The Standard Model
C. Electro-Weak Gauge Theory

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Preface

The original quantum gauge theory, QED, with Abelian gauge group $U(1)_Q$ was generalized by Glashow, Weinberg, Salam, and Ward, to one with $SU(2) \times U(1)_Y$ local symmetry in order to describe also the weak interactions of leptons, such as muon decay. This is often known as electroweak unification, although since the two groups have independent coupling constants the terminology is a little misleading. To agree with nature a number of further requirements were necessary in this electro-weak theory. The enlarged gauge symmetry is spontaneously broken, leading to the Higgs mechanism and some massive gauge fields. The fermions of the theory must be in spin-dependent *chiral* representations. With Iliopoulos and Maiani, Glashow also showed how the electroweak theory could be successfully extended to the hadron bound states of quarks, and hence to a description of processes such as radioactive β-decay.

The starting point for the construction an Electro-Weak gauge theory, that generalizes QED to include the weak interactions, is to identify the appropriate gauge group G and also the corresponding representations under which the fields transform. The gauge group of electromagnetism $U(1)_Q$ should be a subgroup of G, where Q labels electric charge, and the generator of $U(1)_Q$ is associated with the photon (massless because this gauge symmetry is unbroken). One supposes that, just as QED interactions are mediated by photon exchange, weak interactions are also mediated by vector particles. Since the interactions are weak, they must however be very heavy particles whose propagation is therefore suppressed. If they are gauge fields, this heavy mass can be brought about in a renormalizable way via the Higgs mechanism. To agree with nature, it is found that quarks and leptons have to transform in non-trivial representations (multiplets of dimension greater than one) and therefore that G should be non-Abelian.

Since only the electromagnetic and charged-weak current were known about when the electroweak sector was constructed, only the photon and the two

W gauge bosons were required. This suggests that the simplest model might be $SU(2)$ or $O(3)$ broken spontaneously to $U(1)_Q$. However, the conserved charges of the charged-weak current and the electromagnetic current do not form a closed algebra, and hence the required group structure is not present. This could be rectified by adding heavy, as yet undetected, charged and neutral leptons E^+ and N to the family with e^- and ν_e, and forming triplet fermion representations. However, this is not very elegant, and it was difficult to get it to work for fractionally charged quarks. It is also now incompatible with observation.

The other alternative is to increase the possible number of vector bosons. A large amount of detailed experimental data supports the choice $G = SU(2) \times U(1)_Y$, where $SU(2)$ is sometimes referred to as 'weak isospin' and Y is 'weak hypercharge'. Further motivation for this choice of G will become clear later, once the quark and lepton representations are specified. Since G is of dimension 4, one expects three further (massive) vector fields in addition to the (massless) photon. In particular there is a neutral massive vector boson.

1 Gauge Bosons and Spontaneous Symmetry Breaking

Let $i\mathbf{T}$, $\mathbf{T} = (T_1, T_2, T_3)$, be the generators of $SU(2)$, such that $[T_a, T_b] = i\epsilon_{abc}T_c$ and iY be the generator of $U(1)_Y$. The relationship between Y and Q is chosen to be

$$Q = T_3 + Y \ . \tag{1}$$

Let the corresponding (real) gauge fields be $\mathbf{A}_\mu$ and B_μ respectively. Under gauge transformations $g(\alpha(x), \beta(x))$, we have

$$g(\alpha, \beta) = \exp(i\alpha.\mathbf{T}) \exp(i\beta Y) \ , \tag{2}$$

$$\mathbf{A}_\mu(x).\mathbf{T} \to e^{i\alpha(x).\mathbf{T}} \mathbf{A}_\mu(x).\mathbf{T} e^{-i\alpha(x).\mathbf{T}} + \frac{i}{g}(\partial_\mu e^{i\alpha(x).\mathbf{T}}) e^{-i\alpha(x).\mathbf{T}} \ , \tag{3}$$

while the Abelian $U(1)_Y$ gauge field transforms as

$$B_\mu(x) \to B_\mu(x) - \frac{1}{g'}\partial_\mu \beta(x) \ . \tag{4}$$

Note that because of the direct product structure of the gauge group it is necessary to introduce two coupling constants g and g', one for each factor in the gauge group. The existence of two coupling parameters is crucial to the structure of electro-weak theory, although means that the theory is not really fully unified.

We can also define the field strengths for the gauge fields themselves by

$$\mathbf{F}_{\mu\nu}(x) = \partial_\mu \mathbf{A}_\nu(x) - \partial_\nu \mathbf{A}_\mu(x) + g\mathbf{A}_\mu(x) \times \mathbf{A}_\nu(x) \ , \tag{5}$$

where under a gauge transformation $\mathbf{F}_{\mu\nu}(x).\mathbf{T} \to e^{i\alpha(x).\mathbf{T}}\mathbf{F}_{\mu\nu}(x).\mathbf{T}e^{-i\alpha(x).\mathbf{T}}$, and

$$G_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) \ , \tag{6}$$

which is actually invariant under $U(1)_Y$, just as the usual Maxwell electromagnetic field strength is under $U(1)_Q$ gauge transformations. The general renormalizable gauge invariant Lagrangian density is then

$$\mathcal{L}_{\mathrm{gauge}}(x) = -\frac{1}{4}\mathbf{F}^{\mu\nu}.\mathbf{F}_{\mu\nu} - \frac{1}{4}G^{\mu\nu}G_{\mu\nu} \tag{7}$$

1.1 Spontaneous Symmetry Breakdown

The gauge fields defined above correspond to massless vector (spin 1) particles, at least when treated in perturbation theory after quantization. For a theory of electro-weak interactions that agrees with nature, the only allowed massless vector particle is the photon, corresponding to the usual Maxwell gauge field. The remaining vector fields must be given a mass. In order to do this in a manner which ensures that the theory is renormalizable and unitary, we must apply spontaneous symmetry breaking. The easiest way to do this is to introduce an elementary scalar Higgs field ϕ, whose potential $V(\phi)$ is invariant under gauge transformations on ϕ, but is such that its minimum is degenerate. In the ground state of the field theory, the Higgs field is restricted to a subset $\mathbf{V}_{\mathrm{min}}$ on which G acts in a non trivial fashion. In the quantum field theory the vacuum is defined, to lowest order in perturbation theory, by choosing a particular point ϕ_0, belonging $\mathbf{V}_{\mathrm{min}}$, and then expanding about it, as previously discussed.

