Brunel University
Queen Mary, University of London
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Intercollegiate post-graduate course in

High Energy Physics

Paper 2: The Standard Model and

Beyond part 2

Wednesday, 25 January 2017

Time allowed for Examination: **2.5 hours**

Answer four questions out of six (80 marks)

Books and notes may be consulted

1. **Higgs sector of the Glashow Salam Weinberg model [20 marks]**

The kinetic and potential terms relating to the Higgs field in the Standard Model are given by

$$\mathcal{L}_{\Phi} \quad = \quad \mathcal{L}_{\Phi-\text{kinetic}} - V\left(\Phi^{\dagger}\Phi\right) \,,$$

wherein

$$\mathcal{L}_{\Phi-\text{kinetic}} = \left(\boldsymbol{D}_{\mu}\Phi\right)^{\dagger}\left(\boldsymbol{D}^{\mu}\Phi\right) \,, \qquad V\left(\Phi^{\dagger}\Phi\right) = \mu^{2}\Phi^{\dagger}\Phi + \lambda\left(\Phi^{\dagger}\Phi\right)^{2} \,,$$

and

$$\boldsymbol{D}_{\mu} = \partial_{\mu} + ig'\frac{Y_{\phi}}{2}B_{\mu} + ig_{W}\boldsymbol{W}_{\mu} \,, \qquad \Phi = \begin{pmatrix} \phi^{+} \\ \phi^{0} \end{pmatrix} .$$

Inside $\boldsymbol{D}_{\mu}$, B_{μ} is the hypercharge gauge field, Y_{ϕ} the hypercharge of the Higgs doublet Φ, and g' is the hypercharge gauge field coupling constant. Furthermore, $\boldsymbol{W}_{\mu} = \tau^{a}W_{\mu}^{a}$, where W_{μ}^{a} $(a = 1, 2, 3)$, are the three W gauge boson fields, and τ^{a} are the Pauli spin matrices divided by two. g_{W} is the weak coupling constant.

(a) Compute $\boldsymbol{\Sigma}^{\dagger}\boldsymbol{\Sigma}$ in terms of $\Phi^{\dagger}\Phi$, where $\boldsymbol{\Sigma}$ here is the 2×2 *complex matrix*

$$\boldsymbol{\Sigma} = \begin{pmatrix} \phi^{0*} & \phi^{+} \\ -\phi^{+*} & \phi^{0} \end{pmatrix} .$$

Hence show that $V\left(\Phi^{\dagger}\Phi\right)$, as given above, can be written as

$$\tilde{V}\left(\boldsymbol{\Sigma}^{\dagger}\boldsymbol{\Sigma}\right) \quad = \quad \frac{1}{2}\mu^{2}\,\mathrm{Tr}\left[\boldsymbol{\Sigma}^{\dagger}\boldsymbol{\Sigma}\right] + \frac{1}{2}\lambda\,\mathrm{Tr}\left[\boldsymbol{\Sigma}^{\dagger}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\dagger}\boldsymbol{\Sigma}\right] .$$

[7]

(b) Show that $\tilde{V}\left(\boldsymbol{\Sigma}^{\dagger}\boldsymbol{\Sigma}\right)$, given in (a), is invariant under the transformation $\boldsymbol{\Sigma} \to \boldsymbol{\Sigma}' = \boldsymbol{U}_{L}\boldsymbol{\Sigma}\boldsymbol{U}_{R}$, where $\boldsymbol{U}_{L}$ and $\boldsymbol{U}_{R}$ are two different local $SU(2)$ transformations ($\boldsymbol{U}_{L/R} = \boldsymbol{U}_{L/R}^{\dagger}$).

