Black Holes, Vortices and Thermodynamics

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Outline

- Black Hole Thermodynamics
- Vortices
- Black Holes with Vortices

Hawking and Bekenstein in the early 70's conjectured that black holes have thermodynamic properties.

- Black holes have entropy S.
- Black holes have Hawking temperature T_H, consistent with thermodynamic relation between energy, entropy and temperature.

Thermodynamics

- $S = \frac{\mathcal{A}}{4}$ where $\mathcal{A}$ is the area of the event horizon.
- $T_H = \frac{\kappa}{2\pi}$ where κ in the surface gravity of the black hole.

Path Integral Formulation

- In 1976 Hawking and Gibbons demonstrated that these thermodynamic results could be attained via a path integral approach to quantum gravity.
- In this approach one considers expressions of the form

$$Z = \int d[g]d[\phi]e^{iS_E[g,\phi]}$$

- where Z is the partition function, $d[g]$ and $d[\phi]$ are measures of the space of metrics and matter fields respectively and $S_E[g,\phi]$ is the action.

Euclideanisation and Temperature

- For ease of calculation the metric must be Euclideanised i.e. $t \longrightarrow i\tau$ and the metric becomes positive definite.

- Then, by including all metrics that are asymptotically flat and have periodicity of the imaginary time coordinate $\beta = \Delta\tau$, the path integral gives the partition function for a system at temperature $T = \frac{1}{\beta}$.

Topology of Euclidean Black Holes

Euclidean Schwarzschild black hole metric

$$ds^2 = (1 - \frac{r_s}{r})d\tau^2 + (1 - \frac{r_s}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

- Singular at $r = r_s$.
- A change variables of $\rho^2 = (2r_s)^2(1 - \frac{r_s}{r})$, gives

$$ds^2 = \rho^2(\frac{\tau}{2r_s})^2 + d\rho^2 + r_s^2 d\Omega_{II}^2 \qquad \text{as} \qquad r \rightarrow r_s.$$

- $\rightarrow \tau$ must be periodic with period $\beta = 4\pi r_s$
- Singularity is coordinate singularity
- Metric is only defined on $r_s \leq r < \infty$
- The metric has topology $\mathbb{R}^2 \times \mathrm{S}^2$.

Black Hole Partition function

- For black holes the key contributions to the path integral come from geometries that have topology such as this i.e. $\mathbb{R}^2 \times \mathrm{S}^2$.
- For the Schwarzschild black hole including this geometry alone results in the partition function from which the famous results can be derived.

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- A vortex is a non-perturbative, non-trivial solution of the field equations.
- This talk will consider only local Abelian Higgs vortices.
- They can be created during phase transitions.

Abelian Higgs Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}^{\mu}\phi(\mathcal{D}_{\mu}\phi)^{*} - \frac{\lambda}{4}(\phi\phi^{*} - \eta^{2})^{2},$$

$$\phi(x)' = e^{i.e.\Lambda(x)}\phi(x), \qquad A_{\mu}(x)' = A_{\mu}(x) - \partial_{\mu}\Lambda(x),$$

$$\mathcal{D}_{\mu}\phi = \partial_{\mu}\phi + ieA_{\mu}\phi.$$

Vortex Formation

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}^{\mu}\phi(\mathcal{D}_{\mu}\phi)^* - \frac{\lambda}{4}(\phi\phi^* - \eta^2)^2$$

- If $\eta > 0$ then it is the symmetry breaking scale, an energy scale below which $\phi(x)$ acquires a vev$\neq 0$, the symmetry breaks and the theory undergoes a phase transition.
- It is likely that during a transition a non-trivial winding of the phase will appear about some point.
- For this winding to be reconciled at the origin, ϕ must rise up the potential barrier to $\phi = 0$, thus a stable, localised, non-zero energy density appears which forms the vortex core.

Field Configuration

- Finite energy considerations imply that $\phi \to \eta$ as $r \to \infty$ (it's vacuum value) and A_μ must asymptotically be a pure gauge rotation.

