# SMII
# Standard Model II
## Exam 2018

**Answer any Four questions**

**3 Hours**

*The numbers in square brackets in the right-hand margin indicate a provisional allocation of maximum possible marks for different parts of each question.*

[Prof. Robert Thorne]

*(Answer ANY FOUR questions)*
Note: only four answers will be marked

1. (a) Explain what is meant by the matrix $\gamma^5$. What are the properties of this matrix, including its commutation relations with other gamma matrices? **[3]**

   (b) Explain why $g\bar{\psi}(\phi_1 + i\gamma^5\phi_2)\psi$ is a suitable term for an interaction in a Lagrangian for a quantum field theory while $g(\bar{\psi}\psi)^2$ is not. **[4]**

   (c) Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \underline{\phi}) \cdot (\partial^\mu \underline{\phi}) + \frac{1}{2}\mu^2 \underline{\phi} \cdot \underline{\phi} - \frac{\lambda}{4}(\underline{\phi} \cdot \underline{\phi})^2 + \bar{\psi}(i\gamma \cdot \partial)\psi - g\bar{\psi}(\phi_1 + i\gamma^5\phi_2)\psi,$$

   where $\underline{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. Show that this theory is invariant under transformations $\phi_1 \to \cos\alpha\,\phi_1 - \sin\alpha\,\phi_2$, $\phi_2 \to \sin\alpha\,\phi_1 + \cos\alpha\,\phi_2$ and $\psi \to \exp(-i\alpha\gamma^5/2)\psi$. **[5]**

   (d) Explain why the minimum energy states of the system correspond to those with minimum potential energy. Identify the parts of the Lagrangian which correspond to this potential. **[2]**

   (e) Show that the solutions to the classical equations of motion with minimum energy lead to a vacuum which breaks the symmetry spontaneously. Choose a suitable vacuum solution, and show that the fermion field acquires a mass which is $\propto g$. **[6]**

2. (a) Explain what is meant by the orthogonal group $O(3)$, and state how many generators there are. How many would the group $O(4)$ have? **[4]**

(b) An early proposal for the Standard Model was a theory invariant under $O(3)$ gauge transformations, explicitly with the gauge-scalar particle sector defined by the Lagrangian where a three-component "triplet" gauge field $\mathbf{A}_\mu$ is coupled to a real triplet scalar field $\phi$ as below,

$$Ł = -\frac{1}{4}\,\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} + \tfrac{1}{2}(D^\mu\phi)\cdot D_\mu\phi - \frac{1}{8}\lambda\big(\phi^2 - v^2\big)^2,$$
$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + e\,\mathbf{A}_\mu\times\mathbf{A}_\nu\,, \quad D_\mu\phi = \partial_\mu\phi + e\,\mathbf{A}_\mu\times\phi\,.$$

Show that in this theory the $O(3)$ symmetry is broken by a choice of ground state to $O(2)$. **[3]**

Explain what is meant by "would-be Goldstone bosons" and the unitary gauge for the spontaneous broken gauge theory. **[5]**

Find the mass of the one physical scalar field. **[2]**

Using this gauge (i.e. the unitary gauge) rewrite the theory in terms of physical fields and show that the masses of the three gauge bosons are $m_A = \sqrt{2}ev$ for 2 bosons and $m_A = 0$ for one boson. **[6]**

3. (a) The covariant derivative in the electroweak sector of the Standard Model is defined by

$$D^\mu = (\partial^\mu + ig\tfrac{1}{2}\mathbf{A}^\mu(\mathbf{x})\cdot\sigma + iYg'B^\mu(x)),$$

where $g$ is the $SU(2)$ coupling constant $g'$ is the $U(1)_Y$ coupling constant $\sigma_i$ are the Pauli matrices and $Y$ is the hypercharge. Describe the $SU(2)$ and hypercharge representations and quantum numbers for the leptons. (Assume neutrinos are massless in this question.) **[2]**