[7]

(c) The kinetic part of the Higgs Lagrangian can be written as

$$\mathcal{L}_{\Phi-\text{kinetic}} = \frac{1}{2}\mathrm{Tr}\left[\left(\tilde{\boldsymbol{D}}_{\mu}\boldsymbol{\Sigma}\right)^{\dagger}\left(\tilde{\boldsymbol{D}}_{\mu}\boldsymbol{\Sigma}\right)\right] .$$

$\boldsymbol{U}_{L}$ is the usual local $SU(2)_{L}$ gauge transformation, under which

$$\boldsymbol{W}_{\mu} \to \boldsymbol{W}'_{\mu} = \boldsymbol{U}_{L}\boldsymbol{W}_{\mu}\boldsymbol{U}_{L}^{\dagger} - \left(ig_{W}\right)^{-1}\left(\partial_{\mu}\boldsymbol{U}_{L}\right)\boldsymbol{U}_{L}^{\dagger} .$$

Working in the limit that the hypercharge coupling is zero ($g' = 0$) the covariant derivative is

$$\tilde{\boldsymbol{D}}_{\mu}\boldsymbol{\Sigma} = \partial_{\mu}\boldsymbol{\Sigma} + ig_{W}\boldsymbol{W}_{\mu}\boldsymbol{\Sigma} .$$

For $g' = 0$ show that $\tilde{\boldsymbol{D}}_{\mu}\boldsymbol{\Sigma} \to \boldsymbol{U}_{L}\tilde{\boldsymbol{D}}_{\mu}\boldsymbol{\Sigma}$, and hence that $\mathcal{L}_{\Phi-\text{kinetic}}$ is invariant under this transformation.

[6]

2. **Nuclear β-decay [20 marks]**

You should neglect the positron mass throughout this question.

In the limit that the nucleons may be considered non-relativistic, the amplitude for the β-decay of the ^{14}O oxygen isotope to the ^{14}N* nitrogen isotope,

$$^{14}\mathrm{O} \rightarrow ^{14}\mathrm{N}^* + e^+ + \nu_e \,,$$

is given by

$$-i\mathcal{M} = -iG_F\,\bar{u}\left(p_\nu\right)\gamma^0\left(1-\gamma^5\right)v\left(p_e\right)\,\left(2m_N\right)\,,$$

where G_F is the Fermi constant, with p_e and p_ν denoting the positron and neutrino (four) momenta respectively. The nucleon mass is denoted m_N.

(a) Compute $i\mathcal{M}^*$.

[4]

(b) Show that the squared amplitude summed over positron and neutrino spin polarizations reduces to

$$\sum_{\text{spins}} |\mathcal{M}|^2 \quad = \quad 2G_F^2\,\left(2m_N\right)^2\,\mathrm{Tr}\left[\not{p}_\nu\gamma^0\;\not{p}_e\gamma^0\left(1-\gamma^5\right)\right]\,.$$

Remember to neglect the positron mass.

[7]

(c) Using the Dirac trace identity

$$\mathrm{Tr}\left[\not{p}_\nu\gamma^0\;\not{p}_e\gamma^0\left(1-\gamma^5\right)\right] = 8E_eE_\nu - 4p_e.p_\nu\,,$$

determine $\sum_{\text{spins}}|\mathcal{M}|^2$ in terms of G_F, m_N, E_e, E_ν and $\theta_{e\nu}$, where E_e and E_ν denote the energies of the positron and neutrino, with $\theta_{e\nu}$ labeling the angle between the neutrino and positron momenta.

[5]

(d) Including the phase space measure and flux factor the differential decay width can be written

$$d\Gamma \quad = \quad \frac{1}{2\pi^3}\,G_F^2\,\left[1+\cos\theta_{e\nu}\right]\,|\vec{p}_e|^2\,d\,|\vec{p}_e|\,d\cos\theta_{e\nu}\,E_\nu^2 dE_\nu\,\delta\left(E_0-E_e-E_\nu\right)\,,$$

where $|\vec{p}_e|$ is the magnitude of the positron's three-momentum and E_0 is the energy released to the lepton pair. Integrate $d\Gamma$ over E_ν and $\cos\theta_{e\nu}$ to obtain the positron spectrum

$$\frac{d\Gamma}{d\,|\vec{p}_e|} = \frac{G_F^2}{\pi^3}\,|\vec{p}_e|^2\left(E_0-E_e\right)^2\,.$$

[4]

3. **Non-Abelian gauge theory [20 marks]**

Unless otherwise stated you should assume the usual summation convention applies: repeated $SU(N)$ indices are implicitly summed over.