In general, ϕ_0 is not invariant under the action of group transformations belonging to G. Those elements of G which leave ϕ_0 invariant, $h\phi_0 = \phi_0$, define a subgroup $H \subset G$ which is then the unbroken gauge group. The gauge fields linked to the generators of the Lie algebra of H remain massless while those corresponding to the coset G/H gain a mass. In the present case it is necessary to preserve a residual $U(1)_Q$ invariance to ensure that there remains a massless photon. This is ensured by choosing the Higgs field to be in the $T = \frac{1}{2}$ fundamental representation of $SU(2)$ weak iso-spin (i.e., a doublet) and to carry weak hypercharge $Y = \frac{1}{2}$. (In general the irreducible representations of $U(1)_Y$ are one-dimensional and are determined by assigning a particular value to Y). Hence

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \ , \tag{8}$$

where ϕ_1 and ϕ_2 are complex. The covariant derivative for scalars is

$$D_\mu\phi(x) = \left(\partial_\mu + ig\tfrac{1}{2}\mathbf{A}_\mu(x).\sigma + ig'\tfrac{1}{2}B_\mu(x)\right)\phi(x) \ , \tag{9}$$

where $\frac{1}{2}\sigma$ is the $T = \frac{1}{2}$, 2×2 matrix representation of $\mathbf{T}$ for $SU(2)$. $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

A renormalizable Lagrangian density for ϕ that is invariant under the local gauge group $SU(2) \times U(1)_Y$ is then

$$\mathcal{L}_{\rm Higgs}(x) = (D^\mu \phi(x))^\dagger D_\mu \phi(x) - V(\phi(x)) \;, \tag{10}$$

if the potential $V(\phi)$ has the form

$$V(\phi) = F(\phi^\dagger \phi) \;. \tag{11}$$

The Lagrangian defined by eqs.(10,11) is invariant under $SU(2) \times U(1)_Y$ for any value of the weak hyper-charge of ϕ, but choosing $Y = \frac{1}{2}$ is crucial later to allow for coupling of ϕ to the lepton fields. For spontaneous symmetry breakdown the potential V, or F, is assumed to be minimum at points where $\phi^\dagger \phi = \frac{1}{2} v^2$. Since a scalar field has the dimension of inverse length, for renormalizability $V(\phi)$ should be at most quartic in the field ϕ. If, by convention, we choose $V_{\rm min} = 0$, we may take

$$V(\phi) = \tfrac{1}{2}\lambda \left(\phi^\dagger \phi - \tfrac{1}{2} v^2 \right)^2 \;. \tag{12}$$

As a particular ground state which realizes the minimum of $V(\phi)$, we may choose

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \;, \tag{13}$$

where v is real and $v > 0$. All other solutions of the minima condition $\phi^\dagger \phi = \frac{1}{2} v^2$ can be obtained from ϕ_0 by an application of suitable transformations belonging to the symmetry group of V, $SU(2) \times U(1)_Y$. In the quantum field theory of course, ϕ_0 is the vacuum expectation value (VEV) of the Higgs doublet, $\langle 0|\phi|0\rangle = \phi_0$. This is also sometimes referred to as a vacuum *condensate*. With the choice in eq.(13), from eq.(1) we have

$$(\tfrac{1}{2}\sigma_3 + \tfrac{1}{2} 1)\phi_0 = Q\phi_0 = 0 \;. \tag{14}$$

Thus the unbroken subgroup under which the ground state or vacuum is invariant is $U(1)_Q$, generated by iQ. The coupling to the Higgs field then gives masses to all gauge fields other than that corresponding to the photon.

1.2 Physical Degrees of Freedom

The physical fields after spontaneous symmetry breakdown may be identified most easily by using the unitary gauge transformation, to ensure that the Higgs particle is orthogonal to the massless Goldstone boson fields. These Goldstone modes can be regarded as belonging to the coset space $SU(2) \times U(1)_Y / U(1)_Q$ and are effectively absorbed into the gauge fields by the gauge transformation. Applying the unitary gauge transformation

$$\exp(-i\mathbf{n}(x).\sigma + in_3) \quad ; \quad \mathbf{n} = (n_1, n_2, n_3) \;, \tag{15}$$

where $\mathbf{n}$ are matched to the 'would-be' Goldstone fields, we can write, in this gauge,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + H(x)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{16}$$

where H is a real scalar field identified with the Higgs particle, representing the fluctuations of the Higgs field around the ground state value. The choice in eq.(16) is equivalent to imposing three gauge conditions on the Higgs field of the form

$$\phi(x)^\dagger \sigma \phi_0 - {\phi_0}^\dagger \sigma \phi(x) = 0 \,, \qquad \phi(x)^\dagger \phi_0 - {\phi_0}^\dagger \phi(x) = 0 \,, \tag{17}$$

where ϕ_0 is given by eq.(13). Although (17) contains apparently four linear conditions on ϕ, by eq.(14) one is redundant, so there remains one real degree of freedom represented by H in eq.(16). With the definition of the covariant derivative in eq.(9) we then find

$$\begin{aligned} D_\mu \phi &= \frac{1}{\sqrt{2}} \begin{pmatrix} i\,\frac{1}{2} g(v+H)(A_{\mu 1} - iA_{\mu 2}) \\ \partial_\mu H - i\,\frac{1}{2}(v+H)(gA_{\mu 3} - g' B_\mu) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \Big(\partial_\mu H - i\,\frac{g}{2\cos\theta_W}(v+H) Z_\mu \Big) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\,\tfrac{1}{2} g(v+H) W_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \end{aligned} \tag{18}$$

where the Weinberg angle θ_W is defined by

$$\tan\theta_W = \frac{g'}{g} \,, \quad \cos\theta_W = \frac{g}{(g^2 + g'^2)^{\frac{1}{2}}} \,, \tag{19}$$

and we introduced the linear combinations

$$\begin{aligned} W_\mu &= \tfrac{1}{\sqrt{2}} \left(A_{1\mu} - iA_{2\mu} \right) \,, \\ Z_\mu &= \cos\theta_W A_{3\mu} - \sin\theta_W B_\mu \,. \end{aligned} \tag{20}$$

The Higgs Lagrangian which is given by eqs.(10,12) then becomes

$$\mathcal{L}_{\text{Higgs}} = \tfrac{1}{2}\partial^\mu H \partial_\mu H + \tfrac{1}{4} g^2 (v+H)^2 \Big(\frac{1}{\cos^2\theta_W}\, \tfrac{1}{2} Z^\mu Z_\mu + W^{\mu\dagger} W_\mu \Big) - \tfrac{1}{8}\lambda (H^2 + 2vH)^2 \,. \tag{21}$$

The field $H(x)$ represents the degrees of freedom associated with the Higgs boson, whose mass satisfies ${m_H}^2 = \lambda v^2$. Now that the Higgs boson has been observed experimentally we know the mass is 125 GeV.

