Differential field theory and the configuration space method

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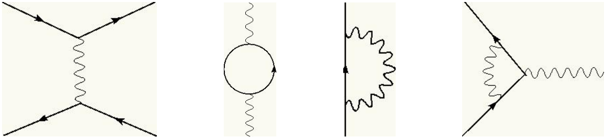
Introduction

- QED Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- Gives rise to a perturbative series in $\alpha_{\mathrm{EM}} \simeq 1/137$.
- The terms are Feynman diagrams:



- The broad question is: Is this a perturbative expansion of a differential equation?

State space as a function space

The Dirac equation in free space

$$\partial_t \psi(t,x) = (-\alpha^j \partial_j - i\beta m)\psi(t,x)$$

is a well-posed *Cauchy problem*: initial data $\psi(0,x)$ uniquely fixes a solution. Easy to see in Fourier space:

$$\tilde{\psi}(t,k) = \exp(-iH(k)t)\ \tilde{\psi}(0,k)$$

where $H(k) = (\alpha^j k_j + \beta m)$ is Hermitian at each $k \in \mathbb{R}^3$. Multiplication in k-space means convolution in real space:

$$\psi(t,x) = \int d^3x'\ S(t, x-x')\psi(0,x')$$

State space as a function space

$$\tilde{\psi}(t,k) = \exp(-iH(k)t)\,\tilde{\psi}(0,k)$$

- Solution ψ is uniquely constrained by initial conditions.
- Solution exists for all initial conditions $\psi(0,\cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$.
- Evolution in time $\psi(0,\cdot) \to \psi(t,\cdot)$ is a unitary transformation on $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Punchline: "state" $\leftrightarrow$ initial data on a given spacelike hypersurface.

$$\begin{aligned} \Lambda^1 &= \text{state space for a single particle} \\ &= \text{space of initial data for Dirac eq.} \\ &= \text{the function space } L^2(\mathbb{R}^3, \mathbb{C}^4) \\ &= \text{the } \textit{one-particle configuration space}. \end{aligned}$$

Photon sector

- Equation of motion (Maxwell eqs) ('Pryce representation')

$$\frac{\partial}{\partial t} f = -iHf$$

where $f = \mathbf{E} + i\mathbf{B}$ and $H = \nabla\times$. [Pryce48, Good57, Hawton99, Raymer05, Smith06].

- Single-photon state space $\Theta^1 = L^2(\mathbb{R}^3, \mathbb{C}^3)$.
- Standard norm on Θ^1 is $\propto$ classical e.m. energy.

$$\langle f, f\rangle = \int d^3x\ f^\dagger(x) f(x) = \int d^3x\ |\mathbf{E}(x)|^2 + |\mathbf{B}(x)|^2$$

H is Hermitian with respect to this norm.

Second quantization

- Two particles: amplitudes $f(x), g(x)$. The tensor product $f \otimes g$ is the function of *pairs* of space coordinates

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2), \qquad f, g \in \Lambda^1.$$

  The square magnitude

$$|(f \otimes g)(x_1, x_2)|^2 = |f(x_1)|^2 |g(x_2)|^2,$$

  gives a *conditional* probability for the locations of each particle.
- Need a vector space: $(f \otimes g + f' \otimes g')$ can generally not be written as $f'' \otimes g''$. (Such states are called *entangled*.)
- Theorem: $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. A general two-particle state is a function $h(x_1, x_2)$.

Statistics and tensor algebra

- Wedge product: antisymmetrised tensor product. e.g. Two single-particle states $f \wedge g = \frac{1}{2}(f \otimes g - g \otimes f)$.
- Fermions: exterior algebra. Bosons: symmetric algebra.

$$\Lambda^p = \mathrm{Alt}(\underbrace{\Lambda^1 \otimes \ldots \otimes \Lambda^1}_{p}), \qquad f \wedge g = \mathrm{Alt}(f \otimes g)$$

$$\Theta^p = \mathrm{Sym}(\underbrace{\Theta^1 \otimes \ldots \otimes \Theta^1}_{p}), \qquad f \odot g = \mathrm{Sym}(f \otimes g)$$

- State space: orthogonal sum of p-particle state spaces

$$\Lambda = \Lambda^0 + \Lambda^1 + \Lambda^2 + \ldots$$
$$\Theta = \Theta^0 + \Theta^1 + \Theta^2 + \ldots$$

Typical element is $f = (f^{(0)}, f^{(1)}, f^{(2)}, \ldots)$ where each $f^{(p)}$ is a p-particle configuration amplitude.