In an $SU(N)$ gauge theory the covariant derivative is given by

$$\boldsymbol{D}_\mu = \partial_\mu + ig\boldsymbol{A}_\mu\,, \qquad \boldsymbol{A}_\mu = \boldsymbol{T}^i A^i_\mu\,,$$

with $\boldsymbol{T}^i$ the i'th generator of $SU(N)$ and A^i_μ its corresponding gauge field $\left(i = 1, ..., N^2 - 1\right)$. Under a local $SU(N)$ transformation $\boldsymbol{U} = \exp\left(-i\alpha^i \boldsymbol{T}^i\right)$ the covariant derivative and the gauge fields change according to

$$\boldsymbol{D}_\mu \to \boldsymbol{D}'_\mu = \boldsymbol{U}\boldsymbol{D}_\mu\boldsymbol{U}^\dagger\,, \qquad \text{and} \qquad \boldsymbol{A}_\mu \to \boldsymbol{A}'_\mu = \boldsymbol{U}\boldsymbol{A}_\mu\boldsymbol{U}^\dagger + \frac{i}{g}\left(\partial_\mu \boldsymbol{U}\right)\boldsymbol{U}^\dagger\,.$$

The generators of $SU(N)$ obey the lie algebra $\left[\boldsymbol{T}^a, \boldsymbol{T}^b\right] = if^{abc}\boldsymbol{T}^c$, where f^{abc} are the group structure constants. The field strength tensor is given by

$$\boldsymbol{F}_{\mu\nu} = -\frac{i}{g}\left[\boldsymbol{D}_\mu, \boldsymbol{D}_\nu\right]\,,$$

where g is the coupling constant of the theory.

(a) Determine how $\boldsymbol{F}_{\mu\nu}$ changes under a local $SU(N)$ transformation, $\boldsymbol{U}$, in terms of the initial $\boldsymbol{F}_{\mu\nu}$ and $\boldsymbol{U}$.

[6]

(b) Hence show that the Yang-Mills Lagrangian, $\mathcal{L}_{\text{YM}} = -\frac{1}{2}\text{Tr}\left[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right]$, is invariant under local $SU(N)$ gauge transformations.

[4]

(c) Show that the field strength tensor, as written above, can be re-expressed as $\boldsymbol{F}_{\mu\nu} = \boldsymbol{T}^c F^c_{\mu\nu}$ where $F^c_{\mu\nu}$ is the *colour stripped* field strength tensor $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}\, A^b_\mu A^c_\nu$.
Hint: make use of the commutator relation for the $SU(N)$ generators.

[10]

4. **Goldstone bosons [20 marks]**

Unless otherwise stated you should assume the usual summation convention applies: repeated indices are implicitly summed over.

Consider a simple scalar field theory, of N *real* scalar fields ϕ_i, $i = 1, 2, ..., N$, with Lagrangian

$$\mathcal{L}\left(\phi_i, \partial_\mu \phi_i\right) = \frac{1}{2}\left(\partial_\mu \phi_i\right)\left(\partial^\mu \phi_i\right) - V\left(\phi_i \phi_i\right) , \qquad V\left(\phi_i \phi_i\right) = \frac{1}{2}\mu^2 \phi_i \phi_i + \lambda \left(\phi_i \phi_i\right)^2 .$$

(a) Given that U is a global $\mathrm{SO}(N)$ transformation matrix, determine / write down $U_{ij} U_{ik}$.

[3]

(b) Compute the effect of the global $\mathrm{SO}(N)$ transformation $\phi_i \to \phi_i' = U_{ij}\phi_j$ on $\phi_i \phi_i$.

[3]

(c) Compute the effect of the global $\mathrm{SO}(N)$ transformation $\phi_i \to \phi_i' = U_{ij}\phi_j$ on $\mathcal{L}$.

[2]

(d) Write the potential in terms of $|\phi| = \sqrt{\phi_i \phi_i}$ and, assuming the parameter μ^2 is negative, show the potential minima has extrema located along $|\phi| = a = \sqrt{-\frac{\mu^2}{4\lambda}}$.