The most important aspect of eq.(21) for the construction of a viable electroweak theory is that it generates mass terms for the vector fields W_μ, Z_μ. These are obtained by identifying the terms quadratic in fields

$$m_W^2 W^{\mu\dagger} W_\mu + \tfrac{1}{2}\, m_Z^2 Z^\mu Z_\mu \,, \tag{22}$$

where

$$m_W^2 = \tfrac{1}{4}\, g^2 v^2 \,, \qquad m_Z^2 = \tfrac{1}{4}\,(g^2 + g'^2) v^2 = \frac{m_W^2}{\cos^2\theta_W} \,. \tag{23}$$

Experimentally $m_Z \approx 91$ GeV, $m_W \approx 80$ GeV, $\sin^2\theta_W \approx 0.23$. In general we write $m_W^2 = \rho m_Z^2 \cos^2\theta_W$, where from above $\rho = 1$ but where radiative corrections lead to ρ nearer to 1.01, which is verified (with a accuracy of 0.5%) by experiment. If a different scalar multiplet were used in spontaneous symmetry breaking, rather than the simplest doublet, ρ would be completely different in general.

As we will see the couplings g and g' are related to particle couplings and are well-determined experimentally. From the values of m_W and m_Z we find that $v \approx 250$ GeV, and this is usually quoted as the scale of spontaneous symmetry breaking, or as the scale of electroweak physics, and hence from $m_H = 125$ GeV $\lambda \approx 1/4$.

The orthogonal combination to Z_μ in eq.(20) given by

$$A_\mu = \sin\theta_W A_{3\mu} + \cos\theta_W B_\mu \ , \tag{24}$$

has no mass term, i.e. there is no term of the form $\frac{1}{2}A^\mu A_\mu$. This is the gauge field for the unbroken $U(1)_Q$ gauge symmetry. Note that the result in eq.(21) is in fact independent of the particular weak hypercharge assignment to the Higgs field ϕ, although the definition of the Weinberg angle θ_W would have to be modified from eq.(19).

Using the definitions in eqs.(20,24), we may now decompose the gauge field Lagrangian in eq.(7) in terms of the physical gauge fields W_μ, Z_μ, A_μ. It is convenient to define

$$F^A{}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \quad F^Z{}_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \ , \quad F^W{}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu \ , \tag{25}$$

and we may write

$$G_{\mu\nu} = \cos\theta_W \, F^A{}_{\mu\nu} - \sin\theta_W \, F^Z{}_{\mu\nu} \tag{26}$$

and for the components of $\mathbf{F}_{\mu\nu}$

$$\begin{aligned} F_{\mu\nu 3} &= \sin\theta_W \, F^A{}_{\mu\nu} + \cos\theta_W \, F^Z{}_{\mu\nu} + ig(W_\mu W_\nu{}^\dagger - W_\nu W_\mu{}^\dagger) \ , \\ \tfrac{1}{\sqrt{2}}(F_{\mu\nu 1} - iF_{\mu\nu 2}) &= d_\mu W_\nu - d_\nu W_\mu \ , \\ & d_\mu = \partial_\mu + ieA_\mu + ig\cos\theta_W \, Z_\mu \ , \end{aligned} \tag{27}$$

where we define

$$e = g\sin\theta_W \ . \tag{28}$$

to be the elementary electric charge. We may now rewrite eq.(7) in the form

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= -\tfrac{1}{2}F^{W\mu\nu\dagger}F^W{}_{\mu\nu} - \tfrac{1}{4}F^{A\mu\nu}F^A{}_{\mu\nu} - \tfrac{1}{4}F^{Z\mu\nu}F^Z{}_{\mu\nu} \\ &\quad + iW^\mu W^{\nu\dagger}\left(e\,F^A{}_{\mu\nu} + g\cos\theta_W F^Z{}_{\mu\nu}\right) \\ &\quad + \tfrac{1}{2}g^2\left(W^2 W^{\dagger 2} - (W.W^\dagger)^2\right) \ . \end{aligned} \tag{29}$$

Since the relations given in eqs.(20,24) between the gauge fields $A_{\mu 3}, B_\mu$ and the physical fields A_μ, Z_μ, which are the natural basis for the mass terms so that they take the form in eq.(22), is just an orthogonal rotation

$$\begin{pmatrix} A_{3\mu} \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \ , \tag{30}$$

the quadratic terms in $\mathcal{L}_{\text{gauge}}$ remain diagonal. Clearly the piece $-\frac{1}{4}F^{A\mu\nu}F^{A}{}_{\mu\nu}$ in $\mathcal{L}_{\text{gauge}}$ represents the usual Lagrangian for the electromagnetic field. There is no coupling between A_μ and Z_μ reflecting that the massive Z particle is neutral, with electric charge zero, like the photon. The complex vector field W_μ is coupled to the electromagnetic gauge field with a coupling e, defined in eq.(28), so that the corresponding spin-1 particles in the quantized theory have charge $\pm e$.

The gauge field and Higgs sector of the Electroweak theory contains 4 fundamental parameters that must be taken from experiment. These can be identified with the Lagrangian parameters (g, g', v, λ) or the physical parameters $(m_Z,\ \sin\theta_W,\ m_H,\ e)$, or some other combination. Many more parameters will be added once leptons and quarks are included in the theory. While we will see that the electroweak theory can accommodate and organize many phenomena, one feels that it is likely to be part of a more fundamental theory, with fewer parameters.

1.3 Massive Vector Bosons

We now discuss some of the general features of massive vector fields. Neglecting its interactions with the other fields, the field Z_μ has a Lagrangian

$$\mathcal{L}_Z = -\tfrac{1}{4}F^{Z\mu\nu}F^{Z}{}_{\mu\nu} + \tfrac{1}{2}m_Z^2 Z^\mu Z_\mu \ . \tag{31}$$

The Lagrangian in eq.(31) gives rise to the equation of motion

$$\partial^\mu F^{Z}{}_{\mu\nu} + m_Z^2 Z_\nu = \partial^2 Z_\nu - \partial_\nu \partial.Z + m_Z^2 Z_\nu = 0 \ . \tag{32}$$

Taking the divergence we find at once that

$$m_Z^2\, \partial.Z = 0 \ . \tag{33}$$

In turn this implies that

$$(\partial^2 + m_Z^2) Z_\nu = 0 \ . \tag{34}$$

In comparing this with the case of massless vector fields, the divergenceless condition eq. (33) would not automatically hold. Unlike the massive case, where the gauge invariance has already been used to fix to unitary gauge, the case of massless vector fields would still have gauge invariance available. This could be used to impose the divergenceless condition (Lorentz gauge), or some other gauge-fixing condition instead.

