The Standard Model
B . Massive Gauge Theories and Spontaneous Symmetry Breaking

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1. Problems With Short Range Interactions
2. Spontaneous Symmetry Breaking
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1 Problems With Short range Interactions

While in principle we now have a basis for quantizing scalar, fermionic and vector fields, this only works in the most naive manner for the simple theory QED with a $U(1)$ symmetry. As already mentioned we have a problem with the strong interaction because the particles are not the fundamental fields. We will deal with this later.

There is also a problem with the weak interaction, which describes such processes as

$$n \to p + e^- + \bar{\nu}_e \tag{1}$$

and

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu. \tag{2}$$

The former also requires dealing with constituent particles, but the latter is an example of an interaction involving fundamental fields.

Phenomenologically there are two intrinsic features of the weak interactions:

1. It is short range in nature, and hence cannot be described by massless vector bosons.

2. It violates parity, i.e. it only involves the left-handed particle, e.g. it couples to currents of the type

$$\begin{aligned} J_l^\lambda &= \bar{\nu}_e \gamma^\lambda (1-\gamma^5) e \\ J_h^\lambda &= \bar{u} \gamma^\lambda (1-\gamma^5) \tilde{d}, \end{aligned}$$

where we use $\tilde{d}$ since there is mixing in the quark sector.

We could try an interaction of the type

$$\mathcal{L}_{int} = -\frac{G_F}{\sqrt{2}} J_\lambda^\dagger J^\lambda + \text{hermitian conjugate}, \tag{3}$$

where J^λ is the charged current and $G_F \sim 10^{-5}\text{GeV}^{-2}$ is the Fermi coupling constant. However this type of interaction runs into two types of problem.

1. Perturbative unitarity. It is a fundamental rule of quantum field theory that

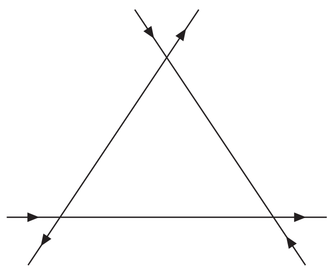
$$\sigma_{tot} \leq k \ln^2(E), \tag{4}$$

where E is the energy of the scattering process, and k is some constant. This is known as the Froissart bound. However, at leading order using the proposed current-current interaction we find

$$\sigma(\nu_\mu e^- \to \mu^- \nu_e) = \frac{G_F^2 s}{\pi}, \tag{5}$$

where s is the usual Mandelstam variable $s = (p_1 + p_2)^2$ which is equal to E^2 in the centre-of-mass frame. Hence, there is a perturbative violation of unitarity. This is not a fundamental problem, it merely implies that perturbation theory becomes useless at energies such that a finite order perturbative calculation breaks the unitarity bound, and this may be true. However, there is another more serious problem.

2. Renormalizability. If we look at a 6 particle scattering amplitude using the current-current interaction we obtain.



$$\sim G_F^3 \int \frac{d^4k}{k^3}$$

We have an integral over the four-momentum from the closed loop, and each of the fermion propagators behaves like $1/k$ in the limit that $|k| \to \infty$. Hence, the contribution to the integral from this region behaves like $G_F^3 \int \frac{d^4k}{k^3}$. This is clearly divergent, and in order to make sense of it we must impose an upper cut-off Λ on the integral, which we usually interpret as the scale of new physics where the theory must be modified. Doing this the integral then behaves like ΛG_F^3, i.e. it is divergent in Λ.

This problem of "ultraviolet" divergences is common to all realistic field theories and requires the procedure known as renormalization. In this one redefines the parameter associated with the lowest order diagram, e.g. a coupling or mass, as the "bare" parameter. This will then have Λ-dependence which cancels that from the loop-correction, resulting in a finite total result. It is this total result which is directly related to a physical process, whereas the "bare" parameter in the Lagrangian is not physically observable, but just a parameter which appears in the calculation of processes. In a renormalizable theory, a suitable Λ-dependent definition of a finite number of bare parameters (e.g. coupling, electron mass, and electron and photon normalization in QED) results in a theory with no ultraviolet divergences, and Λ may be taken to infinity. Physically this means that the theory at scales far below some new physics may be defined in terms of a finite number of parameters with no knowledge of the new physics at higher scales being required.

In the theory with the current-current interaction this procedure does not work. The simplest loop diagram for 6 particle scattering is divergent, and this divergence can only be removed by the introduction of a new 6-point vertex not present in the original proposal of the Lagrangian. This introduces a new parameter into the theory. However, this new 6-point vertex means that the 1-loop 8-point diagram has divergences not removed by renormalization of the parameters already present, and an 8-point vertex must be added to the Lagrangian to remove this divergence. This continues such that an infinite number of parameters is needed to calculate all possible scattering processes, and the theory is said to be non-renormalizable. This means that in order to make real progress in calculations one must know what the new physics is which relates this infinite number of parameters.

An alternative to the current-current interaction, i.e. the model for the new physics at high scales, is to assume we have a very massive vector boson, i.e. the free part of the Lagrangian in the boson sector is

$$\mathcal{L}_W = -\tfrac{1}{2}(\partial_\mu W_\nu^\dagger - \partial_\nu W_\mu^\dagger)(\partial^\mu W^\nu - \partial^\nu W^\mu) + M_W^2 W_\mu^\dagger W^\mu \qquad (6)$$

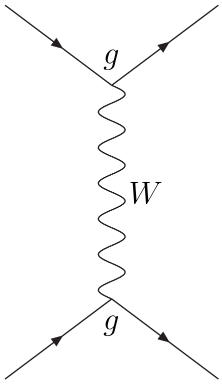
and there is an interaction term of the form

$$\mathcal{L}_{int} = g(J^\mu W_\mu + \text{hermitian conjugate}), \qquad (7)$$

This leads to a gauge boson propagator of the form

$$D_{\mu\nu}(k^2) = -i\frac{g_{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2 + i\epsilon}. \qquad (8)$$

If $k^2 \ll M_W^2$ then $D_{\mu\nu}(k^2) \to i g_{\mu\nu}/M_W^2$. Therefore, in fermion-fermion scattering we obtain diagrams of the form



$$\sim -\frac{g^2}{M_W^2} = -\frac{G_F}{\sqrt{2}}$$

and the four-fermion scattering looks like a current-current point interaction at low energies.