[5]

(e) Assume the vacuum state of the theory is at $\left(\phi_1, \phi_2, ..., \phi_N\right)\big|_{\text{vacuum}} = (0, 0, ..., a)$, i.e. ϕ_N acquires a non-zero vacuum expectation value $\langle \phi_N \rangle_{\text{vacuum}} = a$, with a as given in part (d). Determine, in terms of $\sum_{i=1}^{N-1} \phi_i^2$ and χ^2, only the terms in the potential which are quadratic/bilinear in ϕ_1, ϕ_2,...,ϕ_{N-1} and the shifted field χ, where $\phi_N = \chi + a$. What are the masses of ϕ_1, ϕ_2,...,ϕ_{N-1} and χ?

[7]

5. **Hadron collider cross section [20 marks]**

Throughout this question you should assume that the masses of the colliding hadrons (protons) and final-state leptons are zero.

Theories of Physics beyond the standard model predict the existence of new heavy Z bosons denoted Z'. As with photons and Z bosons, Z' bosons may be produced by a Drell-Yan mechanism at hadron colliders, where, at parton level, a same-flavour quark-antiquark pair ($q\bar{q}$) annihilate to produce a Z', which subsequently decays to a lepton and an antilepton $\left(l\bar{l}\right)$: $q\bar{q} \to Z' \to l\bar{l}$.

Below we denote the colliding hadrons, incident from the $+/-$ z-directions, as $h_\oplus$ and $h_\ominus$ respectively, labelling their corresponding momenta $p_\oplus$ and $p_\ominus$. The fraction of $h_\oplus$'s momentum carried by the colliding quark/antiquark inside it is denoted $\eta_\oplus$ in the hadronic centre-of-mass frame. Similarly, the fraction of $h_\ominus$'s momentum carried by the colliding antiquark/quark inside it is $\eta_\ominus$. The total hadronic centre-of-mass energy squared is denoted $S = (p_\oplus + p_\ominus)^2$. The lepton and antilepton momenta are denoted p_l and $p_{\bar{l}}$ respectively.

(a) Using momentum conservation, $\eta_\oplus p_\ominus + \eta_\ominus p_\ominus = p_l + p_{\bar{l}}$, determine $m_{l\bar{l}}^2$ in terms of $\eta_\oplus$, $\eta_\ominus$, and S, where $m_{l\bar{l}}$ denotes the invariant mass of the lepton-antilepton system.

[3]

(b) Determine the rapidity of the lepton pair, $y_{l\bar{l}}$, in terms of $\eta_\oplus$ and $\eta_\ominus$, in the hadronic centre-of-mass frame.

[3]

(c) Determine the momentum fractions $\eta_\oplus$ and $\eta_\ominus$ in terms of $m_{l\bar{l}}$, $y_{l\bar{l}}$ and S.

[2]

(d) Compute the flux factor, F, in terms of $m_{l\bar{l}}$, where F is in general given by

$$F = 4\sqrt{\left(\left(\eta_\oplus p_\oplus\right).\left(\eta_\ominus p_\ominus\right)\right)^2 - \left(\eta_\oplus p_\oplus\right)^2\left(\eta_\ominus p_\ominus\right)^2}\,.$$

[3]

(e) The two-body phase space factor for the final-state leptons is

$$d\text{LIPS} \quad = \quad \frac{1}{16\pi}\,d\cos\hat{\theta}\,,$$

where $\hat{\theta}$ is the polar angle of the final-state lepton in the $q\bar{q}$ centre-of-mass frame. Given that the matrix element for $q\bar{q} \to Z' \to l\bar{l}$, summed over fermion spins, can be written in the form

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 128\pi\, m_{l\bar{l}}^2\, N_C^2 \left[\frac{3}{8}\left(1+\cos^2\hat{\theta}\right)\hat{\sigma}_T^{q\bar{q}} + \frac{1}{2}\cos\hat{\theta}\,\hat{\sigma}_{FB}^{q\bar{q}}\right],$$

where $N_C = 3$ is the number of colours in QCD, and $\hat{\sigma}_T^{q\bar{q}}$, $\hat{\sigma}_{FB}^{q\bar{q}}$ are constants depending only on the quark and lepton flavours, determine the total *partonic cross section*, $\hat{\sigma}^{q\bar{q}}$, in terms of these variables, integrated over all $\hat{\theta}$.