As already discussed, the propagator for the massive vector boson is

$$\tilde{D}_{\mu\nu}(p) = \frac{i}{p^2 - m_Z^2 + i\epsilon}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\right) \ . \tag{35}$$

Also, when expressed in terms of annihilation and creation operators for states of definite momentum the (real) vector field Z_μ becomes

$$Z_\mu(x) = \sum_{p,\lambda}\left(a(p,\lambda)\epsilon_\mu(p,\lambda)e^{-ip.x} + a(p,\lambda)^\dagger \epsilon_\mu(p,\lambda)^* e^{ip.x}\right) \ , \tag{36}$$

where the summation is over 4-momenta that satisfy the mass-shell condition eq. (34)

$$p^2 = m_Z^2 \ , \tag{37}$$

and the λ summation is over the three linearly independent polarization vectors which satisfy eq. (33)

$$p.\epsilon(p,\lambda) = 0 \quad \text{and} \quad \epsilon(p,\lambda)^*.\epsilon(p,\lambda') = \delta_{\lambda\lambda'} \ . \tag{38}$$

(We have chosen the vectors orthonormal by convention). If we specialize to the rest frame of the massive vector particle state, then $p = (m_Z, 0, 0, 0)$ and $\epsilon(p,\lambda)$ has the form $(0, \epsilon(\lambda))$ where $\{\epsilon(\lambda)\}$ are three orthonormal 3-vectors. By writing it in terms of all available second rank tensors, using dimensional analysis, and contracting both sides with the linearly independent set $\{p, \epsilon(p,\lambda)\}$, we can verify the useful identity

$$\sum_\lambda \epsilon_\mu(p,\lambda)\epsilon_\nu(p,\lambda)^* = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \ . \tag{39}$$

After quantization, the annihilation and creation operators satisfy the commutation relations

$$[a(p,\lambda), a(p',\lambda')^\dagger] = 2E(\mathbf{p})\delta_{pp'}\delta_{\lambda\lambda'} \ . \tag{40}$$

A Z particle state of polarization λ and momentum p is then $a(p,\lambda)^\dagger|0\rangle$.

The charged vector boson field $W_\mu(x)$ can be treated in an almost identical way. The propagator is exactly the same, with the mass m_Z replaced by the mass m_W. Because it is a complex field, the mode expansion becomes

$$W_\mu(x) = \sum_{p,\lambda} \left(a(p,\lambda)\epsilon_\mu(p,\lambda)e^{-ip.x} + b(p,\lambda)^\dagger \epsilon_\mu(p,\lambda)^* e^{ip.x} \right) \ , \tag{41}$$

with $a^\dagger$ and $b^\dagger$ creating particles W^+ and antiparticles W^- respectively.

Note that for low energy processes, such as occur in weak decays, the mass of the vector boson m_Z or m_W is very large relative to the momentum components $\{p_\mu\}$ involved. In this case it is appropriate to make the approximation in which the momentum is neglected. In the neutral Z boson case for example

$$\tilde{D}_{\mu\nu}(p) \sim \frac{g_{\mu\nu}}{m_Z^2}. \tag{42}$$

The small factor $1/m_Z^2$ will suppress the gauge boson propagation in internal lines and explains why the weak interactions are 'weak'.

2 Lepton Representations

For simplicity, we initially restrict the leptons solely to the electron and its associated neutrino which may be assumed, for the remainder of this section, to be massless and purely left-handed. The weak interactions violate parity, with the left handed and right handed fields belonging to different representations of the gauge group. For the electromagnetic and weak interactions to be treated on a combined basis there has to be a close connection between the neutrino, which interacts only through weak processes, and the electron, which in addition is charged under $U(1)_Q$. The standard method for achieving such a relationship in quantum field theory is to combine the fields for the related particles into a multiplet that forms a non-trivial representation of the appropriate symmetry group. If $\psi(x)$ is a field multiplet with weak hyper-charge Y, then it transforms under a local $SU(2) \times U(1)_Y$ gauge transformation according to

$$\psi(x) \to e^{i\alpha(x).\mathbf{T}+i\beta(x)Y}\psi(x) \ , \tag{43}$$

where $\mathbf{T}$ belongs to the appropriate representation of $SU(2)$.

In the present case the neutrino and the left-handed electron are put into an $SU(2)$ doublet

$$L(x) = \begin{pmatrix} \nu_e(x) \\ e_L(x) \end{pmatrix} \ , \tag{44}$$

which forms a two dimensional $T = \frac{1}{2}$ representation of $SU(2)$ weak iso-spin, where $\mathbf{T} \to \frac{1}{2}\sigma$. The members of a symmetry multiplet must have the same numbers of degrees of freedom, so only the *left* chiral component of the electron field is linked to the naturally left-handed neutrino field. The *right* chiral component of the electron field is taken as a *weak iso-singlet*, in the trivial $T = 0$ singlet representation representation of $SU(2)$, which can be written as

$$R(x) = e_R(x) \ , \tag{45}$$

with $\mathbf{T} \to \mathbf{0}$. Under $SU(2)$ gauge transformations

$$L(x) \to e^{\frac{1}{2}i\alpha.\sigma}L(x) \ , \quad \bar{L}(x) \to \bar{L}(x)e^{-\frac{1}{2}i\alpha.\sigma} \ , \tag{46}$$

and $R(x)$ is invariant

$$R(x) \to R(x) \ , \qquad \bar{R}(x) \to \bar{R}(x) \ . \tag{47}$$

Acting on the left-handed doublet and right-handed singlet fields defined in eq.(44) and eq.(45), since the neutrino has charge $Q = 0$ and the electron charge $Q = -1$, it is easy to see that

$$QL(x) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} L(x) \ , \qquad QR(x) = -R(x) \ . \tag{48}$$

Acting on L clearly

$$Q = \tfrac{1}{2}\sigma_3 - \tfrac{1}{2}1 \ . \tag{49}$$

Hence from eq. (1) we must have $Y = -\frac{1}{2}$ for L and $Y = -1$ for R.

The gauge covariant derivative for $\psi(x)$ depends on gauge fields $\mathbf{A}_\mu, B_\mu$ and has the form

$$D_\mu\psi(x) = (\partial_\mu + ig\mathbf{A}_\mu(x).\mathbf{T} + ig' B_\mu(x)Y)\,\psi(x)\ , \tag{50}$$

where the vector gauge fields transform as in eqs. (3)(4). With these transformation properties for the vector fields, the covariant derivatives transform in the same way as the multiplet itself in eq.(43). Thus

$$D_\mu\psi(x) \to e^{+i\alpha(x).\mathbf{T}+i\beta(x)Y} D_\mu\psi(x)\ , \tag{51}$$

Using the hyper-charge assignments for the various multiplets we see that the covariant derivatives for the lepton fields are then

$$\begin{aligned} D_\mu L(x) &= \left(\partial_\mu + ig\tfrac{1}{2}\mathbf{A}_\mu(x).\sigma - ig'\tfrac{1}{2}B_\mu(x)\right) L(x)\ , \\ D_\mu R(x) &= \left(\partial_\mu - ig' B_\mu(x)\right) R(x)\ . \end{aligned} \tag{52}$$

The kinetic term for the lepton fields can then be written in gauge invariant form as

$$\mathcal{L}_{\text{lept.}}(x) = \bar{L}(x) i\gamma^\mu D_\mu L(x) + \bar{R}(x) i\gamma^\mu D_\mu R(x)\ , \tag{53}$$

with covariant derivatives defined as in eq.(52).

