The Standard Model
Weak Interactions

R.S. Thorne

Preface

Despite its many parameters, the electroweak gauge theory leads to predictions of unprecedented scope and accuracy in particle physics. The parameters may be measured to very high accuracy in a few processes (quark masses have a reasonable uncertainty due to the confinement of quarks in hadrons), and then the relative weakness of the couplings allows very accurate predictions of other processes to be made with only low order calculations.

By considering processes in the electroweak gauge theory in the limit of low momentum, one can obtain an effective theory where dynamics of the heavy gauge fields are neglected. The weak interaction Lagrangian density is

$$\mathcal{L}_W = -\frac{g}{2\sqrt{2}}\left(J^\mu W_\mu + J^{\mu\dagger} W^\dagger_\mu\right) - \frac{g}{2\cos\theta_W} J^\mu_n Z_\mu \qquad (1)$$

where the currents J and J_n are defined in terms of elementary quarks and leptons in section C. If we expand in powers of W and Z fields and consider matrix elements $\langle i|S|f\rangle$ where no W or Z particles occur in the initial or final states, then we need only keep even powers of the currents in the expansion

$$\begin{aligned}\langle i|S|f\rangle \;=\; \langle i|\mathcal{T}\Big\{1 - \frac{1}{8}g^2 \int \mathrm{d}^4x\mathrm{d}^4x' \Big(J^\mu(x)^\dagger D^W_{\mu\nu}(x-x')J^\nu(x') \\ + \frac{1}{\cos^2\theta_W} J^\mu_n(x) D^Z_{\mu\nu}(x-x') J^\nu_n(x')\Big) + \ldots\Big\}|f\rangle, \qquad (2)\end{aligned}$$

where

$$\begin{aligned}\langle 0|\mathcal{T}\{W_\mu(x)^\dagger W_\nu(x')\}|0\rangle &= D^W_{\mu\nu}(x-x') \\ \langle 0|\mathcal{T}\{Z_\mu(x)^\dagger Z_\nu(x')\}|0\rangle &= D^Z_{\mu\nu}(x-x') \qquad (3)\end{aligned}$$

are the vector field propagators. If in any process the components of the momenta for virtual W's and Z's satisfy $|p^\mu| \ll m_W, m_Z$, then we can make the approximation

$$D^W_{\mu\nu}(x) \sim \frac{i}{m_W^2}\, g_{\mu\nu}\, \delta^4(x)\ ,\ D^Z_{\mu\nu}(x) \sim \frac{i}{m_Z^2}\, g_{\mu\nu}\, \delta^4(x)\ . \tag{4}$$

To leading order in m_Z^{-1} and m_W^{-1} we may then use an effective Lagrangian density

$$\mathcal{L}_{\mathrm{Weff}} = -\frac{G_F}{\sqrt{2}}\left(J^\mu(x)^\dagger J_\mu(x) + \rho J^\mu_n(x)^\dagger J_{n\mu}(x)\right)\ ,\ \rho = \frac{m_W^2}{\cos^2\theta_W m_Z^2}\ , \tag{5}$$

where $G_F = \sqrt{2}g^2/8m_W^2 \approx 10^{-5}\mathrm{GeV}^{-2}$ is called the Fermi constant and $\rho = 1$ in the Standard Model. The charged current-current interaction in eqn.(5) was proposed in a similar form by Fermi, as long ago as 1932, to describe weak interactions. Thus, we see that weak interactions are strictly only weak when the energies involved are less than $G_F^{-1/2}$. In principle the dimensionless gauge coupling g is $O(1)$. In fact, even for QED, e is not a particularly small number; perturbation theory in e works so well principally because the actual expansion parameter is the fine structure constant $\alpha \equiv e^2/4\pi$, and $1/4\pi$ is also a small number. However $e = g\sin\theta_W \approx 0.22g$, and in practice $\alpha_W > \alpha$, and at scales $\gg M_W$ the weak interaction is rather stronger than the electromagnetic interaction. Below we will consider some of the low energy applications of the charged current-current approximation, which for many processes is all we really need.