Relativistic covariance

- Intrinsic (no t dependence) representation of Poincaré algebra on Λ^1:

$$\begin{aligned}
\mathcal{P}_j &= \partial_j \\
\mathcal{J}_j &= -\epsilon_{jkl} x_k \partial_l + J_j \\
\mathcal{P}_0 &= -iH = -\alpha_k \partial_k - i\beta m \\
\mathcal{K}_j &= x_j \mathcal{P}_0 + K_j = -x_j \alpha_k \partial_k - i x_j \beta m + K_j
\end{aligned}$$

where

$$J_j = \frac{-1}{2}\begin{pmatrix} i\sigma_j & 0 \\ 0 & i\sigma_j \end{pmatrix}, \qquad K_j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}$$

- Unitary rep on Λ^1 extends multiplicatively to higher tensor powers Λ^p. (= Operators on p-particle amplitudes $\psi(x_1, \ldots, x_p)$.)
- Tensor-algebraic implementation of second quantization is relativistic.

Dynamics on Λ

- Notation: *p*-particle Hamiltonian

$$\frac{\partial}{\partial t}\left(f_1 \wedge f_2\right) = \left(-iHf_1\right) \wedge f_2 + f_1 \wedge \left(-iHf_2\right)$$
$$\equiv -iH\left(f_1 \wedge f_2\right).$$

- In general, simply write

$$\frac{\partial}{\partial t}F = -iHF \qquad\qquad F \in \Lambda$$

Steps towards Electrodynamics

- Couple multi-photon states to a classical current $j \in \Theta^1$. (e.g. Dirac $j^k = \psi^\dagger \alpha^k \psi$.)

$$\frac{\partial}{\partial t} F = -iHF + j \odot F \qquad (1)$$

with $j \in \Theta^1$ a classical current.

- Classical limit: *coherent states* (Glauber 1963)

$$F = \mathrm{Coh}\, f = \sum_{p=0}^{\infty} \frac{1}{p!} \underbrace{f \odot \ldots \odot f}_{p \text{ copies}} \qquad (f \in \Theta^1)$$

satisfy (1) if f satisfies classical Maxwell eqs.

$$\frac{\partial f}{\partial t} = -iHf + j$$

Perturbative solution

Equation of motion:

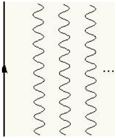
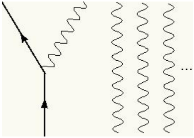
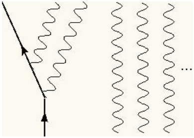
$$\frac{\partial}{\partial t} F = -iHF + j \odot F$$
$$= -iH_{\text{free}} F + -iH_{\text{int}} F$$

Solution by *time-ordered exponential*:

$$\begin{aligned}
F(t) &= e^{-itH_{\text{free}}} F(0) \\
&\quad + -i \int_0^t dt' \; e^{-i(t-t')H_{\text{free}}} H_{\text{int}} e^{-it'H_{\text{free}}} F(0) \\
&\quad + (-i)^2 \int_0^t dt' \int_0^{t'} dt'' \; e^{-i(t-t')H_{\text{free}}} H_{\text{int}} e^{-i(t-t'')H_{\text{free}}} H_{\text{int}} e^{-it''H_{\text{free}}} F(0) \\
&\quad + \cdots
\end{aligned}$$

Perturbative solution

- Perturbative series:

 $+$  $+$  $+ \dots$

- Positive-energy projection recovers infinite series of Feynman diagrams.
- Equality of perturbative solution of differential equation with perturbative series of Feynman diagrams.

Negative energy problem.