2.1 Lepton Mass

Since the left handed and right handed lepton fields transform differently under both $SU(2)$ and $U(1)_Y$, there is no possibility of directly adding any mass term for the electron to eq.(53) which is compatible with invariance under the gauge group (the electron has mass $m_e \approx 0.5$ MeV.) Mass terms for leptons may be generated via spontaneous symmetry breaking and interaction with the Higgs field as follows. If the $T = \frac{1}{2}$ doublet ϕ has weak hypercharge $Y = \frac{1}{2}$, then there is a unique renormalizable and gauge invariant (Yukawa type) coupling

$$\mathcal{L}_{\text{lept},\phi}(x) = -\sqrt{2}\lambda_e \left[\bar{L}(x)\phi(x)R(x) + \bar{R}(x)\phi(x)^\dagger L(x)\right]\ . \tag{54}$$

With the choice of unitary gauge, when the Higgs field takes the form in eq.(16), this becomes

$$\mathcal{L}_{\text{lept},\phi} = -\lambda_e(v+H)\left[\overline{e}_L e_R + \overline{e}_R e_L\right] = -\lambda_e(v+H)\,\overline{e}e\ . \tag{55}$$

The terms quadratic in fields are identified with mass terms, so that

$$m_e = \lambda_e v\ . \tag{56}$$

The mass of the electron is thus determined by the coupling of the Higgs field to the lepton fields and by the vacuum expectation value v of the Higgs field, which sets the basic mass scale of the theory. It is important to recognize that the full Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{lept}} + \mathcal{L}_{\text{lept},\phi} + \mathcal{L}_{\text{Higgs}}\ , \tag{57}$$

as given by eqs.(7,53,54,10,12), contains all terms allowed by renormalizability and $SU(2)\times U(1)_Y$ gauge invariance of the Lagrangian for the fields e, ν_e, ϕ, with the assumed representations of $SU(2)$ and weak hypercharge Y.

2.2 Lepton Electro-Weak Interactions

All these interactions arise from the gauge invariant kinetic part of the leptonic Lagrangian, as given in eq.(53),

$$\begin{aligned}\mathcal{L}_{\text{lept.}} &= \bar{L}(x)i\gamma^\mu\partial_\mu L(x) + \bar{R}(x)i\gamma^\mu\partial_\mu R(x) \; - g\bar{L}\gamma^\mu\tfrac{1}{2}\sigma L.\mathbf{A}_\mu + g'(\tfrac{1}{2}\bar{L}\gamma^\mu L + \bar{R}\gamma^\mu R)\,B_\mu \\ &= \bar{L}(x)i\gamma^\mu\partial_\mu L(x) + \bar{R}(x)i\gamma^\mu\partial_\mu R(x) \; - \frac{g}{2\sqrt{2}}(J^\mu W_\mu + J^{\mu\dagger}{W_\mu}^\dagger) \\ & -ej^\mu_{\text{e.m.}}A_\mu - \frac{g}{2\cos\theta_W}J^\mu_n Z_\mu \,. \end{aligned} \tag{58}$$

The couplings of the charged vector bosons W arise from the terms involving $A_{1\mu}$ and $A_{2\mu}$. Using the definition of W_μ in eq.(20), it is easy to see that

$$J^\mu = 2\bar{L}\gamma^\mu\sigma_+ L = \overline{\nu}_e\gamma^\mu(1-\gamma_5)e \;, \tag{59}$$

using the definition of the lepton doublet L in eq.(44) and

$$\sigma_+ = \tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;. \tag{60}$$

J^μ is called the charged (lepton) weak current, and is involved in the traditional processes of weak interaction where there is a change in charge, such as β-decay.

Using the results in eq.(30) for $A_{\mu 3}, B_\mu$ in terms of the electromagnetic gauge field A_μ and the massive neutral vector field Z_μ, and also the definition of e in eq.(28), we may find expressions for the electromagnetic current $j^\mu_{\text{e.m.}}$ and also the weak neutral current J^μ_n. For the former it is easy to obtain

$$j^\mu_{\text{e.m.}} = \bar{L}\gamma^\mu\tfrac{1}{2}(\sigma_3 - 1)L - \bar{R}\gamma^\mu R = -\overline{e}\gamma^\mu e \;, \tag{61}$$

which is of course the required form for the contribution to the electromagnetic current arising from the electron Dirac field. Unlike the charged weak current it is purely vector-like.

For the neutral current we can similarly read off the required contributions from the electron, neutrino fields, using eq.(19) to eliminate g',

$$\begin{aligned} J^\mu_n &= \bar{L}\gamma^\mu(\cos^2\theta_W\sigma_3 + \sin^2\theta_W 1)L + 2\sin^2\theta_W\bar{R}\gamma^\mu R \\ &= \tfrac{1}{2}\Big[\overline{\nu}_e\gamma^\mu(1-\gamma_5)\nu_e - \overline{e}\gamma^\mu(1-\gamma_5-4\sin^2\theta_W)e\Big] \;. \end{aligned} \tag{62}$$

The discovery in the 1970's of interactions involving the weak neutral current, which had not been observed when the electroweak theory was first proposed, was an important confirmation of this gauge theory. The neutral current allows the Z to decay into $\overline{\nu}_e\nu_e$ or e^+e^-, and the charged current allows the W to decay into $\overline{\nu}_e e$. In the 1980's, long after the basic structure of electroweak theory was well established by relatively low energy experiments, the heavy W and Z particles were detected experimentally by their decays. Today they can be produced copiously in accelerator experiments and are routinely used to probe for physics beyond the Standard Model.

2.3 Family Replication

There are two more known sets of leptons $(\mu, \nu_\mu), (\tau, \nu_\tau)$ which form identical representations under $SU(2) \times U(1)_Y$ as (e, ν_e), but with different masses for μ and τ ($m_\mu = 105.7$ MeV, $m_\tau = 1.777$ GeV). Again, we will define the Standard Model to have massless neutrinos ν_μ, ν_τ. The gauge interactions are universal with respect these three 'families' or 'generations' in that if

$$L_1 = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \ , \ L_2 = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \ , \ L_3 = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \ , \tag{63}$$

and

$$R_1 = e_R \ , \ R_2 = \mu_R \ , \ R_3 = \tau_R \ , \tag{64}$$

then eqn. (53) becomes

$$\mathcal{L}_{\text{lept.}}(x) = \bar{L}_N i\gamma^\mu D_\mu L_N + \bar{R}_N i\gamma^\mu D_\mu R_N \ , \tag{65}$$

where D_μ is given by (52) and $N \in \{1, 2, 3\}$ is a family index.