Simplest Field Configuration for Vortices

$$\phi = \eta X(r) e^{ik\theta}, \qquad \begin{cases} X(0) = 0, \\ X(r) \to 1, \quad r \to \infty. \end{cases}$$

$$A_\mu = \frac{1}{e}(P(r) - k)\partial_\mu \theta, \qquad \begin{cases} P(0) = k, \\ P(r) \to 0, \quad r \to \infty. \end{cases}$$

Field Equations

- This form simplifies the field equations for variables $X(r)$ and $P(r)$.

Field Equations In Minkowski Background

$$X'' = \frac{-X'}{r} + \frac{P^2 X}{r^2} + \frac{\lambda\eta^2}{2}(X^2 - 1)X,$$
$$P'' = \frac{P'}{r} + 2e^2\eta^2 X^2 P.$$

- These coupled, second order, ordinary differential equations can be solved numerically.

Field Distributions

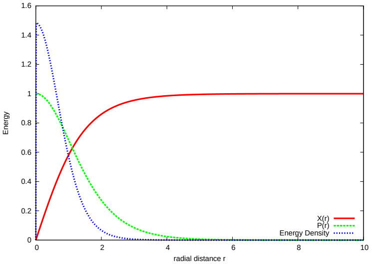

Figure: Field distribution for k=1 and k=2 vortices

Vortices and Gravity

To include gravity:

- The Minkowski metric must be replaced by $g_{\mu\nu}$, the general metric.
- The field equations must now include components of the metric and be coupled to the Einstein equations. Giving more differential equations of more variables.

These equations have been solved for a vortex in an otherwise flat spacetime and give an interesting result.

- The geometry of the spacetime outside the core is locally identical to Minkowski but not globally.
- The effect of the vortex is to introduce a 'deficit angle' making the spacetime that of a snub-nosed cone.

$$\Delta = 8\pi G\mu$$

where Δ is the deficit angle, G is Newtons constant and μ is the vortex mass per unit length.

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Vortex on a Black Hole

- The temperature of a black hole depends on the periodicity, β, of the imaginary temporal coordinate.
- The gravitational effect of a vortex on the surrounding space time is to reduce the period of the dimension in which its phase lies.
- Therefore, one might expect that a vortex on a black hole configured such that its phase lies in the temporal direction may effect the temperature of a black hole.

Set up

We now consider an Abelian Higgs Lagrangian with General Euclidean Schwarzschild metric

Lagrangian and Metric

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}^{\mu}\phi(\mathcal{D}_{\mu}\phi)^{*} + \frac{\lambda}{4}(\phi\phi^{*} - \eta^{2})^{2},$$
$$ds^{2} = A^{2} d\tau^{2} + A^{-2} dr^{2} + C^{2}(d\theta^{2} + sin^{2}\theta d\phi^{2}).$$

Field Configuration

$$\phi = \eta X(r) e^{ik\frac{2\pi\tau}{\beta}},$$
$$A_{\mu} = \frac{2\pi}{\beta e}(P(r) - k)\partial_{\mu}\tau = \frac{2\pi}{\beta e}(P_{\mu} - k\partial_{\mu}).$$

This configuration ensures cylindrical symmetry about τ which leads to $A = A(r)$ and $C = C(r)$.

Field Equations

Field Equations

Varying ϕ and A_μ gives

$$\frac{1}{C^2}(C^2P')' = 2e^2\eta^2\frac{X^2P}{A^2}$$
$$\frac{1}{C^2}(C^2A^2X')' = \frac{P^2X4\pi^2}{A^2\beta^2} + \frac{\lambda\eta^2}{2}X(X^2-1).$$

Varying $g^{\mu\nu}$ gives the Einstein equations, which for this case are:

$$C'' = 4\pi G\frac{C}{A^2}(T^0_0 - T^r_r)$$
$$((A^2)'C^2)' = 8\pi GC^2(2T^\theta_\theta + T^r_r - T^0_0)$$
$$\frac{(A^2)'C'}{C} - \frac{1}{C^2}(1 - A^2C'^2) = 8\pi GT^r_r$$

Where T^i_i are components of the energy-momentum tensor.