- Dirac Hamiltonian:

$$H(k) = \alpha^j k_j + \beta m$$

  and $H^2 = E^2 = k^2 + m^2$. Orthogonal eigenspaces with eigenvalues $\pm E$.
- (electron) $\otimes$ (positron) may have eigenvalue 0.
- Possibility of cascade to infinitely many electron-positron pairs!

Response: Had no problem in the photon sector. $H = \nabla\times$ also has $\pm E$ eigenvalues.

- Perturbative solution demonstrates that p-particle component $F^{(p)}$ cannot diverge in finite time (factorially suppressed at large p).

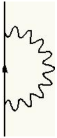
  [Houseman in prep]

Steps towards Electrodynamics

- Dynamics presented above *only increase* particle-number.
- $F \mapsto j \odot F$ is not Hermitian: the equation of motion will break unitarity.
- Try including the Hermitian conjugate $F \mapsto j \rfloor F$

$$\frac{\partial}{\partial t} F = -iHF + j \odot F - j \rfloor F.$$

- Second order perturbative contribution includes loop-like term.



- Current due to two-lepton state $\psi(x_1, x_2)$ can describe e^+e^- annihilation.

Perturbative expansions of differential equations

- Can perturbative expansions of differential equations generate factorially-growing complexity? Yes, even as ODE: *Lindstedt algorithm* for classical 3-body problem [Siegel43,Gallavotti07]. Relationship to multiscale analysis, renormalization.
- Remark: $SU(n)$ gauge symmetry consistent with L^2-norm.

Conclusion

- Concrete characterisation of state space.

$$\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(p)}, \ldots) \in \Lambda$$

- States are superpositions of configuration space amplitudes.

$$\psi^{(p)}_{i_1 i_2 \ldots i_p}(t; \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$$

- Looking for modifications to free equations of motion, consistent with underlying tensor structure, that may generate field theory perturbatively.

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Extras

Current due to multiple leptons

- Current due to one lepton $\psi(x)$:

$$j^k(x) = \psi^\dagger(x)\alpha^k\psi(x).$$

- Current due to two leptons $\psi^{(2)}(x_1, x_2)$:

$$j^k(x) = \mathrm{Tr}\,\alpha^j \int \psi^{(2)}(x, x')\psi^{(2)}(x, x')^\dagger d^3x' + \mathrm{Tr}\,C\alpha^j\psi^{(2)}(x, x)\psi^{(0)\dagger}$$

where $C = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}$ (= charge conj. matrix).

- Special case: two spatially-separated lepton states: $\psi^{(2)} = \phi \wedge \chi$.

$$j^k(x) = \phi^\dagger\alpha^k\phi + \chi^\dagger\alpha^k\chi.$$

Current due to multiple leptons

$$j^k(x) = \mathrm{Tr}\,\alpha^j \int \psi^{(2)}(x,x')\psi^{(2)}(x,x')^\dagger d^3x' + \mathrm{Tr}\,C\alpha^j\psi^{(2)}(x,x)\psi^{(0)\dagger}$$

Second term couples a two-lepton state to the vacuum.

- Vanishes if $\psi^{(2)}$ represents $\mathrm{e}^-\mathrm{e}^-$ or $\mathrm{e}^+\mathrm{e}^+$. Non-zero only for $\mathrm{e}^+\mathrm{e}^-$
- First-order perturbation may give rise to $\mathrm{e}^+\mathrm{e}^- \to \gamma$. (Check: needs to vanish on-shell.)
- Second-order perturbation may give rise to $\mathrm{e}^+\mathrm{e}^- \to 2\gamma$. (Need to specify dynamics on lepton sector [Ruijsenaars77,Marecki04]). Will it give loop diagrams?

Anticommutation relations

- These tensor algebras carry a natural representation of the commutation relations, independent of choice of dynamics.
- Creation operator / exterior product:

$$a_g^\dagger h = g \wedge h$$

- Annihilation operator / interior product: Define as Hermitian adjoint of $a_g^\dagger$, that is the unique operator a_g for which

$$\langle f, a_g h\rangle = \langle a_g^\dagger f, h\rangle \qquad \text{for all } f, h \in \Lambda$$

- Notation: [Sternberg64]

$$a_g h = g \rfloor h$$

Anticommutation relations

Let $f, g \in \Lambda^1$.