The interaction with Higgs scalars ϕ generalizes to

$$\mathcal{L}_{\text{lept},\phi} = -\sqrt{2}[\bar{L}_N f_{NM} \phi R_M + \bar{R}_M f^*_{NM} \phi^\dagger L_N \ ] \ , \tag{66}$$

where f_{NM} are components of a general 3×3 matrix in family space. The quadratic terms in fields, which may be picked out by replacing $\phi \to \phi_0$, define a mass matrix for leptons. The matrix f can be diagonalized, with real positive eigenvalues for the masses, by a bi-unitary transformation $f \to V^\dagger f U$, with $U, V \in U(3)$. In this diagonal basis we then have $R \to U^\dagger R$ and $L \to V^\dagger L$. However, this transformation is an invariance of eqn. (65), so we can, without loss of generality, identify the fields appearing in L and R representations of the electroweak gauge group as the mass eigenstates.

It is one of the great mysteries of particle physics why there are replicated families and why they have their particular pattern of masses. In the Standard Model, the lepton masses (i.e. the Yukawa couplings to the Higgs field) are parameters that must be taken from experiment.

3 Quark Representations

The electro-weak coupling of gauge fields to hadrons is similar to that for leptons when it is assumed that it is sufficient to use a Lagrangian involving the fundamental quark fields, from which hadrons are formed. There are three known families of quark fields $(u', d'), (c', s'), (t', b')$ forming the same $SU(2)$ representations as the leptons except that all fields have right-handed parts in singlet representations. The prime on the fields is to distinguish these representations of the gauge symmetry from the flavour fields u, d, s, c, t, b which eventually will diagonalize the mass matrix and occur as physical particles. Unlike the case for leptons (with no right-handed neutrinos), with quarks one must make this distinction. Thus, left-handed quarks occur in the $T = \frac{1}{2}$ $SU(2)$ representations

$$L_1 = \left(\begin{array}{c} u' \\ d' \end{array} \right)_L \quad , \; L_2 = \left(\begin{array}{c} c' \\ s' \end{array} \right)_L \quad , \; L_3 = \left(\begin{array}{c} t' \\ b' \end{array} \right)_L \quad , \tag{67}$$

where ${u'}_L = \frac{1}{2}(1 - \gamma_5)u'$ etc.. Similarly, right-handed quarks occur in the $T = 0$ $SU(2)$ representations

$$R_{1+} = {u'}_R \quad , \quad R_{2+} = {c'}_R \quad , \quad R_{3+} = {t'}_R \tag{68}$$

$$R_{1-} = {d'}_R \quad , \quad R_{2-} = {s'}_R \quad , \quad R_{3-} = {b'}_R \tag{69}$$

To obtain the correct observed quark electric charges, the hypercharge assignments are

$$L_N \quad : \quad Y = \frac{1}{6} \to Q_{u'} = \frac{2}{3} \, , Q_{d'} = -\frac{1}{3}$$

$$R_{N+} \quad : \quad Y = \frac{2}{3} = Q_{u'} \tag{70}$$

$$R_{N-} \quad : \quad Y = -\frac{1}{3} = Q_{d'} \tag{71}$$

The gauge invariant quark kinetic Lagrangian density is then

$$\mathcal{L}_{\text{quark}}(x) = \bar{L}_N i\gamma^\mu D_\mu L_N + \bar{R}_{N+} i\gamma^\mu D_\mu R_{N+} + \bar{R}_{N-} i\gamma^\mu D_\mu R_{N-} \tag{72}$$

with D_μ given by (51) in the appropriate representation and $N \in \{1, 2, 3\}$.

3.1 Quark Mass

When we assume the most general form for the coupling of the $T = \frac{1}{2}, Y = \frac{1}{2}$ Higgs field ϕ to the quark fields, it is important to recognize that from ϕ it is possible to form a conjugate field ϕ^c, which transforms under $SU(2) \times U(1)_Y$ as a weak iso-doublet with a weak hyper-charge $Y = -\frac{1}{2}$. This is defined by

$$\phi^c = i\sigma_2 \phi^* = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left(\begin{array}{c} {\phi_1}^\dagger \\ {\phi_2}^\dagger \end{array} \right) = \left(\begin{array}{c} {\phi_2}^\dagger \\ -{\phi_1}^\dagger \end{array} \right) . \tag{73}$$

The hyper-charge Y of ϕ^c is obvious and the $SU(2)$ transformation properties can be seen by using

$$i\sigma_2 \, \sigma^* \, i\sigma_2 = \sigma, \qquad i\sigma_2 \, \sigma^* = -\sigma \, i\sigma_2 \; , \tag{74}$$

with the result that

$$i\sigma_2 \left[e^{\frac{i}{2}\alpha.\sigma} \right]^* = e^{\frac{i}{2}\alpha.\sigma} i\sigma_2 \; , \tag{75}$$

i.e. ϕ^c transforms in the same way under $SU(2)$ as ϕ. Note that the vacuum expectation value of ϕ^c, taking v to be real, from eq.(13) is,

$$\phi_0{}^c = \frac{1}{\sqrt{2}} \left(\begin{array}{c} v \\ 0 \end{array} \right) . \tag{76}$$

The general gauge invariant expression for the coupling of the Higgs field to the quark fields is then of the form

$$\mathcal{L}_{\text{quark},\phi} = -\sqrt{2}[\bar{L}_N f^-_{NM} \phi R_{M-} + \bar{L}_N f^+_{NM} \phi^c R_{M+} + \text{h.c.}] \;, \tag{77}$$

where $f^{\pm}$ are 3×3 complex matrices acting in family space. When the Higgs field is replaced by its vacuum expectation values, then this becomes a mass (matrix) term

$$\mathcal{L}_{\text{quark},m} = -\begin{pmatrix} \bar{u}'_L & \bar{c}'_L & \bar{t}'_L \end{pmatrix} m^+ \begin{pmatrix} u'_R \\ c'_R \\ t'_R \end{pmatrix} - \begin{pmatrix} \bar{d}'_L & \bar{s}'_L & \bar{b}'_L \end{pmatrix} m^- \begin{pmatrix} d'_R \\ s'_R \\ b'_R \end{pmatrix} + \text{h.c.} \tag{78}$$

where $m^{\pm} = vf^{\pm}$. The mass matrices $m^{\pm}$ can be diagonalized by bi-unitary transformations $V^\dagger_{\pm} m^{\pm} U_{\pm}$ with positive real eigenvalues

$$V^\dagger_+ m^+ U_+ = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \;, \quad V^\dagger_- m^- U_- = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \;. \tag{79}$$

Experimentally one finds $m_u \approx 1 - 5$ MeV, $m_d \approx 3 - 9$ MeV, $m_s \approx 75 - 170$ MeV, $m_c \approx 1.3 - 1.5$ GeV, $m_b \approx 4.6 - 4.9$ GeV, $m_t \approx 173.3 \pm 0.8$ GeV. The large percentage uncertainty in these results reflects to some extent the ambiguity in extracting quark masses from experiment, since quarks are permanently confined in hadron bound states.