(b) Using

$$W^{+\mu} = \frac{1}{\sqrt{2}}(A_1^\mu + iA_1^\mu), \qquad W^{-\mu} = \frac{1}{\sqrt{2}}(A_1^\mu - iA_1^\mu),$$

show in detail that the interaction of the $W^+$ boson with the lepton fields may be written as

$$\mathcal{L}_{W+lep} = -\frac{g}{2\sqrt{2}}W^{+\mu}\bar{\nu}_l\gamma_\mu(1-\gamma^5)l,$$

for each family of leptons. **[3]**

(c) Write the corresponding term for the $W^-$ boson. **[1]**

(d) Explain why the corresponding interaction term is more complicated for quarks by referrring to any distinction between weak and mass eigenstates for the fermions. **[3]**

(e) Show that at low energies it is equivalent to use an effective Lagrangian density

$$\mathcal{L}_{\text{Weff}} = -\frac{G_F}{\sqrt{2}}(J^\mu(x)^\dagger J_\mu(x))$$

where $G_F = \sqrt{2}g^2/8m_W^2$. **[3]**

(f) Consider the decay

$$\pi^-(p) \to e^-(k) + \bar{\nu}_e(q).$$

The matrix element for this decay is

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}}\langle e^-(k)\,\bar{\nu}_e(q)|\bar{e}\gamma^\alpha(1-\gamma_5)\nu_e|0\rangle\,\langle 0|J_\alpha^{\text{had.}}|\pi^-(p)\rangle.$$

By considering the leptonic part of the appropriate symmetry property of the pion explain very briefly why only the axial part of the hadronic current contributes to $\langle 0|J_\alpha^{\text{had.}}|\pi^-(p)\rangle$. **[2]**

(g) By considering this matrix element prove that the decay rate vanishes in the limit $m_e \to 0$. **[3]**

(h) Explain physically why the matrix element must vanish in this limit. **[3]**

4. (a) The covariant derivative in the electroweak sector of the Standard Model is defined by

$$D^\mu = (\partial^\mu + ig\tfrac{1}{2}\mathbf{A}^\mu(\mathbf{x}) \cdot \sigma + iYg'B^\mu(x)),$$

where $g$ is the $SU(2)$ coupling constant $g'$ is the $U(1)_Y$ coupling constant $\sigma_i$ are the Pauli matrices and $Y$ is the hypercharge. Using

$$\begin{pmatrix} A_3^\mu \\ B^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix},$$

and $g'/g = \tan\theta_W$, show that the interaction of the $Z$ boson with the fermion fields may be written as

$$\mathcal{L}_{Zf} = -\frac{g}{2\cos\theta_w}\sum_i Z^\mu \bar{\psi}^i \gamma_\mu (c_v^i - c_a^i\gamma^5)\psi^i,$$

where the sum is over fermion species. **[6]**

(b) Using the decomposition of the fermions in one generation into their $SU(2)$ and hypercharge representations find the values of $c_v^i$ and $c_a^i$ for the four types of fermion. **[4]**

(c) $M$ is the matrix element for $Z \to \bar{\psi}\psi$. Neglecting fermion masses we find that

$$\sum_{\lambda,s_1,s_2} |M|^2 = \frac{g^2 M_Z^2}{\cos^2\theta_W}\mathrm{Tr}\big(\gamma\cdot p_2\gamma^\mu(c_v^i - c_a^i\gamma^5)\gamma\cdot p_1\gamma^\nu(c_v^i - c_a^i\gamma^5)\big)\sum_\lambda \epsilon_\mu^*(q,\lambda)\epsilon_\nu(q,\lambda),$$

where $q$ is the $Z$ momentum and $p_1, p_2$ that of the fermion and antifermion, and the sum is over initial polarizations and final spins. Working in the $Z$ rest frame show that **[7]**

$$\sum_{\lambda,s_1,s_2} |M|^2 = \frac{g^2 M_Z^2((c_v^i)^2 + (c_a^i)^2)}{\cos^2\theta_W}.$$

(d) Assuming that neutrinos in any possible family are effectively massless, what information did the first measurement of the total width, and hence decay rate, of the $Z$ boson provide? **[3]**

[You may use $\sum_\lambda \epsilon_\mu^*(q,\lambda)\epsilon_\nu(q,\lambda) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}$, and

$$\mathrm{Tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \qquad \mathrm{Tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4i\epsilon_{\alpha\beta\gamma\delta},$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor. ]