1 Leptonic Processes

The lepton contribution to the weak charged current is

$$J_\alpha(x) = \overline{\nu}_e(x)\gamma_\alpha(1-\gamma_5)e(x) + \overline{\nu}_\mu(x)\gamma_\alpha(1-\gamma_5)\mu(x) + \overline{\nu}_\tau(x)\gamma_\alpha(1-\gamma_5)\tau(x)\ . \tag{6}$$

For processes involving only leptons in initial and final states we may restrict to just this form. The current $J_\alpha(x)$ can be decomposed into a vector part under parity transformations that is denoted by $V_\alpha(x)$

$$V_\alpha(x) = \overline{\nu}_e(x)\gamma_\alpha e(x) + \dots\ , \tag{7}$$

and an axial vector part $A_\alpha(x)$

$$A_\alpha(x) = \overline{\nu}_e(x)\gamma_\alpha\gamma_5 e(x) + \dots\ , \tag{8}$$

so that

$$J_\alpha(x) = V_\alpha(x) - A_\alpha(x)\ . \tag{9}$$

Hence the effective theory is sometimes referred to as the $V - A$ theory. This violates parity and charge conjugation but is CP invariant. Such an interaction with the leptonic current in eq.(6) gives rise to the decay processes

$$\mu^\pm \to e^\pm + \nu_e(\overline{\nu}_e) + \overline{\nu}_\mu(\nu_\mu)\ , \tag{10}$$

and similar decays for the τ-lepton, as well as neutrino electron scattering.

We now briefly review the formalism for calculating decay rates in order to apply it to the example of μ decay. Suppose $\mathcal{L}_I(x)$ is a Lagrangian density which gives rise to a coupling between a single particle state $|p\rangle$, of mass m, $p^2 = m^2$. We may calculate the decay rate Γ (probability of decay per unit time) to first order in $\mathcal{L}_I$ in terms of the differential decay rate given by

$$\mathrm{d}\Gamma = \frac{1}{2m}\,\mathrm{d}\rho_f\,|\mathcal{M}|^2\ , \tag{11}$$

where

$$\mathcal{M} = \langle f|\mathcal{L}_I|p\rangle\ . \tag{12}$$

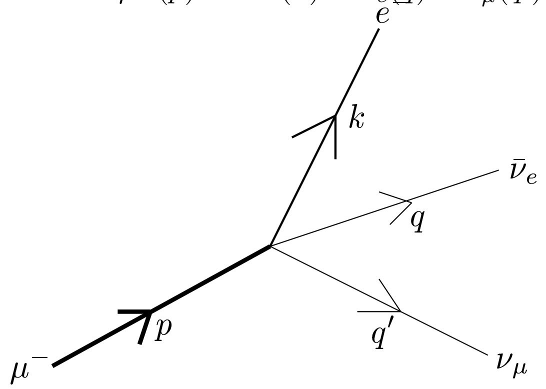
The measure $\mathrm{d}\rho_f$, called the phase space element, is defined, for $|f\rangle$ composed of particles with momenta p_r, by

$$\mathrm{d}\rho_f = \prod_r \left[\frac{\mathrm{d}^3p_r}{(2\pi)^3 2E_{\mathbf{p}_r}}\right] (2\pi)^4\delta^4(P_f - p)\ ,\quad P_f = \sum_r p_r\ . \tag{13}$$

The differential decay rate for a particular decay process is then defined by summing or integrating eq.(11) over all final states. For the total decay rate Γ, all types of decay are summed over. If the decaying particle has spin, but experimentally only decays of unpolarized particles are measured, then the rate should be summed over final state spins and averaged over the initial spin.