The fields corresponding to mass eigenstates are then obtained from the change of basis

$$U^\dagger_+ \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_R \equiv \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R \;, \quad U^\dagger_- \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_R \equiv \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R \;, \tag{80}$$

and

$$V^\dagger_+ \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L \equiv \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L \;, \quad V^\dagger_- \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L \equiv \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \;. \tag{81}$$

3.2 Quark Electro-Weak Interactions

The quark interactions with gauge fields can be extracted from eqn. (72), and takes a similar form to that for leptons eqn.(65)

$$\mathcal{L}^I_{\text{quark}}(x) = -\frac{g}{2\sqrt{2}}(J^\mu W_\mu + J^{\mu\dagger} W_\mu{}^\dagger) - e j^\mu_{\text{e.m.}} A_\mu - \frac{g}{2\cos\theta_W} J^\mu_n Z_\mu \,. \tag{82}$$

The quark contribution to the charged weak current is then

$$J^\mu \;=\; 2\bar{L}_N \gamma^\mu \sigma_+ L_N$$

$$
\begin{aligned}
&= 2\left(\begin{array}{ccc}\overline{u}' & \overline{c}' & \overline{t}'\end{array}\right)_L \gamma^\mu \left(\begin{array}{c} d' \\ s' \\ b' \end{array}\right)_R \\
&= \left(\begin{array}{ccc}\overline{u} & \overline{c} & \overline{t}\end{array}\right) \gamma^\mu (1-\gamma_5) V \left(\begin{array}{c} d \\ s \\ b \end{array}\right) , \qquad (83)
\end{aligned}
$$

where $V = V_+^\dagger V_-$ is called the Cabbibo-Kobayshi-Maskawa (CKM) *mixing matrix*. Because of V, interaction vertices involving the charged weak current can change the flavour of quarks.

A mixing matrix does not occur for leptons in the Standard Model (with massless neutrinos). If we redefine the charged leptons in the charged weak interaction part of the Lagrangian so that they are expressed in terms of mass eigenstates, as we have done above for the quarks, we introduce a matrix equivalent to V_- in (83). However, we do not get a matrix equivalent to $V_+^\dagger$ because we did not need to define neutrino mass eigenstates. The matrix we obtain by choosing to use the charged lepton mass eigenstates, V_- may simply be absorbed into the definition of the neutrinos, defining them as the neutrinos that interact via the charged current with the charged lepton mass eigenstates. Since the neutrinos are all degenerate with zero mass, these weak eigenstates are automatically also mass eigenstates. If right-handed neutrinos were to exist and neutrinos had masses, then one would also expect an independent mixing matrix in the lepton sector.

The quark electromagnetic current is

$$
\begin{aligned}
j^\mu_{\text{e.m.}} &= \bar{L}_N \gamma^\mu \left(\tfrac{1}{2}\sigma_3 + \frac{1}{6}\right) L_N + \frac{2}{3}\bar{R}_{N+}\gamma^\mu R_{N+} - \frac{1}{3}\bar{R}_{N-}\gamma^\mu R_{N-} \\
&= -\frac{1}{3}\left(\begin{array}{ccc}\overline{d}' & \overline{s}' & \overline{b}'\end{array}\right)\gamma^\mu \left(\begin{array}{c} d' \\ s' \\ b' \end{array}\right) + \frac{2}{3}\left(\begin{array}{ccc}\overline{u}' & \overline{c}' & \overline{t}'\end{array}\right)\gamma^\mu \left(\begin{array}{c} u' \\ c' \\ t' \end{array}\right) \\
&= -\frac{1}{3}\left(\begin{array}{ccc}\overline{d} & \overline{s} & \overline{b}\end{array}\right)\gamma^\mu \left(\begin{array}{c} d \\ s \\ b \end{array}\right) + \frac{2}{3}\left(\begin{array}{ccc}\overline{u} & \overline{c} & \overline{t}\end{array}\right)\gamma^\mu \left(\begin{array}{c} u \\ c \\ t \end{array}\right) . \qquad (84)
\end{aligned}
$$

Lastly, the weak neutral current is

$$
\begin{aligned}
J^\mu_n &= \bar{L}_N \gamma^\mu \left(\cos^2\theta_W \sigma_3 - \frac{1}{3}\sin^2\theta_W\right) L_N + \bar{R}_{N+}\gamma^\mu \left(-\frac{4}{3}\sin^2\theta_W\right) R_{N+} \\
&\quad + \bar{R}_{N-}\gamma^\mu \left(\frac{2}{3}\sin^2\theta_W\right) R_{N-} \\
&= \frac{1}{2}\left(\begin{array}{ccc}\overline{u} & \overline{c} & \overline{t}\end{array}\right)\gamma^\mu \left(1-\gamma_5 - \frac{8}{3}\sin^2\theta_W\right)\left(\begin{array}{c} u \\ c \\ t \end{array}\right) \\
&\quad - \frac{1}{2}\left(\begin{array}{ccc}\overline{d} & \overline{s} & \overline{b}\end{array}\right)\gamma^\mu \left(1-\gamma_5 - \frac{4}{3}\sin^2\theta_W\right)\left(\begin{array}{c} d \\ s \\ b \end{array}\right) . \qquad (85)
\end{aligned}
$$

One notes that there is no flavour mixing matrix in the electromagnetic and weak neutral currents. In the latter case, this absence of flavour changing neutral currents is called the Glashow-Illiopoulos-Maiani (GIM) mechanism, and is an important prediction of the electroweak theory that is borne out by experiment.

Once again, the Electroweak gauge theory can accommodate but not predict the quark masses and any further parameters in the CKM mixing matrix V.

3.3 CKM Matrix

The above formalism could be applied to a theory with any number of quark families. For N generations the unitary matrix V is $N \times N$ so would contain at most N^2 parameters. The $2N$ quark fields contain $2N$ unobservable complex phases, but the current J^μ is invariant under a common phase transformation on all the quark fields $V_\pm = e^{i\alpha}$. Thus $2N - 1$ complex phases in the matrix V are physically unobservable, leaving $N^2 - 2N + 1$ parameters in general.