Hence, this approach looks plausible. However, given that we invoked gauge symmetry to get a sensible theory of vector bosons in the last section we have to ask if we are allowed to simply add a term $M_W^2 W_\mu^\dagger W^\mu$ to the Lagrangian, since it clearly violates gauge symmetry.

It is actually allowed for the photon. Consider the Lagrangian

$$\mathcal{L}_g = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} M_A^2 A^\mu A_\mu. \tag{9}$$

The equation of motion for this is

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = \frac{\partial \mathcal{L}}{\partial A^\nu} \tag{10}$$

which leads to

$$\partial_\mu F^{\mu\nu} + M_A^2 A^\nu = 0. \tag{11}$$

Differentiating again and using the asymmetry of $F^{\mu\nu}$ we automatically obtain $M_A^2 \partial_\nu A^\nu = 0$. Hence, $\partial_\nu A^\nu = 0$ and we can eliminate one degree of freedom. In momentum space this becomes the condition $p^\mu \epsilon_\mu = 0$ on the polarization vector. In the rest frame of the particle $p = (m_A, 0)$ so we must have $\epsilon = (0, \underline{\epsilon})$ and it is the timelike photon polarization which is eliminated by using the equations of motion. This means that the field A_μ describes a particle with three real degrees of freedom, and has the mode expansion

$$A_\mu(\underline{x}, t) = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{1}{2E_{\underline{k}}} \sum_{\lambda=1,2,3} [\epsilon_\mu(\lambda, \underline{k}) a(\lambda, \underline{k}) e^{-ik\cdot x} + \epsilon_\mu^*(\lambda, \underline{k}) a^\dagger(\lambda, \underline{k}) e^{ik\cdot x}] \tag{12}$$

where the creation and annihilation operators satisfy

$$[a(\lambda, \underline{k}'),), a^\dagger(\lambda', \underline{k})] = \delta_{\lambda\lambda'} 2E_{\underline{k}} (2\pi)^3 \delta^3(\underline{k} - \underline{k}'). \tag{13}$$

So in this case the theory is quantized correctly.

We might worry about problems with renormalizability, since as $k^2 \to \infty$ the propagator $D_{\mu\nu}(k^2) \to i(k^2/M_A^2)/k^2 = i/M_A^2$ rather than the $1/k^2$ usually required for renormalizability from power counting. However, in fact there is no problem. This is because this bad high-momentum behaviour only comes from the part of the propagator of the form

$$\frac{k_\mu k_\nu/M_A^2}{k^2 - M_A^2 + i\epsilon}. \tag{14}$$

The photon only couples to fermions via the term $A^\mu J_\mu$, where $J_\mu = -e\bar{\psi}\gamma_\mu\psi$, and J_μ is conserved, i.e. $\partial^\mu J_\mu = 0$ or in momentum space $k^\mu J_\mu = 0$. Therefore, the above contribution to the propagator makes no contribution in any Feynman diagram, and renormalizability can be maintained.

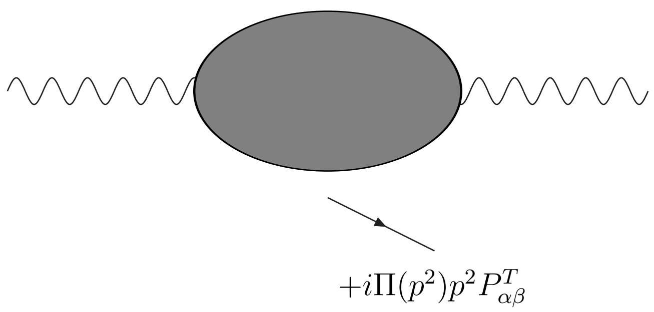
Therefore, there is NOTHING at all wrong with a massive $U(1)$ gauge theory, but the $U(1)$ symmetry is needed for all but the mass term in order to eliminate the timelike degrees of freedom and guarantee unitarity.

In order to obtain a charged current in the weak interaction and hence at least two distinct gauge fields we need a non-Abelian symmetry. In a non-Abelian theory we have more complicated couplings of the gauge field W_μ than just that to the conserved fermionic current. In general the $\frac{k_\mu k_\nu/M_A^2}{k^2-M_A^2+i\epsilon}$ part of the unitarity inspired propagator does contribute, and renormalizability is lost. A modification of the propagator to a more acceptable form as far as renormalizability is concerned, e.g. $-ig_{\mu\nu}/(k^2 - M_W^2 + i\epsilon)$ leads to explicit breaking of gauge symmetry and to the existence of negative norm timelike modes. Therefore, non-Abelian massive gauge theories do not exist. (The equation of motion for the non-Abelian theory would be $D_\mu F^{\mu\nu} + M_W^2 W^\nu = 0$, and does not lead to the elimination of the timelike mode in the same way as for the photon.)

There is one possible way around the problem of how to give the non-Abelian gauge boson a mass. Consider a general non-Abelian gauge theory quantum gauge theory with bare propagator

$$D^0_{\mu\nu} = -i\frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2 + i\epsilon} = -i\frac{P^T_{\mu\nu}}{p^2 + i\epsilon}. \tag{15}$$

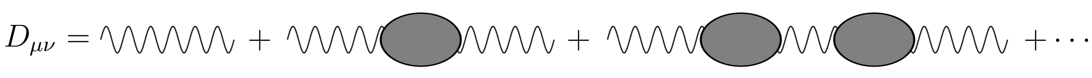
This will receive quantum corrections of the form



where the hatched oval represents any higher order correction, e.g. it could be a closed loop of fermions. We only need consider the transverse part since the longitudinal part will give zero when contracted with $P^T_{\mu\nu}$. Under this correction

$$D^0_{\mu\nu} \to D^0_{\mu\nu} - i\Pi(p^2)\frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2 + i\epsilon} = D^0_{\mu\nu}(1 + \Pi(p^2)). \tag{16}$$

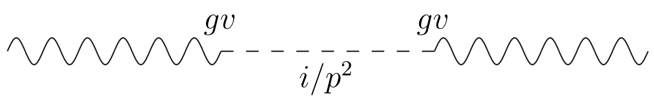
There will be an infinite number of such *bubble* corrections, forming a geometric series.