Consider now the decay of the μ. Since $m_\mu \sim 100$ MeV in this case, we may use the effective weak Lagrangian $\mathcal{L}_{\mathrm{Weff}}$ in eqn.(5) with the leptonic charged weak current eqn.(6). Choose the momenta of the particles so that

$$\mu^-(p) \to e^-(k) + \overline{\nu}_e(q) + \nu_\mu(q'). \tag{14}$$



Mu decay to electron and neutrinos

The matrix element for the process, suppressing spin labels, is

$$\mathcal{M} = \langle e^-(k)\,\overline{\nu}_e(q)\,\nu_\mu(q')|\mathcal{L}_{\mathrm{Weff}}|\mu^-(p)\rangle\ . \tag{15}$$

In this process the Dirac fields can be regarded as free so that

$$\begin{aligned}\mathcal{M} &= -\frac{G_F}{\sqrt{2}}\langle e^-(k)\,\overline{\nu}_e(q)|\overline{e}\gamma^\alpha(1-\gamma_5)\nu_e|0\rangle\langle\nu_\mu(q')|\overline{\nu}_\mu\gamma_\alpha(1-\gamma_5)\mu|\mu^-(p)\rangle \\ &= -\frac{G_F}{\sqrt{2}}\,\overline{u}_e(k)\gamma^\alpha(1-\gamma_5)v_{\nu_e}(q)\,\overline{u}_{\nu_\mu}(q')\gamma_\alpha(1-\gamma_5)u_\mu(p)\ .\end{aligned} \tag{16}$$

To calculate the transition rate we need to compute the sum over spins of $|\mathcal{M}|^2$, the labels λ for which have been suppressed above. These can be calculated using $\sum_\lambda u(p,\lambda)\overline{u}(p,\lambda) = \gamma.p + m$ and $\sum_\lambda v(p,\lambda)\overline{v}(p,\lambda) = \gamma.p - m$ if $p^2 = m^2$. We find

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{G_F^{\,2}}{2} \sum_{\text{spins}} [\overline{u}_e(k)\gamma^\alpha(1-\gamma_5)v_{\nu_e}(q)\bar{v}_{\nu_e}(q)\gamma^\beta(1-\gamma_5)u_e(k)] \\
&\qquad \times[\overline{u}_{\nu_\mu}(q')\gamma_\alpha(1-\gamma_5)u_\mu(p)\overline{u}_\mu(p)\gamma_\beta(1-\gamma_5)u_{\nu_\mu}(q')] \\
&= \frac{G_F^{\,2}}{2} S_1^{\alpha\beta} S_{2\,\alpha\beta}\ , \qquad (17)
\end{aligned}$$

where, assuming the neutrinos have zero mass,

$$\begin{aligned}
S_1^{\alpha\beta} &= \mathrm{tr}\left\{(\gamma.k + m_e)\gamma^\alpha(1-\gamma_5)\gamma.q\gamma^\beta(1-\gamma_5)\right\}\ , \qquad (18) \\
S_{2\,\alpha\beta} &= \mathrm{tr}\left\{(\gamma.p + m_\mu)\gamma_\beta(1-\gamma_5)\gamma.q'\gamma_\alpha(1-\gamma_5)\right\}\ .
\end{aligned}$$

Note that we have swapped the order of the α and β labels in the second line because we have permuted the muon spinor contribution to the beginning of the trace. Using the standard rules for traces of γ-matrices

$$\begin{aligned}
\mathrm{tr}\left\{\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right\} &= 0 \quad n \text{ odd} \qquad (19) \\
\mathrm{tr}\left\{\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right\} &= 4i\epsilon^{\mu\nu\rho\sigma} \qquad (20) \\
\mathrm{tr}\left\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right\} &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \qquad (21)
\end{aligned}$$

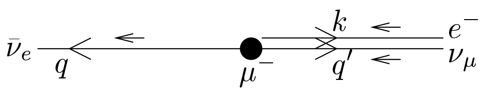
we find

$$\begin{aligned}
S_1^{\alpha\beta} &= 8\left\{k^\alpha q^\beta + k^\beta q^\alpha - k.q\, g^{\alpha\beta} + i\epsilon^{\alpha\beta\sigma\rho}k_\sigma q_\rho\right\}\ , \\
S_{2\,\alpha\beta} &= 8\left\{p_\beta {q'}_\alpha + p_\alpha {q'}_\beta - p.q' g_{\alpha\beta} - i\epsilon_{\alpha\beta\lambda\tau}p^\lambda {q'}^\tau\right\}\ . \qquad (22)
\end{aligned}$$