5. (a) Under charge conjugation we assume $\psi(x) \longrightarrow \psi^C(x)$ , $\psi^C(x) = C\bar{\psi}(x)^t$ , with $t$ denoting transpose, and that $\bar{\psi}(x) \to -\psi^t(x)C^{-1}$. Explain why a current interaction

$$J^\mu V_\mu = \bar{\psi}(x)\gamma^\mu(1-\gamma^5)\psi(x)V_\mu,$$

is not invariant under parity or charge conjugation separately but is invariant under the combined transformation. **[6]**

(b) Under a time reversal transformation $\hat{T}\psi(x)\hat{T}^{-1} = B^{-1}\psi(x_T)$ and $\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B$ where $B = \gamma_5 C$ and $\hat{T}$ is an antilinear transformation, i.e. it takes the complex conjugate of numbers. Show that $B(\gamma^{0*}, -\gamma_i^*)B^{-1} = (\gamma^0, \gamma_i)$, and that the above current interaction is invariant under time reversal. **[4]**

(c) The $K^0$ and its anti-particle $\bar{K}^0$ are pseudoscalar mesons with dominant quark structure $\bar{s}d$ and $\bar{d}s$. Under $CP$ we can define

$$\hat{C}\hat{P}|K^0\rangle = |\bar{K}^0\rangle\,, \qquad \hat{C}\hat{P}|\bar{K}^0\rangle = |K^0\rangle\,.$$

The mass eigenstates of the system are the eigenvectors of the matrix

$$M = \begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\bar{K}^0\rangle \\ \langle \bar{K}^0|H'|K^0\rangle & \langle \bar{K}^0|H'|\bar{K}^0\rangle \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\,,$$

where $H'$ is an effective Hamiltonian arising from weak processes that mix $|K^0\rangle$ and $|\overline{K}^0\rangle$, and $M_{11} = M_{22}$. Draw a Feynman diagram representing one such mixing process. **[2]**

(d) Show that if $H'$ is not invariant under $CP$ then $M_{12} \neq M_{21}$. **[3]**

(e) Show that the mass eigenstates are equal to the $CP = +1$ and $-1$ eigenstates

$$|K_1{}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle + |\bar{K}^0\rangle\Big)\,, \qquad |K_2{}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle - |\bar{K}^0\rangle\Big)\,,$$

up to small corrections proportional to **[5]**

$$\epsilon = \frac{\sqrt{M_{12}} - \sqrt{M_{21}}}{\sqrt{M_{12}} + \sqrt{M_{21}}}.$$

[Under parity transformations $P$

$$\psi(x) \to \gamma^0\psi(x_P) \qquad \bar{\psi}(x) \to \bar{\psi}(x_P)\gamma_0 \qquad V_\mu(x) \to V^\mu(x_P).$$

Under charge conjugation $C$ $V_\mu(x) \to -V_\mu(x)$, and under time reversal $V_\mu(x) \to V^\mu(x_T)$.

You may assume $C(\gamma^\mu)^t C^{-1} = -\gamma^\mu$ and $C^\dagger = C^{-1}$.]

6. (a) Briefly justify the fact that the renormalized coupling in any quantum field theory $g(\mu^2)$ is a function of the renormalization scale and satisfies a renormalization group equation of the general form

$$\frac{dg}{d\ln\mu^2} = \sum_{n=i}^{\infty} b_n g^n,$$

where $b_n$ are constant coefficients unique to a particular theory. **[7]**

(b) At lowest order this equation can be written in the form

$$\frac{d\alpha}{d\ln\mu^2} = b_0\alpha^2,$$

where $\alpha = g^2/4\pi$, find the solution subject to boundary conditions $\alpha = \alpha_0$ at $\mu^2 = \mu_0^2$ for $b_0 = \pm\beta_0$, where $\beta_0$ is a positive constant. If $b_0$ is negative show that the solution may be rewritten as **[4]**

$$\alpha(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)}.$$

(c) $b_0$ is negative for the strong QCD coupling. Explain what happens as $\mu^2 \to \infty$ and for small $\mu^2$, and discuss briefly what results this produces in strong interaction physics. Roughly what size should $\Lambda$ be assuming $\beta_0$ is of order unity? **[5]**

(d) Discuss the behaviour of the coupling with $\mu^2$ for positive $b_0$. This is the case for the Higgs self-coupling. Using the relationship between this coupling and the Higgs mass explain what implication a very large Higgs mass would have had on the reliability of the Standard model for very large $\mu^2$. **[4]**

**END OF PAPER**