We may sum this series (a so-called Dyson resummation) obtaining

$$D_{\mu\nu} = D^0_{\mu\nu}\frac{1}{1 - \Pi(p^2)}. \tag{17}$$

In usual circumstances $\Pi(p^2) = A + Bp^2 + \mathcal{O}(p^4)$, and all it leads to is a finite wavefunction renormalization (i.e. an alteration of the residue at the pole at $p^2 = 0$). However, if the gauge boson is able to couple to a massless scalar particle as below



then we can obtain $\Pi(p^2) \sim g^2v^2/p^2$. in this case

$$D^0_{\mu\nu} \to -i\frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2(1 - g^2v^2/p^2)} = -i\frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2 - g^2v^2 + i\epsilon}, \tag{18}$$

and the gauge boson acquires a mass.

From this admittedly rather heuristic example above we see that a massive vector boson can be obtained from the interaction with an exactly massless scalar particle. But massless scalars are unnatural. The mass of a scalar field is completely arbitrary and hence we need some explanation of why we should have exactly massless scalars to provide our non-Abelian gauge bosons with mass, as well as a precisely formulated mechanism for the way in which they do so. Fortunately both are provided by the process known as spontaneous symmetry breaking.

2 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking occurs when the physics theory under question has some underlying symmetry, but the ground state is degenerate and a particular choice of ground state or vacuum does not respect this symmetry of the Lagrangian or Hamiltonian. In classical physics, there are many instances when the ground state is not unique. For example, in a ferromagnetic material, the interactions between neighbouring atomic spins depends only on their relative orientation. The lowest energy state of the material corresponds to all spins aligned in the same direction. There are an infinite number of possible directions for this alignment. Generally the material (or at least domains within it) picks a particular direction which, of course, breaks the rotational symmetry.

In quantum field theory the same phenomena can occur when the vacuum state is not unique. To uncover the particle content and kinds of particle interaction in a quantum field theory, one constructs a Fock space by expanding fields about their vacuum values. This involves choosing a vacuum state, in which fields take a definite value, that minimizes the energy. In general the vacuum state chosen is not invariant under all the symmetries of the original Lagrangian. Neither will the interactions of particle states obtained by expanding fields around particular vacuum values be invariant under these symmetries. This is called spontaneous symmetry breaking and it plays a fundamental role in the Standard Model. However, we will discuss the process in stages, building up to the ultimate result of massive vector bosons in non-Abelian gauge theories.

2.1 Discrete Symmetries

To illustrate this in field theory, we first consider the simplest case of a Lagrangian density for a single real scalar field ϕ,

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \phi \, \partial_\mu \phi - V(\phi) \,, \tag{19}$$

which is invariant under the Z_2 symmetry (corresponding to the group with elements $\{1, -1\}$ under multiplication),

$$\phi \leftrightarrow -\phi \,. \tag{20}$$

The assumption of symmetry under (20) requires

$$V(\phi) = V(-\phi) \,, \tag{21}$$

and as a typical field theory example we may take

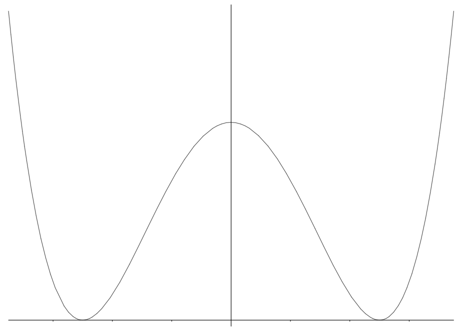
$$V(\phi) = \tfrac{1}{2}\, m^2 \phi^2 + \tfrac{1}{4!}\, g\phi^4 \,, \qquad g > 0 \,, \tag{22}$$

with the condition on the coupling g necessary to ensure that $V(\phi)$ is bounded below. For $m^2 > 0$, classically the minimum of V uniquely occurs at $\phi_0 = 0$, which is invariant under (20).

However a different picture emerges if $m^2 < 0$. In this case by addition of a constant we might rewrite $V(\phi)$ in the form

$$V(\phi) = \tfrac{1}{4!}\, g(\phi^2 - v^2)^2 \,, \tag{23}$$

which has the form the form of a double well, shown below, and is designed so that $V(\phi_0) = 0$.



In the ground state, which minimizes energy, there are now two possibilities classically, $\phi_0 = \pm v$, and in the quantum field theory there are expected to be two vacua $|0_\pm\rangle$ such that

$$\langle 0_\pm|\phi(x)|0_\pm\rangle = \pm v_R \,, \tag{24}$$

with v_R some renormalised value, including quantum corrections, of the constant v. For the two vacua it is possible to construct two independent Hilbert spaces of states $\mathcal{H}_\pm$ by the application of field operators to $|0_\pm\rangle$.

To identify the particle content and set up a perturbative expansion for this theory it is necessary to choose a particular vacuum value for the field. Let us shift the field

$$\phi = v + f \,, \tag{25}$$

so that the Lagrangian density defined by (19) and (23) becomes

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu f \partial_\mu f - \tfrac{1}{6} g (v^2 f^2 + v f^3 + \tfrac{1}{4} f^4) \,. \tag{26}$$

Given that v is just a fixed number, f then describes a massive scalar field, of mass squared $gv^2/3$, with further cubic and quartic interactions. However, these two interactions do not depend on independent free parameters and will still be related in a simple manner after renormalization.

Classically, or in quantum perturbation theory to lowest order, $\langle 0|f|0\rangle = 0$. (There are quantum corrections to this result which will make it non zero from Feynman diagrams with one external line. One must then shift f by quantum corrections in order to maintain zero VEV (vacuum expectation value)). The Fourier modes of f then have the interpretation of creation and annihilation operators for massive scalar particles. However, the field theory of the scalar

f has no Z_2 symmetry $f \to -f$. It will be described by just two renormalized parameters v_R and g_R however.