- 1. $\{a_f^\dagger, a_g^\dagger\} = 0$
  Let $h \in \Lambda$. e.g. [Sternberg64]

$$f \wedge g \wedge h = -g \wedge f \wedge h$$
$$a_f^\dagger a_g^\dagger h = -a_g^\dagger a_f^\dagger h$$

- 2. $\{a_f, a_g\} = 0$
  Similarly

$$f \rfloor (g \rfloor h) = -g \rfloor (f \rfloor h)$$
$$a_f a_g h = -a_g a_f h.$$

Anticommutation relations

Let $f, g \in \Lambda^1$.

- 3. $\{a_f, a_g^\dagger\} = \langle f, g\rangle$
  Let $h \in \Lambda$.

$$f \rfloor (g \wedge h) = (f \rfloor g) \wedge h - g \wedge (f \rfloor h)$$
$$a_f a_g^\dagger h = \langle f, g\rangle h - a_g^\dagger a_f h$$

- Compare with

$$\{a_x, a_{x'}^\dagger\} = \delta^3(x - x')$$

corresponding to the limits $f \to \delta^3(x')$ and $g \to \delta^3(x)$.

Relativistic covariance

- p-particle state $\psi \in \Lambda^p$: space representation

$$\psi(t; \mathbf{x}_1, \ldots, \mathbf{x}_p).$$

- Treats t on a different footing to $\mathbf{x}$. Doesn't that break relativity?

  [e.g. Chen10, Haller60]

No, definitely not in the free theory and we'll discuss why.

Poincaré algebra

- Poincare group generates symmetry transformations corresponding to relativistic covariance of a theory. Generators:

$$\{\underbrace{P_0, P_1, P_2, P_3}_{\text{translation}}, \underbrace{J_1, J_2, J_3}_{\text{space rotation}}, \underbrace{K_1, K_2, K_3}_{\text{Lorentz boost}}\}$$

- Lie algebra (specify commutator $[\cdot,\cdot]$):

$[\cdot,\cdot]$	P_0	P_j	J_j	K_j
P_0	0	0	0	P_j
P_i	0	0	$\epsilon_{ijk} P_k$	$\delta_{ij} P_0$
J_i	0	$\epsilon_{ijk} P_k$	$\epsilon_{ijk} J_k$	$\epsilon_{ijk} K_k$
K_i	$-P_j$	$-\delta_{ij} P_0$	$\epsilon_{ijk} K_k$	$-\epsilon_{ijk} J_k$

[Foldy56].

Relativistic covariance of Fock space

- Intrinsic representation of Poincaré algebra on Λ^1 (no t dependence):

$$\begin{aligned}
\mathcal{P}_j &= \partial_j \\
\mathcal{J}_j &= -\epsilon_{jkl} x_k \partial_l + J_j \\
\mathcal{P}_0 &= -iH = -\alpha_k \partial_k - i\beta m \\
\mathcal{K}_j &= x_j \mathcal{P}_0 + K_j = -x_j \alpha_k \partial_k - i x_j \beta m + K_j
\end{aligned}$$

  where

$$J_j = \frac{-1}{2} \begin{pmatrix} i\sigma_j & 0 \\ 0 & i\sigma_j \end{pmatrix}, \qquad K_j = \frac{1}{2} \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}$$

- Differs from [Foldy56, Chakrabarti66, Bialynicki-Birula87, Hawton01].

Poincaré algebra

Example:

$$
\begin{aligned}
[\mathcal{P}_i, \mathcal{K}_j] &= [\partial_i, x_j \mathcal{P}_0 + K_j] \\
&= [\partial_i, x_j \mathcal{P}_0] \\
&= [\partial_i, x_j] \mathcal{P}_0 \\
&= \delta_{ij} \mathcal{P}_0
\end{aligned}
$$

as required.