Contracting, and using the symmetry of the first three elements and antisymmetry of the last (so that contractions between them yield zero) we obtain

$$S_1^{\alpha\beta} S_{2\,\alpha\beta} = 256\, p.q\, k.q'\ . \qquad (23)$$

A consistency check for the result provided by eqs.(17,23) for $|\mathcal{M}|^2$ can be found by considering the case when all the 3-momenta are along the same direction $\hat{\mathbf{z}}$ in the limit $m_e \to 0$. If the initial μ is at rest and assuming the final electron and ν_μ are moving parallel to $\hat{\mathbf{z}}$, then $k \propto q'$, for $m_e = 0$, so that $|\mathcal{M}|^2 = 0$. This is essential for angular momentum conservation since as the e, ν_μ are left handed and the $\overline{\nu}_e$, which moves in direction $-\hat{\mathbf{z}}$, is right handed, the total spin along $\hat{\mathbf{z}}$ is $-\frac{3}{2}$, which is incompatible with the initial μ having spin $\frac{1}{2}$.



Collinear decay of a mu forbidden by conservation of angular momentum

According to eq.(11) the decay rate Γ of the muon is given by

$$\Gamma = \frac{1}{2m_\mu}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\frac{\mathrm{d}^3q}{(2\pi)^3 2E_{\mathbf{q}}}\frac{\mathrm{d}^3q'}{(2\pi)^3 2E_{\mathbf{q}'}}(2\pi)^4\delta^4(p-k-q-q')\,\tfrac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\ , \tag{24}$$

where we average over the initial muon spin states. We have then

$$\Gamma = \frac{G_F^{\,2}}{8m_\mu\pi^5}\int \frac{\mathrm{d}^3k}{E_{\mathbf{k}}}\frac{\mathrm{d}^3q}{E_{\mathbf{q}}}\frac{\mathrm{d}^3q'}{E_{\mathbf{q}'}}\,\delta^4(p-k-q-q')\,p.q\,k.q'\ . \tag{25}$$

We evaluate this phase space integral by first integrating over the neutrino momenta q, q' since these are unobserved, and because the masslessness leads to some simplifications. If we introduce a momentum $Q = q + q' = p - k$, $Q^2 = 2q.q' > 0$ with also $Q^0 > 0$ then, since for massless neutrinos $q^0 = |\mathbf{q}|$, $q'^0 = |\mathbf{q}'|$, the essential integral becomes

$$I_{\mu\nu}(Q) = \int \frac{\mathrm{d}^3q}{|\mathbf{q}|}\frac{\mathrm{d}^3q'}{|\mathbf{q}'|}\,\delta^4(Q-q-q')\,q_\mu q'_\nu\ . \tag{26}$$

Lorentz invariance and dimensions requires that this has the form

$$I_{\mu\nu}(Q) = a\,Q_\mu Q_\nu + b\,g_{\mu\nu}Q^2\ . \tag{27}$$

To calculate a, b we may contract $I_{\mu\nu}(Q)$ with $g^{\mu\nu}$ and also $Q^\mu Q^\nu$ which then gives the equations

$$a + 4b = \tfrac{1}{2}I\ ,\quad a + b = \tfrac{1}{4}I\ ,\qquad I = \int \frac{\mathrm{d}^3q}{|\mathbf{q}|}\frac{\mathrm{d}^3q'}{|\mathbf{q}'|}\,\delta^4(Q-q-q')\ . \tag{28}$$

The Lorentz-invariant integral I can be easily evaluated in the centre of mass frame $Q = (Q_0, \mathbf{0})$

$$I = \int \frac{\mathrm{d}^3q}{|\mathbf{q}|^2}\,\delta(Q_0 - 2|\mathbf{q}|) = 4\pi\int \mathrm{d}q\,\delta(Q_0 - 2q) = 2\pi\ , \tag{29}$$

and hence

$$a = \tfrac{1}{3}\pi\ ,\qquad b = \tfrac{1}{6}\pi\ . \tag{30}$$

Using eqs.(27,30) then eq.(25) becomes

$$\Gamma = \frac{G_F^{\,2}}{3m_\mu(2\pi)^4}\int \frac{\mathrm{d}^3k}{E_{\mathbf{k}}}\,\Big(2p.(p-k)\,k.(p-k) + p.k\,(p-k)^2\Big)\ . \tag{31}$$