It is interesting to note that the scenario just described is valid in quantum field theory but it fails in ordinary quantum mechanics. This is because for a similar potential in quantum mechanics there will be tunnelling between the two vacua which connects them and which will result in the true lowest energy states being two superpositions of the two naive vacuum states which would be parity eigenstates with parity $\pm$. In quantum field theory, one is essentially dealing with a quantum mechanics problem at each point in space, with x replaced by $\phi(\mathbf{x})$. Tunnelling between the two ground states requires tunnelling at every point in space. If we neglect the coupling between $\phi(\mathbf{x})$ at different points, the total amplitude is given by

$$\langle 0_- | 0_+ \rangle \sim e^{-CN} \,. \tag{27}$$

where N is the number of points in space and $e^{-C} < 1$ is the quantum mechanics result. If space is a continuum, obviously this amplitude is zero.

The vacuum field in the example above is translationally invariant. However, in $\mathcal{H}_+$ there may be states $|\psi\rangle$ which are essentially identical to $|0_-\rangle$ for some finite region $\mathbf{V}$, i.e.

$$\langle \psi | \phi(0,\mathbf{x}) | \psi \rangle = -v_R \quad \text{for} \quad \mathbf{x} \in \mathbf{V} \,, \quad \langle \psi | \phi(0,\mathbf{x}) | \psi \rangle \to v_R \quad \text{as} \quad |\mathbf{x}| \to \infty \,. \tag{28}$$

Such states are similar to a bubble inside which the theory looks like that represented by the space $\mathcal{H}_-$. The state $|\psi\rangle$ however has energy greater than (either) ground state by an amount which is at least proportional to the area of the boundary of the region $\mathbf{V}$, where the field ϕ changes sign. If the original Z_2 symmetry of (19) were explicitly broken, for example by adding a non-symmetric term to lift the vacuum degeneracy so that $|0_-\rangle$ has a lower energy density, then it is possible for the bubble to grow indefinitely. The gain in energy proportional to the volume of $\mathbf{V}$ can compensate the energy involved in the boundary in that case. This type of situation occurs in certain cosmological models where symmetries at high temperatures are spontaneously broken at some critical temperature as the universe cools.

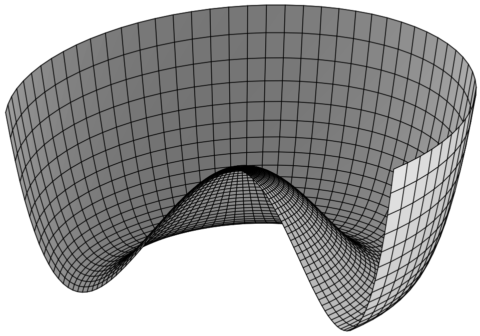
The above description of spontaneous symmetry breakdown for Z_2 generalizes straightforwardly to any discrete symmetry group of order N, but spontaneous symmetry breaking is more interesting when we consider continuous symmetries.

2.2 Continuous Symmetries

We may also consider a continuous symmetry group which may undergo spontaneous symmetry breakdown. As a simple illustration we first consider an n component scalar field theory with real fields $\phi = (\phi_1, \dots, \phi_n)$. Defining $\phi^2 \equiv \phi{\cdot}\phi = \sum_r \phi_r \phi_r$ we postulate a Lagrangian density.

$$\mathcal{L} = \tfrac{1}{2} \partial^\mu \phi {\cdot} \partial_\mu \phi - V(\phi) \,, \quad V(\phi) = \tfrac{1}{8} g (\phi^2 - v^2)^2 \,, \quad g > 0 \,. \tag{29}$$

For $n = 2$ the potential has the form shown,



This Lagrangian is clearly invariant under the symmetry group $O(n)$ which rotates the n-vector ϕ. It is also evident that the classical ground state corresponding to the minimum of the potential in (29) is given by $\phi = \phi_0$ for any ϕ_0 such that

$$ {\phi_0}^2 = v^2 \, . \tag{30} $$

This defines an $n-1$ dimensional sphere, S^{n-1} (for $n = 2$ the classical ground states lie on a circle). At any point on the S^{n-1} defined by (30) there are 'flat' directions along which the potential energy remains unchanged. This has important physical consequences.

We can see this explicitly by expanding ϕ about a particular point on S^{n-1}, for example

$$ \phi_0 = (0, \ldots, 0, v) \, , \tag{31} $$

so that

$$ \phi = (\phi_\perp, v + f) \, , \quad \phi_\perp = (\phi_1, \ldots, \phi_{n-1}) \, . \tag{32} $$

The potential now becomes

$$ V(\phi) = \tfrac{1}{2} g v^2 f^2 + \tfrac{1}{2} g v \, ({\phi_\perp}^2 + f^2) f + \tfrac{1}{8} g \, ({\phi_\perp}^2 + f^2)^2 \, . \tag{33} $$

There is a quadratic term for the field f, but the $n-1$ fields $\phi_\perp$ have no quadratic contribution so that the frequencies of these modes for small fluctuations around ϕ_0 are zero. The quadratic terms in a Lagrangian, after any linear terms have been removed by shifting the fields, determine the particle masses in the associated quantum field theory, so that in this example there are $n-1$ massless fields after spontaneous symmetry breakdown (choice of a particular vacuum (31)) and one massive field f. The fields which are massless are called Goldstone modes. The symmetry group $O(n)$ for this situation is then reduced to an $O(n-1)$ symmetry that leaves the vacuum ϕ_0 invariant. The $O(n-1)$ group acts only on the first $n-1$ components $\phi_\perp$ of ϕ and thus remains a symmetry of (33).