Relativistic covariance of Fock space

- $\mathcal{P}_0$ integrates to finite translation forward in time,

$$\frac{d}{dt}\psi = \mathcal{P}_0\psi = -iH\psi$$

- $\mathcal{K}_j$ integrates to finite Lorentz boost (= change of inertial frame).

$$\frac{d}{d\eta}\psi = \mathcal{K}_j\psi$$
$$\frac{\partial}{\partial\eta}\psi(x) = (-x_j\alpha_k\partial_k - ix_j\beta m + K_j)\psi(x)$$

- $\psi \mapsto e^{\eta\mathcal{K}_j}\psi$ is also a unitary transformation. [e.g. Houseman, in prep]

Relativistic covariance of Fock space

- Translation in time on Λ^1...

$$f \mapsto e^{-itH} f$$

- ... and on products

$$f_1 \wedge f_2 \mapsto (e^{-itH} f_1) \wedge (e^{-itH} f_2)$$

  with linear extension to Λ^2.
- Similarly for boosts or any other Poincaré transformation.

Punchline: Poincaré rep on Λ^1 extends to Poincaré rep on Λ^p. Therefore, tensor-algebraic implementation of second quantization is consistent with special relativity.

Energy in the classical limit

Departure from unitarity?

$$\begin{aligned}\frac{d}{dt}\langle F, F\rangle &= 2\,\mathrm{Re}\langle j \odot F, F\rangle \\ &= 2\,\mathrm{Re}\langle F, j\rfloor F\rangle\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\tfrac{1}{2}\log F, F &= \mathrm{Re}\langle j, f\rangle \\ &= \int d^3x\,\mathbf{j}(x)\cdot\mathbf{E}(x)\end{aligned}$$

Energy in the classical limit

$$
\begin{aligned}
\langle F, F \rangle &= \langle \mathrm{Coh}\, f, \mathrm{Coh}\, f \rangle \\
&= \sum_{p=0}^{\infty} \frac{1}{(p!)^2} \langle \underbrace{f \odot \ldots \odot f}_{p}, \underbrace{f \odot \ldots \odot f}_{p} \rangle \\
&= \sum_{p=0}^{\infty} \frac{1}{(p!)^2} p! \langle f, f \rangle^p \\
&= e^{\langle f, f \rangle}
\end{aligned}
$$

Therefore, when F is a coherent state,

$$\text{Classical energy} = \tfrac{1}{2} \log \langle F, F \rangle$$

Projection operators

Dirac Hamiltonian:

$$H(k) = \alpha^j k_j + \beta m$$

and $H^2 = E^2 = k^2 + m^2$. Orthogonal eigenspaces with eigenvalues $\pm E$. Projection operators:

$$\tilde{\psi}(k) \mapsto \tilde{P}_{\pm} k \tilde{\psi}(k), \qquad \tilde{P}_{\pm}(k) = \frac{\pm H(k) + E}{2E}$$

project onto $\pm$ energy subspaces. Multiplication in k-space means convolution in real space:

$$(P_{\pm}\psi)(t, x) = \int d^3x' \; P_{\pm}(t, x - x')\psi(0, x')$$

Example Hamiltonian [Haller60]:

$$H_{\text{free}} = \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \omega_{\mathbf{k}}$$
$$H_{\text{int}} = \sum_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} + a_{\mathbf{k}}) V_k$$

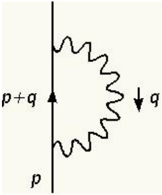

Figure: A contribution to the electron self energy.

$$
\begin{aligned}
\tilde{M}(p) &= \int \frac{d^4 q}{(2\pi)^4} \left(-ie\gamma^{\mu}\right) \frac{\not{p} + \not{q} - m}{(p+q)^2 - m^2 + i\epsilon} \left(-ie\gamma^{\nu}\right) \frac{g_{\mu\nu}}{q^2 + i\epsilon} \\
&= \int \frac{d^4 q}{(2\pi)^4} (-ie)^2 \, \gamma^{\mu} \, \tilde{S}^F(p+q) \, \gamma_{\mu} \, \tilde{D}^F(q)
\end{aligned}
$$