In the muon rest frame the integral can be reduced to one over the electron energy E using the result that $p.k = m_\mu E$ and $\mathrm{d}^3k/E_{\mathbf{k}} \to 4\pi|\mathbf{k}|\mathrm{d}E$. At this point it is also convenient to take advantage of the fact that $m_e/m_\mu = 0.0048 \ll 1$ to neglect the electron mass so that eq.(31) becomes

$$\Gamma = \frac{2G_F^{\,2}m_\mu}{3(2\pi)^3}\int_0^{\frac{1}{2}m_\mu}\mathrm{d}E\,E^2(3m_\mu - 4E)\ , \tag{32}$$

which is easily evaluated to give the final result for the muon decay rate

$$\Gamma_{\mu^- \to e^- + \overline{\nu}_e + \nu_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \ . \tag{33}$$

The muon lifetime is measured to be $\tau_\mu = 2.1970 \times 10^{-6}$sec and, as this is the inverse of the decay rate, therefore $\Gamma_{\mu^- \to e^- + \overline{\nu}_e + \nu_\mu} = 0.2996 \times 10^{-18}$GeV (the muon has only one decay channel) where we have used for the conversion 1 GeV$^{-1} = 6.582 \times 10^{-25}$sec. Inserting the experimental numbers in eq.(33), with $m_\mu = 105.658$ MeV, we would find $G_F = 1.1638 \times 10^{-5}\text{GeV}^{-2}$. Including small quantum corrections the current experimental result is

$$G_F = 1.1664 \times 10^{-5}\text{GeV}^{-2} \ . \tag{34}$$

On replacing m_μ by m_τ in eq.(33) we obtain the estimate for the purely leptonic decays of the τ

$$\Gamma_{\tau^- \to e^- + \overline{\nu}_e + \nu_\tau} \sim \Gamma_{\tau^- \to \mu^- + \overline{\nu}_\mu + \nu_\tau} = 0.405 \times 10^{-12}\text{GeV} \ , \tag{35}$$

since there are no new parameters to be determined. Experimentally the τ decays 18% of the time into each of these channels. The lifetime of the τ is 0.295×10^{-12}sec so the total decay rate $\Gamma_\tau = 2.23 \times 10^{-12}$GeV and 18% of this total decay rate is $\sim 0.402 \times 10^{-12}$GeV which is very close to the estimate in eq.(35). This is therefore strong evidence that the same weak coupling constant controls all leptonic weak interactions.

2 Semi-Leptonic Processes

The β-decay of the neutron,

$$n \to p + e^- + \overline{\nu}_e \ , \tag{36}$$

is referred to as a *semi-leptonic* process because it involves hadrons, the bound states of quarks, as well as leptons. The neutron state when expanded in terms of Fock space states contains arbitrarily large numbers of quarks (and also gluons that will be introduced later). However, the β-decay of the neutron is then regarded as being induced by the β-decay of one of the d-quarks in the neutron:

$$d \to u + e^- + \overline{\nu}_e \ . \tag{37}$$

These characteristics are shared by weak decays of other hadrons, such as π decays

$$\pi^\pm \to e^\pm + \nu_e(\overline{\nu}_e) \ , \quad \pi^\pm \to \mu^\pm + \nu_\mu(\overline{\nu}_\mu) \ . \tag{38}$$