Hence, if we only had one family there would be no free parameters. For two families V would have 1 parameter, which is interpreted as an angle of rotation. In the real case of three families, the CKM matrix V has 4 relevant parameters. A real orthogonal 3×3 matrix is determined by 3 angles, so in general V must contain a complex phase. There are many ways of choosing the 4 parameters. A version of the CKM matrix is

$$V \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_1 c_3 & s_1 c_3 & s_3 e^{-i\delta} \\ -s_1 c_2 - c_1 s_2 s_3 e^{i\delta} & c_1 c_2 - s_1 s_2 s_3 e^{i\delta} & s_2 c_3 \\ s_1 s_2 - c_1 c_2 s_3 e^{i\delta} & -c_1 s_2 - s_1 c_2 s_3 e^{i\delta} & c_2 c_3 \end{pmatrix}, \tag{86}$$

using the convention that $c_i = \cos\theta_i$ and $s_i = \sin\theta_i \quad i = 1, 2, 3$. The 4 parameters are then the three angles $\theta_1,\ \theta_2,\ \theta_3$ and one complex phase δ. To verify the unitarity of the CKM matrix it is easiest to note that it can be written as a product of three obviously unitary matrices each of which describes the mixings between two families.

$$V \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & 0 & s_3 e^{-i\delta} \\ 0 & 1 & 0 \\ -s_3 e^{i\delta} & 0 & c_3 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{87}$$

The presence of the phase $\delta \neq 0$ in the matrix shows that in general it cannot be reduced to purely real form for three families of quarks. We will see that this lack of reality corresponds to a breakdown of CP invariance or equivalently of T invariance in the Lagrangian (82). In this picture, clearly at least three families are necessary in order to have CP violation.

Independent of any specific parameterization it is a major experimental challenge to determine the elements of the matrix V, up to the phase arbitrariness $V_{rs} \sim e^{i(\theta_r - \phi_s)} V_{rs}$. The magnitudes are $|V_{ud}| \approx 0.9743$, $|V_{us}| \approx 0.225$, $|V_{cs}| \approx 0.9734$, $|V_{tb}| \approx 0.999$, $|V_{cb}| \approx 0.040$, $|V_{ub}| \approx 0.009$. Also $\delta \approx 1.2$ rads.

One notes that the elements reduce in magnitude as one moves away from the diagonal. From the unitarity of V, $V^\dagger V = I$, there is a condition on the elements in the first and third columns of V of the form

$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0 \ . \tag{88}$$

The three complex numbers in eq.(88) form a closed triangle. Non-closure of this 'unitarity triangle' would indicate physics beyond the Standard Model.

For many low energy applications it is sufficient to ignore the heavy t and b quarks, and consider a 2-family approximation. Then the unitary matrix V can be restricted to a real orthogonal 2×2 matrix depending solely on an angle θ_C

$$V = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \ . \tag{89}$$

$\theta_C \approx 13^o$ is called the Cabibbo angle. The form of this two family matrix inspires a common parameterization of the CKM matrix, known as the Wolfenstein parameterization.

$$V \equiv \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{90}$$

where $A \approx 0.81$, $\lambda \approx \sin\theta_C = 0.226$, $\rho \approx 0.13$, $\eta \approx 0.35$ and each term is accurate up to a relative correction of λ^2. This parameterization illustrates the hierarchy in the mixing, i.e. $\theta_1 \gg \theta_2 \gg \theta_3$. Taking our unitarity triangle relationship (88), and dividing by $-A\lambda^3 = V_{cb}^* V_{cd}$ we obtain

$$-(\rho + i\eta) + 1 - (1 - \rho - i\eta) = 0 = \frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}} + 1 + \frac{V_{tb}^* V_{td}}{V_{cb}^* V_{cd}}. \tag{91}$$

This is often illustrated as overleaf. The test of whether this triangle does indeed close, i.e. measurements of the relevant CKM matrix elements, is currently a very active topic at colliders involving mesons with bottom quarks. There is no evidence to the contrary yet, but the accuracy is at the few % level.

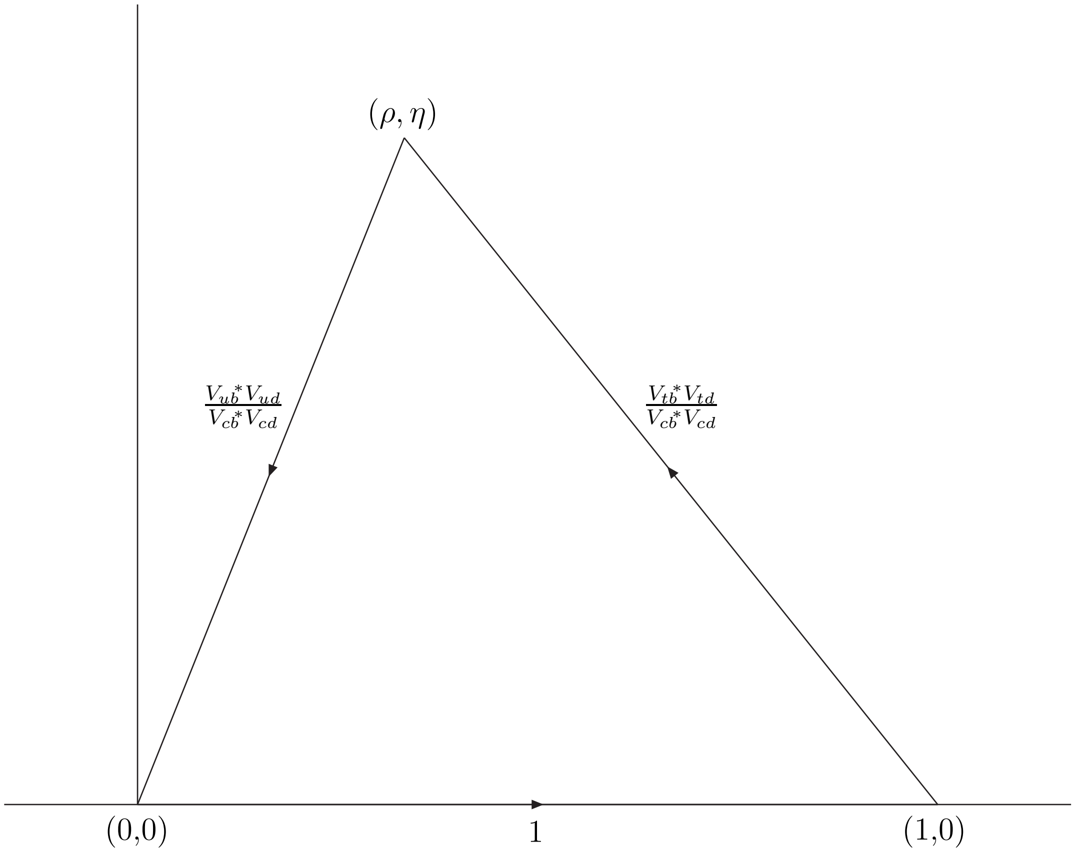

Figure 1: Diagrammatic representation of unitarity triangle.