A more general discussion of spontaneous symmetry breakdown can be developed which is applicable to any field theory in which the Lagrangian is invariant under a continuous symmetry group G but the ground state is invariant under a subgroup H. We assume a Lagrangian density with a multi-component scalar field ϕ and where the potential V is assumed to be invariant so that,

$$V(U(g)\phi) = V(\phi) \quad \text{for all} \quad g \in G\,. \tag{34}$$

Classically spontaneous symmetry breakdown arises when the ground state is not a single point invariant under G, but is a non trivial manifold,

$$\Phi_0 = \{\phi_0 \,:\, V(\phi_0) = V_{\min}\}\,. \tag{35}$$

For any point $\phi_0 \in \Phi_0$ we may define its stability group $H \subset G$ by

$$U(h)\phi_0 = \phi_0 \quad \text{for all} \quad h \in H\,. \tag{36}$$

2.3 Goldstone's Theorem - Classical

The Goldstone theorem states that, in a quantum field theory when spontaneous symmetry breakdown of a continuous symmetry occurs, there are zero mass particles, Goldstone bosons, whose numbers are determined by the dimensions of G and H.

At the classical level this amounts to counting the number of zero frequency modes for small oscillations around the classical ground state. We prove the result for real scalar field theory of a scalar with components ϕ_r. To demonstrate the result we first recast (34) in infinitesimal form,

$$V(\phi + \delta\phi) = V(\phi) \quad \text{for} \quad \delta\phi = iT_a\chi_a\phi\,, \quad a = 1, \ldots \dim G\,, \tag{37}$$

where T_a are the $\dim G$ generators of the Lie algebra of G in the representation defined by ϕ and χ_a are some infinitesimal parameters. We use a summation convention for any repeated indices. The generators are hermitian.

(37) can obviously be rewritten as

$$\frac{\delta}{\delta\phi_r} V(\phi)\, (T_a\phi)_r = 0\,. \tag{38}$$

The frequencies of the oscillations of the field around the ground state are determined by the eigenvalues of the matrix formed by the second derivatives of V evaluated at the minimum. Choosing an arbitrary point $\phi_0 \in \Phi_0$ this 'mass matrix' is then defined by

$$\mathcal{M}_{sr} = \frac{\delta^2}{\delta\phi_s\delta\phi_r} V(\phi)\Big|_{\phi=\phi_0}\,. \tag{39}$$

Now from (38) we have at $\phi = \phi_0$.

$$\frac{\delta^2}{\delta\phi_s\delta\phi_r} V(\phi)\, (T_a\phi)_r + \frac{\delta}{\delta\phi_r} V(\phi)\, (T_a)_{rs} = 0\,, \tag{40}$$

and since at a minimum the first derivatives of V must be zero we have

$$\mathcal{M}_{sr}(T_a\phi_0)_r = 0\,. \tag{41}$$

Thus $T_a\phi_0$ is a zero frequency eigenvector for the matrix $\mathcal{M}$.

To count the number of such zero eigenvectors we first note that if t_i is a generator of the Lie algebra of H, which is the stability group at $\phi_0 \in \Phi_0$, then

$$t_i\phi_0 = 0\,, \quad i = 1, \ldots \dim H\,. \tag{42}$$

If G is compact and semi-simple (as is the case for internal symmetry groups of interest in particle physics) we may choose a basis for the Lie algebra such that

$$T_a = (t_i, T_{\hat{a}})\,, \tag{43}$$

with $T_{\hat{a}}$ orthogonal to t_i, which corresponds to $\mathrm{tr}(t_i T_{\hat{a}}) = 0$. With this result, it is clear from (41) and (42) that there are $\dim G - \dim H$ linearly independent eigenvectors $T_{\hat{a}}\phi_0$ with zero eigenvalues for the matrix $\mathcal{M}$.

If we apply this counting to the example given with $G = O(n)$, $H = O(n-1)$ then

$$\dim O(n) - \dim O(n-1) = \tfrac{1}{2}n(n-1) - \tfrac{1}{2}(n-1)(n-2) = n-1\,, \tag{44}$$

which is the correct number of Goldstone modes in this case. The group H is the manifest unbroken symmetry group of the theory after spontaneous symmetry breakdown.

3 Higgs Mechanism

3.1 Abelian

When spontaneous symmetry breaking occurs in gauge field theory, Goldstone modes can be absorbed into the definition of a massive vector field, as we have already demonstrated in an approximate fashion. This is usually called the Higgs effect or Higgs mechanism. It can maintain the renormalizability and unitarity achieved for a vector theory with exact gauge invariance while introducing the mass.

The reinterpretation of a massless Goldstone boson as the conversion of a massless vector theory (2 degrees of freedom) to a massive vector theory (3 degrees of freedom) relies intrinsically on the local nature of the gauge symmetry. Considering a particle as a localized disturbance in a field, a global transformation of the field by definition cannot remove this disturbance. However, for a local transformation one may match the transformation function $\chi(x)$ to the local field disturbance so as to cancel it out. Since it is a symmetry transformation, the particle that corresponded to that local disturbance in the field is not a physical degree of freedom.

We have already seen something similar in QED where there is a vector photon A_μ which apparently has four components and where the process of choosing a particular gauge transformation function $\chi(x)$ to eliminate two unphysical degrees of freedom is called *gauge fixing*. This can be done by specifying $\chi(x)$, but is is more usual to put some condition on the fields, such as Lorentz gauge condition $\partial_\mu A^\mu = 0$.

In a spontaneously broken gauge theory the choice of which is the true vacuum is equivalent to a choice of gauge. The Goldstone bosons then correspond to transformations into the other degenerate vacuum states, and are transitions into states not consistent with the original gauge choice. This shows that the Goldstone bosons are unphysical and are often called "Goldstone ghosts". We first illustrate this for a $U(1)$ gauge theory with a complex scalar field and then analyse the general case for a non-Abelian gauge theory.

We start with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + (D^\mu \phi)^* D_\mu \phi - V(\phi^*\phi)\,, \tag{45}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad D_\mu\phi = \partial_\mu \phi + ieA_\mu \phi\,. \tag{46}$$

This is invariant under local $U(1)$ gauge transformations where

$$A_\mu \to A_\mu - \frac{1}{e}\,\partial_\mu \chi\,, \qquad \phi \to e^{i\chi}\phi\,, \tag{47}$$

for arbitrary space-time dependent $\chi(x)$. The covariant derivative is constructed so that under (47), $D_\mu\phi \to e^{i\chi} D_\mu \phi$, so that gauge invariance of (45) is trivially evident.