The π^+ for example, is a state predominantly composed of $u\overline{d}$ and its decay can be understood as the process $u + \overline{d} \to e^+ + \nu_e$. Such processes can be accommodated in the charged current-current approximation for weak interactions by the inclusion of the additional quark part in the charged weak current

$$J^\alpha(x) = J_\alpha(x)^{\text{lept.}} + J_\alpha(x)^{\text{had.}} \ , \tag{39}$$

where ${J_\alpha}^{\text{lept.}}$ is as in eq.(6) and, provided we may treat the quarks as free

$$\begin{aligned} J_\alpha(x)^{\text{had.}} &= V_\alpha(x)^{\text{had.}} - A_\alpha(x)^{\text{had.}} \\ V_\alpha(x)^{\text{had.}} &= \overline{u}(x)\gamma_\alpha\,(V_{ud}d(x) + V_{us}s(x) + V_{ub}b(x)) + \ldots\{\overline{c}\ \&\ \overline{t}\} \\ A_\alpha(x)^{\text{had.}} &= \overline{u}(x)\gamma_\alpha\gamma_5\,(V_{ud}d(x) + V_{us}s(x) + V_{ub}b(x)) + \ldots\{\overline{c}\ \&\ \overline{t}\}\ . \end{aligned} \tag{40}$$

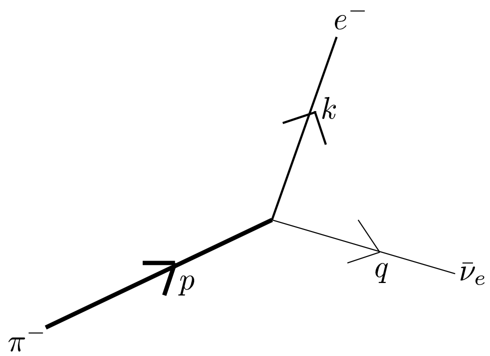
V_{ud} etc. are the appropriate elements of the CKM matrix describing flavour mixing in the weak charged current interactions. Since we are considering low energies, it is usually sufficient neglect the heavy quarks with masses above 1 GeV and use $V_{ud} \approx \cos\theta_C$, $V_{us} \approx \sin\theta_C$ only . Only the cross terms $J^{\alpha\text{lept.}\dagger}{J_\alpha}^{\text{had.}}$+h.c. in eq.(5) are of course relevant for semi-leptonic processes.

It is not really correct just to consider the free quarks taking place in the process, and some additional strong interaction (nonperturbative) physics input is required for the conversion of e.g. $d \to u + e^- + \bar{\nu}_e$ into a full β-decay calculation. This introduces some uncertainty, but high accuracy can sometimes still be achieved, especially if we compare rates where the nonperturbative contributions cancel.

2.1 Charged Pion Decay to Two Leptons

As a simple example of a semi-leptonic process, we consider the decay

$$\pi^-(p) \to e^-(k) + \overline{\nu}_e(q). \tag{41}$$



Pi decay to electron and neutrino

To lowest order the leptons can be taken as non-interacting, so the basic matrix element for the first of these processes is

$$\begin{aligned} \mathcal{M} &= \langle e^-(k)\,\overline{\nu}_e(q)|\mathcal{L}_{\text{Weff}}|\pi^-(p)\rangle \\ &= -\frac{G_F}{\sqrt{2}}\langle e^-(k)\,\overline{\nu}_e(q)|\overline{e}\gamma^\alpha(1-\gamma_5)\nu_e|0\rangle\,\langle 0|J^{\text{had.}}_\alpha|\pi^-(p)\rangle \\ &= \frac{G_F}{\sqrt{2}}\overline{u}_e(k)\gamma^\alpha(1-\gamma_5)v_{\nu_e}(q)\,\langle 0| - A^{\text{had.}}_\alpha|\pi^-(p)\rangle\ , \end{aligned} \tag{42}$$

where because of parity only the axial vector A_α has a non zero matrix element between the pseudoscalar π^- and the vacuum. This last matrix element could be evaluated exactly if one knew the detailed form of the π^- state in terms of quark Fock states. In practice this is a formidable non-perturbative problem, though approximate results can be used. However, one can parameterize the result in a simple way. Neglecting the heavy quarks, by Lorentz covariance since the pion is spinless this matrix element must have the form

$$\langle 0| \cos\theta_C \overline{u}\gamma_\alpha\gamma_5 d|\pi^-(p)\rangle = i\cos\theta_C\sqrt{2}F_\pi p_\alpha \ , \tag{43}$$

