of negative helicity, for a wave in the z-direction:

$$\begin{aligned} \nu_{\alpha L}(z, t) &= e^{-iE(t-z)} f_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \nu_{\alpha R} &= 0 \end{aligned}$$

where f_α are constants.

Introducing the neutrino mass modifies these solutions:

$$\begin{aligned} \nu_{\alpha L}(z, t) &= e^{-iE(t-z)} f_\alpha(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \nu_{\alpha R}(z, t) &= e^{-iE(t-z)} y_\alpha(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

substituting into the Dirac equation:

$$i\frac{d}{dz}f_\alpha(z) - M_{\alpha\beta}g_\beta(z) = 0$$
$$(2E - i\frac{d}{dz})g_\gamma(z) - m^*_{\alpha\gamma}f_\alpha(z) = 0$$

For energies much greater than the masses we can neglect the $-i\frac{d}{dz}g_\gamma(z)$ term compared to $2Eg_\gamma(z)$

$$\therefore g_\gamma(z) = \frac{m^*_{\alpha\gamma}f_\alpha(z)}{2E}$$

and we then have:

$$i\frac{d}{dz}f_\beta(z) = M_{\beta\gamma}m^*_{\alpha\gamma}\frac{f_\alpha(z)}{2E}$$

Diagonalising the mass matrices gives:

$$i\frac{d}{dz}f_\beta(z) = U^{L*}_{\beta i}U^L_{\alpha i}f_\alpha(z)\frac{m_i^2}{2E}$$

to solve, we construct linear combinations (ignore L superscript on U terms)

$$\begin{aligned} f_i(z) &= U_{\alpha i}f_\alpha(z) \\ \Rightarrow i\frac{d}{dz}f_i(z) &= iU_{\alpha i}\frac{d}{dz}f_\alpha(z) \\ &= iU_{\alpha i}U^*_{\alpha j}U_{\beta j}\frac{M_j^2}{2E}f_\beta(z) \\ &= \delta_{ij}U_{\beta j}\frac{m_j^2}{2E}f_\beta(z) \\ &= \frac{m_i^2}{2E}f_\beta(z) \end{aligned}$$

These have simple solutions

$$f_i(z) = e^{-i\frac{M_i^2}{2E}z}f(0)$$

so the neutrino wave function is

$$\nu_i(z,t) = e^{-iEt+i\frac{(E-M_i^2)}{2E}z}f_i(0)$$

The state has energy, E, and momentum, $p_i = E - \frac{m_i^2}{2E}$. For $m_i^2 \ll E^2$, $p_i^2 = E^2 - m_i^2$

at $z = 0$ a neutrino, type α, is created. The neutrino wavefunction, ν_α, is a linear superposition of mass eigenstates ν_i with

$$f_i(0) = U_{\alpha i}f_\alpha(0)$$

different eigenstates propagate with different phases so that the neutrino type changes with z

$$\begin{aligned} f_\beta(z) &= U^*_{\beta i} f_i(z) \\ &= U^*_{\beta i} e^{-i\frac{-m_i^2}{2E}z} U_{\alpha i} f_\alpha(0) \end{aligned}$$

the probability of a transition after distance, D, is $P_D(\nu_\alpha \to \nu_\beta)$

$$\begin{aligned} P_D(\nu_\alpha \to \nu_\beta) &= |U^*_{\beta i} e^{-i\frac{-m_i^2}{2E}z} U_{\alpha i}|^2 \\ &= \sum_{ij} U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j} e^{-i\frac{\Delta m^2_{ij} D}{2E}} \end{aligned}$$

under interchange of i and j, Re(U^*UUU^*) is symmetric and Im(U^*UUU^*) is anti-symmetric

$$\begin{aligned} \therefore P_D(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} &Re\left\{U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right\} \sin^2\left(\frac{\Delta m^2_{ij} D}{4E}\right) \\ &+ 2\sum_{i>j} Im\left\{U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right\} \sin^2\left(\frac{\Delta m^2_{ij} D}{2E}\right) \end{aligned}$$

in the Weinberg-Salam model right-handed neutrinos exist as SU(2) singlets. Masses are introduced as for u-type quarks by coupling to the Higgs field.

So far we have considered neutrinos as Dirac particles but they could be Majorana i.e there is no distinction between neutrinos and anti-neutrinos so:

$$\begin{aligned} \nu_R &= (i\sigma^2)\nu_L^* \\ \nu_L &= -(i\sigma^2)\nu_R^* \end{aligned}$$

so either field can be derived from the other and only one needs to appear in the theory.

Neutrino-less double beta decay $0\nu\beta\beta$

$$(A, Z) \to (A, Z+2) + 2e^-$$

This is a process that can be used to determine whether neutrinos are Majorana or not, $0\nu\beta\beta$ can come from a variety of sources, however its observation would imply that nature contains at least one Majorana term and then the neutrino mass eigenstates must be the Majorana neutrinos.