For the theory described by the Lagrangian (45), there are two phases with very different physics for which the natural physical variables are completely different.

1. The minimum of $V(\phi^*\phi)$ occurs at $\phi^*\phi = 0$; for instance

$$V(\phi^*\phi) = \tfrac{1}{2} m^2 \phi^*\phi + \tfrac{1}{2} g (\phi^*\phi)^2\,, \quad m^2, g > 0\,. \tag{48}$$

In this case, in the classical theory the ground state is $\phi_0 = 0$ and in the quantum theory we expect a unique vacuum state $|0\rangle$. The gauge field couples to a conserved current j^μ whose corresponding charge $Q = \int \mathrm{d}^3x\, j^0$ is conserved, generating a $U(1)$ symmetry with $Q|0\rangle = 0$. In a perturbative expansion, the theory describes spinless charged particles, with to lowest order a mass m and charges $\pm e$, interacting with massless photons. The physical degrees of freedom are then 2 for the field A_μ, corresponding to the two photon polarization states after removal of gauge degrees of freedom, and 2 for the field ϕ, corresponding to the two charge states.

2. The minimum of $V(\phi^*\phi)$ occurs away from the origin at $\phi_0^*\phi_0 = \tfrac{1}{2}v^2$; for instance we might take

$$V(\phi^*\phi) = \tfrac{1}{2} g (\phi^*\phi - \tfrac{1}{2}v^2)^2\,. \tag{49}$$

In this case the $U(1)$ gauge symmetry is broken by a specific choice of the ground state. To derive the physical consequences in this situation, it is convenient to rewrite the fields if $\phi \neq 0$ in the form

$$A_\mu = B_\mu - \frac{1}{e}\,\partial_\mu \theta/v\,, \qquad \phi = \frac{1}{\sqrt{2}}\,(v+f)e^{i\theta/v}\,, \tag{50}$$

with f, θ real. It is sensible to write the scalar field in this form because expanding about the vacuum we obtain

$$\phi = \frac{1}{\sqrt{2}}(v + f + i\theta + \cdots), \tag{51}$$

i.e. f and θ represent the excitations about the vacuum. If the vacuum expectation value were zero writing $\phi = \frac{1}{\sqrt{2}}\,f e^{i\theta/\mu}$ would result in only $\frac{1}{\sqrt{2}}\,f$ plus interaction terms between f and θ - θ would not be an excitation about the vacuum in this case, and hence not a mass eigenstate. Under the action of gauge transformations in (47) it is easy to see that

$$\theta/v \to \theta/v + \chi\,, \tag{52}$$

while B_μ, f are gauge invariant. Using

$$D_\mu\phi = \frac{1}{\sqrt{2}}\,e^{i\theta/v}\Big(\partial_\mu f - ie\,B_\mu(v+f)\Big)\,, \tag{53}$$

we may rewrite the Lagrangian in (45) in the form

$$\mathcal{L} = -\tfrac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}e^2(v+f)^2 B^\mu B_\mu + \tfrac{1}{2}\partial^\mu f\,\partial_\mu f - \tfrac{1}{8}\,g(2vf+f^2)^2\,. \tag{54}$$

For small fluctuations around the ground state given by $f, B_\mu = 0$ we may restrict this to just the quadratic terms giving

$$\mathcal{L}_{\text{quadratic}} = -\tfrac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}e^2v^2\,B^\mu B_\mu + \tfrac{1}{2}\partial^\mu f\,\partial_\mu f - \tfrac{1}{2}gv^2 f^2\,, \tag{55}$$

which results in the "free" equations of motion,

$$\partial^\mu F_{\mu\nu} + e^2v^2 B_\nu = 0 \;\Rightarrow\; \begin{cases} \partial^\nu B_\nu = 0 \\ (\partial^2 + e^2v^2)B_\nu = 0 \end{cases}, \qquad (\partial^2 + gv^2)f = 0\,. \tag{56}$$

Thus B_μ represents a massive vector field describing spin 1 particles with mass M_B, $M_B{}^2 = e^2v^2$ at lowest order, which has therefore 3 degrees of freedom ($J_Z = -1, 0, +1$), while the field f describes spinless particles of mass m_f, $m_f{}^2 = gv^2$. Unlike the case of spontaneous symmetry breakdown of a purely global continuous symmetry, there are no massless modes. In a sense, the photon absorbs the Goldstone boson so as to ensure it has the right degrees of freedom to give a massive spin 1 particle. It was possible to rewrite the theory just in terms of gauge invariant variables, so that θ disappeared from the Lagrangian in (54). The field $\theta(x)$ is the 'would-be' Goldstone field in this case, but by a suitable gauge transformation as in (52) we could transform it to zero; this is known as unitary gauge choice. Equivalently, we could impose a gauge condition on the fields

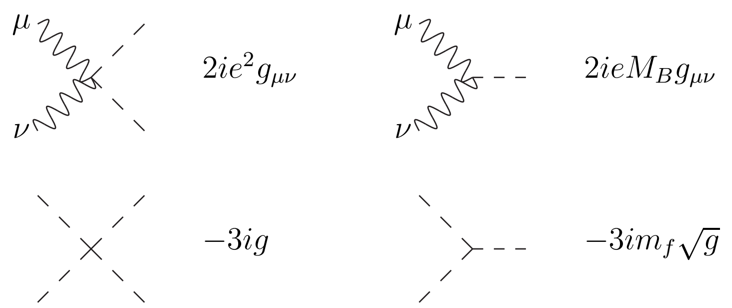
$$\phi = \phi^* \,, \tag{57}$$

which makes ϕ real and hence $\theta = 0$. The degree of freedom corresponding to θ is not lost however, since in the unitary gauge the field B_μ becomes a massive 3-component field. There are four physical degrees of freedom before and after spontaneous symmetry breakdown.