[5]

(f) Determine the total $h_\oplus\left(p_\oplus\right) + h_\ominus\left(p_\ominus\right) \to Z' \to l\bar{l}$ hadronic cross section, $\sigma_{h_\oplus h_\ominus \to Z' \to l\bar{l}}$, in the form of an integral over $y_{l\bar{l}}$ and $m_{l\bar{l}}$; you may use the relation $d\eta_\oplus d\eta_\ominus = \frac{1}{S}\,dm_{l\bar{l}}^2 dy_{l\bar{l}}$.

[4]

6. **Decay width of the top quark [20 marks]**

Throughout the question you should assume that the b-quark mass is zero and CKM element $V_{tb} = 1$.

In the following we denote the t, b and W particle momenta respectively as p_t, p_b, p_W.

(a) Draw the Feynman diagram for the two-body decay of a top quark into a b-quark and a W^+ boson, labelling the external particles by their momenta p_t, p_b, p_W. Take care to label the flow of fermion number appropriately with an arrow.

[2]

(b) The amplitude for the top quark decay to a b-quark and *longitudinal* W^+ boson is

$$-i\mathcal{M}_0 = \frac{-ig_W}{\sqrt{2}}\,\epsilon_L^{\mu}(p_W)\,j_{\mu}^{tb}, \qquad j_{\mu}^{tb} = \bar{u}(p_b)\,\gamma_{\mu}\frac{1}{2}(1-\gamma_5)\,u(p_t)\,,$$

where g_W is the weak coupling constant. Show that the complex conjugate of the amplitude is given by (recall $\gamma_{\mu}^{\dagger} = \gamma_0\gamma_{\mu}\gamma_0$)

$$i\mathcal{M}_0^* = \frac{ig_W}{\sqrt{2}}\,\epsilon_L^{\nu *}(p_W)\,j_{\nu}^{tb\,*}, \qquad j_{\nu}^{tb\,*} = \bar{u}(p_t)\,\gamma_{\nu}\frac{1}{2}(1-\gamma_5)\,u(p_b)\,.$$

[4]

(c) Summing the amplitude squared over fermion spins (t and b) one obtains

$$\sum_{t,b\,\text{spins}} |\mathcal{M}_0|^2 \;=\; \frac{g_W^2}{2}\,\epsilon_L^{\mu}(p_W)\,\epsilon_L^{\nu *}(p_W)\;\mathrm{Tr}\left[\not{p}_b\gamma_{\mu}\;\not{p}_t\gamma_{\nu}\frac{1}{2}(1-\gamma_5)\right].$$

The longitudinal polarization vector $\epsilon_L^{\mu}(p_W)$ of the W^+ boson, can be written as

$$\epsilon_L^{\mu}(p_W) \;=\; \frac{1}{m_W}\left(p_W^{\mu} - \frac{m_W^2}{p_W{\cdot}p_b}\,p_b^{\mu}\right).$$

Show that $\sum_{t,b\,\text{spins}} |\mathcal{M}_0|^2$ can be reduced to

$$\sum_{t,b\,\text{spins}} |\mathcal{M}_0|^2 \;=\; \frac{1}{2}\frac{g_W^2}{m_W^2}\;\mathrm{Tr}\left[\not{p}_b\;\not{p}_W\;\not{p}_t\;\not{p}_W\frac{1}{2}(1-\gamma_5)\right].$$

[Hint: ignoring the mass of the b-quark $\not{p}_b\;\not{p}_b = p_b^2 = 0$].

[6]

(d) Using the identity

$$\mathrm{Tr}\left[\not{p}_b\;\not{p}_W\;\not{p}_t\;\not{p}_W\frac{1}{2}(1-\gamma_5)\right] = 2p_b{\cdot}p_W\,m_t^2$$

compute $\sum_{t,b\,\text{spins}} |\mathcal{M}_0|^2$ in terms of g_W, m_t and x, where $x \equiv m_W^2/m_t^2$.

[5]

(e) The squared matrix element summed over *all* fermion spins and gauge boson polarizations is

$$\sum_{\text{spins,pols}} |\mathcal{M}|^2 \;=\; \frac{1}{2}\,g_W^2\,m_t^2\,\frac{1}{x}\,\left[1 + x - 2x^2\right].$$

Determine the branching fraction of top quark decays to longitudinally polarized W^+ bosons and b-quarks in the limit $m_t \gg m_W$.

[3]