Boundary conditions

These coupled, ordinary differential equations must be solved simultaneously along with the boundary conditions specified by finite energy constraints and regularity of the metric at the horizon.

Boundary Conditions

$$C(r_s) = r_s$$

$$A(r_s) = 0 \qquad A(r_s)^2 = \frac{1}{r_s}$$

$$X(r_s) = 0 \qquad X(\infty) = 1$$

$$P(r_s) = 1 \qquad P(\infty) = 0.$$

- The problem complicated by the 'mixed type' boundary conditions.
- The method used involves a Runge-Kutta algorithm on the equations for the gravity fields and successive under-relaxation on the matter fields, repeated on successive iterations.

Results

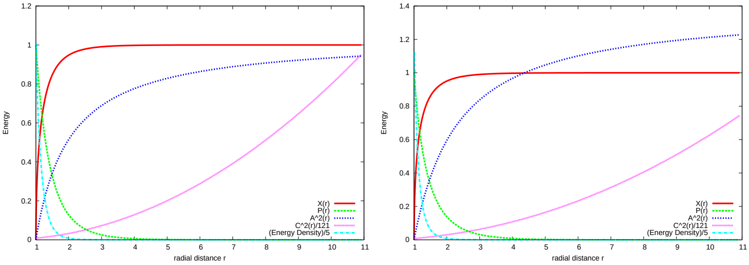
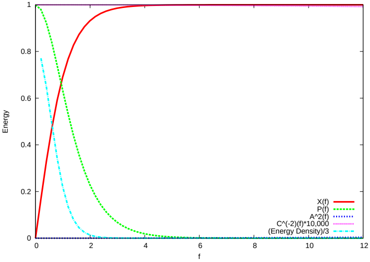
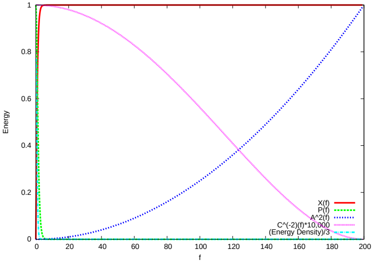


Figure: Field distribution for G=0.0 and G=0.02 vortices

- Caveat: There is a small numerical artefact in these solutions (not shown here) which needs some further investigation to ascertaining its origin.

Results





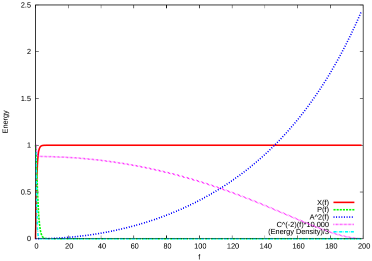


Figure: Field distribution for G=0.0 (close up), G=0.0 and G=0.02 vortices (coordinates appear flat at the horizon).

Conclusions

Key observations:

- Gravity fields are asymptotically Schwarzschild.
- A^2's asymptotic value is increased by the presence of the gravitating vortex.

If we look at the asymptotic Schwarzschild where A^2 has been multiplied by a constant λ.

$$ds^2 = \lambda^2 A^2 d\tau^2 + \frac{1}{\lambda^2} A^{-2} dr^2 + \frac{1}{\lambda^2} C^2 (d\theta^2 + sin^2\theta d\phi^2).$$

This factor can only be absorbed by a rescaling of $d\tau \rightarrow \frac{d\tau}{\lambda}$ and $dr \rightarrow \lambda dr$

- Therefore the period at infinity, $\tilde{\beta} = \frac{\Delta\tau}{\lambda} = \frac{\beta}{\lambda}$, is reduced and the temperature of the black hole $\tilde{T}_H = \frac{1}{\tilde{\beta}}$ is increased.

Summary

- Verified numerically that the presence of a vortex on a Euclidean Schwarzschild black hole increased the temperature of the system.
- This supports previous work of my supervisor and collaborators when looking analytically at the extreme case of a thin weakly gravitating vortex on a black hole.
- These results apply to the more general case of thicker and stronger gravitating vortices.
- This may well have important implications on other current work in the field.