The interaction part of the Lagrangian is

$$\mathcal{L}_{\rm int} = \frac{e^2}{2} B^\mu B_\mu f^2 + e M_B B^\mu B_\mu f - \frac{g}{8} f^4 - \tfrac{1}{2} m_f \sqrt{g} f^3 \tag{58}$$

leading to the vertices below. Note that while there are four of these, once we know the masses, they are described in terms of only two parameters, e and g, reflecting the memory retained of the original symmetry.



3.2 Renormalizability and Unitarity

The propagator for the massive vector boson field is now

$$D_{\mu\nu}(k^2) = -i\Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_B^2}\Big)\frac{1}{k^2 - M_B^2 + i\epsilon}. \tag{59}$$

This behaves like $k^2/(k^2 M_B^2) \to 1/M_B^2$ as $k^2 \to \infty$, rather than $1/k^2$ usually required for renormalizability. Hence we have lost manifest renormalizability. (Note that we previously discovered that a propagator of the form (59) was acceptable in an Abelian gauge theory, though not in a non-Abelian theory. However, with the spontaneous symmetry breaking it is no longer obvious that B_μ couples only to a conserved current, and even manifest $U(1)$ renormalizability has been lost.)

Hence, it is unclear whether we have really gained anything. The answer to this is yes because the underlying symmetry is preserved. To see this we construct the theory in a slightly different manner. This time we let the scalar field be

$$\phi = \frac{1}{\sqrt{2}}(v + f + i\varphi), \tag{60}$$

where f and φ both have zero vacuum expectation value. Now we find that

$$\begin{aligned}(D^\mu\phi)^* D_\mu\phi \rightarrow \tfrac{1}{2}\partial^\mu f\,\partial_\mu f \quad &+ \quad \tfrac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi + \tfrac{1}{2}e^2v^2A^\mu A_\mu + evA^\mu\partial_\mu\varphi \\ -eA^\mu(\varphi\partial_\mu f - f\partial_\mu\varphi) \quad &+ \quad e^2vfA^\mu A_\mu + \tfrac{1}{2}A^\mu A_\mu(f^2+\varphi^2). \end{aligned} \qquad (61)$$

This contains terms we saw for f and the vector field before, but also terms in φ, particularly the bilinear term $evA^\mu\partial_\mu\varphi$, which under integration by parts $\rightarrow -M_A\varphi\partial_\mu A^\mu$, defining $M_A = ev$. This interaction mixes the φ and A^μ fields (in particular the longitudinal component), making the mass eigenstates unclear.

We can eliminate this bilinear term by gauge fixing. We introduce the gauge fixing term

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A\varphi)^2 \qquad (62)$$

to the Lagrangian, which is similar to the usual covariant gauge fixing. Multiplying out (62) we obtain a term $M_A\varphi\partial_\mu A^\mu$ which cancels the previous mixing term and results in a propagator for the gauge boson

$$D_{\mu\nu}(k^2) = -i\Big(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\Big)\frac{1}{k^2 - M_A^2 + i\epsilon}, \qquad (63)$$

which now has the physical pole and an additional unphysical ξ-dependent pole.

The scalar field φ also has the propagator

$$-i\frac{1}{p^2 - \xi M_A^2}, \qquad (64)$$

and has an unphysical pole at the same place as the gauge field.

In the limit $\xi \rightarrow \pm\infty$ the Goldstone boson mass becomes infinite and it decouples from the theory. In this limit the gauge boson propagator

$$\rightarrow -i\Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_A^2}\Big)\frac{1}{k^2 - M_A^2 + i\epsilon}. \qquad (65)$$

In this limit we regain the manifestly unitary theory of the last section. However, if, e.g. $\xi = 1$, known as the Feynman gauge,

$$D_{\mu\nu}(k^2) = -i\frac{g_{\mu\nu}}{k^2 - M_A^2 + i\epsilon}, \qquad D_\varphi(p^2) = -i\frac{1}{p^2 - M_A^2}, \qquad (66)$$

and we have manifest renormalizability (this is true for any finite ξ – the Landau gauge $\xi = 0$ is more physical), but an unphysical vector field and a Goldstone ghost, both of which have interaction vertices. However, the result of any calculation for a physical process is independent of ξ, and we can prove unitarity in the unitary gauge and renormalizability in a renormalizable gauge (finite ξ). Usually one uses the unitary gauge at tree-level to get rid of unwanted unphysical particles, but uses the renormalizable gauge in loop calculations in order to have the more conventional form of the propagator.

3.3 Non-Abelian

For the gauge theory corresponding to local symmetry G the gauge field Lagrangian density is

$$\mathcal{L}_{\rm gauge} = -\tfrac{1}{4}\, F^{\mu\nu}.F_{\mu\nu}\,. \tag{67}$$

If the symmetry G is not spontaneously broken, since (67) contains no mass terms $A_a^\mu A_{\mu a}$, there are $\dim G$ massless vector particles in the theory, generalizations of the photon. We will now see that the Higgs mechanism just discussed generalizes to the non Abelian case.

The scalar contribution to the Lagrangian density is

$$\mathcal{L}_\phi = \tfrac{1}{2}\, (D^\mu\phi)^\dagger{\cdot}D_\mu\phi - V(\phi^\dagger\phi)\,, \tag{68}$$

where the covariant derivative is

$$D_\mu\phi = \partial_\mu\phi + ig\, A_{\mu a}T_a\phi\,. \tag{69}$$

We assume that the potential determines a ground state corresponding to spontaneous symmetry breakdown of the group G, i.e. the minimum of the potential occurs for non zero $\phi_0 \in \Phi_0$ as before. With the gauge group G reduced at any point on Φ_0 to invariance under a subgroup H we first need to consider $\dim G - \dim H$ gauge conditions which maintain local gauge invariance for H. In the unitary gauge we restrict the scalar fields ϕ by making them satisfy,

$$\phi^\dagger T_a\phi_0 - \phi_0^\dagger T_a\phi = 0. \tag{70}$$

As in (43), the generators and gauge fields can be decomposed into those belonging to the Lie algebra of H and those which are orthogonal,

$$T_a = (t_i, T_{\hat a})\,, \qquad A_{\mu a} = (\mathbf{A}_{\mu i}, A_{\mu\hat a})\,. \tag{71}$$

Since, for t_i the generators of H,

$$t_i\phi_0 = 0\,, \tag{72}$$

and t_i is hermitian, (70) is automatically satisfied for these generators and we may restrict (70) to just $\phi^\dagger T_{\hat a}\phi_0 - \phi_0^\dagger T_{\hat a}\phi = 0$. Any ϕ can be arranged to satisfy (70) by applying a suitable gauge transformation $\phi \to U\phi$. Then we may generally write

$$\begin{aligned} \phi &= U^{-1}(\phi_0 + f)\,, \qquad\qquad f^\dagger T_{\hat a}\phi_0 - \phi_0^\dagger T_{\hat a} f = 0\,, && (73)\\ A^\mu_{\hat a}T_{\hat a} &= U^{-1}B^\mu_{\hat a}T_{\hat a}U + \frac{i}{g}(\partial^\mu U^{-1})U && (74)\\ U &= \exp\left(i\eta_{\hat a}(x)/v\, T_{\hat a}\right), && (75) \end{aligned}$$

where v is the mass scale of symmetry breaking. The fields $\eta_{\hat a}(x)$ are the ($\dim G - \dim H$) would-be Goldstone modes.

The unitary gauge choice corresponds to making a gauge transformation U that precisely matches the space-time variation of these fields:

$$\begin{aligned} A^\mu_{\hat a} &\rightarrow B^\mu_{\hat a} \\ \phi &\rightarrow (\phi_0 + f) \\ \mathbf{A}^\mu_i &\rightarrow \mathbf{A}^\mu_i \end{aligned} \tag{76}$$

It is clear from (73) and the hermiticity of T_a that the new ϕ satisfies (70). Note that one is still free to make H gauge transformations.

With this decomposition and (73) the covariant derivative defined in (69) reduces to

$$D_\mu \phi = \Delta_\mu f + ig\, B_{\mu\hat a} T_{\hat a}(\phi_0 + f)\,, \quad \Delta_\mu f = \partial_\mu f + ig\, \mathbf{A}_{\mu i} t_i f\,, \tag{77}$$

where $\Delta_\mu f$ is the H covariant derivative and $\mathbf{A}_\mu$ the corresponding gauge field.

In $(D^\mu\phi)^\dagger \cdot D_\mu\phi$ there are terms of the form

$$ig[\partial^\mu f^\dagger B_{\mu\hat a} T_{\hat a} \phi_0 - \phi_0^\dagger T_{\hat a} B_{\mu\hat a} \partial^\mu f] \tag{78}$$

which are defined to be zero from the form of the gauge fixing. Hence, this gauge fixing removes the bilinear terms which couple f and $B_{\mu\hat a}$. This means the Lagrangian in (68) is now

$$\begin{aligned} \mathcal{L}_\phi \;=\; & \tfrac{1}{2}\,(\Delta^\mu f)^\dagger \cdot \Delta_\mu f + \tfrac{1}{2} g^2\, B^\mu{}_{\hat a} B_{\mu\hat b}\, (T_{\hat a}(\phi_0 + f))^\dagger \cdot (T_{\hat b}(\phi_0 + f)) \\ & + [ig(\Delta^\mu f)^\dagger B_{\mu\hat a}\,(T_{\hat a} f) + g^2 \mathbf{A}_{\mu i} f^\dagger t_i B_{\mu\hat a}\,(T_{\hat a}(\phi_0)) + \text{herm. conj.}] \\ & - V((\phi_0 + f)^\dagger(\phi_0 + f))\,. \end{aligned} \tag{79}$$

Although the complete theory is described by $\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_\phi$, the physical particle states can be identified from the quadratic terms only

$$\begin{aligned} \mathcal{L}_{\text{quadratic}} \;=\; & -\tfrac{1}{4}(t_i \partial_\mu \mathbf{A}_{\nu i} - t_i \partial_\nu \mathbf{A}_{\mu i})(t_i \partial^\mu \mathbf{A}^\nu_i - t_i \partial^\mu \mathbf{A}^\nu_i) \\ & -\tfrac{1}{4}(T_{\hat a} \partial_\mu B_{\nu\hat a} - T_{\hat a} \partial_\nu B_{\mu\hat a})(T_{\hat a} \partial^\mu B^\nu_{\hat a} - T_{\hat a} \partial^\nu B^\mu_{\hat a}) \\ & + \tfrac{1}{2}\,\partial^\mu f^\dagger \cdot \partial_\mu f - \tfrac{1}{2} f^\dagger \cdot (\mathcal{M} f) + \tfrac{1}{2} M_{\hat a\hat b} B^\mu{}_{\hat a} B_{\mu\hat b}\,, \end{aligned} \tag{80}$$

with the matrix $\mathcal{M}$ defined as in (39) and

$$M_{\hat a\hat b} = g^2 (\phi_0^\dagger T_{\hat a}) \cdot (T_{\hat b} \phi_0)\,. \tag{81}$$

Thus, M has $\text{dim}G - \text{dim}H$ positive eigenvalues, which are the masses squared of the vector bosons $B^\mu_{\hat a}$. There are $\text{dim}H$ massless vector particles $\mathbf{A}^\mu_i$, each of which requires gauge-fixing, and scalar 'Higgs particles' f, which by virtue of (73) are orthogonal to zero eigenvalue eigenvectors $T_{\hat a}\phi_0$ of $\mathcal{M}$. It is important to note that the number of Higgs particles in f depends upon the dimension of the representation chosen for ϕ.

The renormalizable gauge can be defined in the case of the non-Abelian theory in an exactly analogous manner to the Abelian theory. Hence, both renormalizability and unitarity can be proven for massive non-Abelian gauge theories which acquire their masses via spontaneous symmetry breaking and the Higgs mechanism. This now allows us to proceed with defining the electroweak sector of the Standard Model.