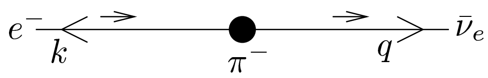
where the pion weak decay constant F_π is a parameter computable in principle from the pion wavefunction. Importantly F_π is a processes independent parameter, only depending on the weak decay of the pion, not on the final state. From eqs.(42) and (43) we find

$$\mathcal{M} = iG_F F_\pi \cos\theta_C\, \overline{u}_e(k)\gamma.p(1-\gamma_5)v_{\nu_e}(q) \ . \tag{44}$$

If we take into account the fact that $p = k + q$ and make use of the results $\overline{u}_e(k)\gamma.k = \overline{u}_e(k)m_e$ and $\gamma.q v_{\nu_e}(q) = 0$ for massless neutrinos we find

$$\mathcal{M} = iG_F F_\pi m_e \cos\theta_C\, \overline{u}_e(k)(1-\gamma_5)v_{\nu_e}(q) \ , \tag{45}$$

so that the matrix element vanishes if $m_e = 0$. This is a consequence of angular momentum conservation since in this limit the electron has negative helicity while the anti-neutrino has positive helicity which, in their centre of mass frame, add up to a component of spin or angular momentum -1 along the electron direction of motion which is incompatible with an initial spinless pion.



Decay of pion forbidden by angular momentum for zero electron mass

If we sum the modulus squared of the matrix element over the final electron, neutrino spins we get

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 2\left(G_F F_\pi m_e \cos\theta_C\right)^2 \text{tr}\left\{(\gamma.k + m_e)(1-\gamma_5)\gamma.q\right\} \ , \tag{46}$$

where the factor of 2 appears from moving one of the factors of $(1-\gamma_5)$ through the trace, i.e. $(1-\gamma_5)^2 = 2(1-\gamma_5)$. We can evaluate this using the trace theorems

$$\text{tr}\left\{\gamma.k\gamma.q\right\} = 4k\cdot q \tag{47}$$

$$\text{tr}\left\{\gamma^5\gamma.k\gamma.q\right\} = 0 \tag{48}$$

$$\text{tr}\left\{\gamma^\mu\right\} = \text{tr}\left\{\gamma^5\gamma^\mu\right\} = 0. \tag{49}$$

This results in

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 2\left(G_F F_\pi m_e \cos\theta_C\right)^2 4k\cdot q \ . \tag{50}$$

But

$$p = k + q, \qquad \to \qquad p^2 = k^2 + 2k \cdot q + q^2. \tag{51}$$

Therefore,

$$k.q = \frac{1}{2}(m_\pi^2 - m_e^2) \ . \tag{52}$$

From the general formula eq.(11), the decay rate is

$$\Gamma = \frac{1}{2m_\pi} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\mathbf{q}}} (2\pi)^4 \delta^4(p - k - q) \sum_{\text{spins}} |\mathcal{M}|^2 \ . \tag{53}$$

Therefore from eqs.(46,52) and $E_{\mathbf{q}} = |\mathbf{q}| = |\mathbf{k}|$

$$\Gamma = 2\left(G_F F_\pi m_e \cos\theta_C\right)^2 \frac{1}{m_\pi}(m_\pi^2 - m_e^2) \frac{1}{(4\pi)^2} \int \frac{\mathrm{d}^3 k}{E_{\mathbf{k}} |\mathbf{k}|} \delta(m_\pi - E_{\mathbf{k}} - |\mathbf{k}|) \ , \tag{54}$$

where $E_{\mathbf{k}} = (m_e^2 + \mathbf{k}^2)^{\frac{1}{2}}$ and we have restricted the integral to the π^- rest frame. The remaining integral may be evaluated giving

$$\Gamma_{\pi^- \to e^- + \overline{\nu}_e} = \frac{m_\pi}{4\pi} m_e^2 \left(1 - \frac{m_e^2}{m_\pi^2}\right)^2 G_F^2 F_\pi^2 \cos^2\theta_C \ . \tag{55}$$

If we were to do the calculation for muons we would find

$$\Gamma_{\pi^- \to \mu^- + \overline{\nu}_\mu} = \frac{m_\pi}{4\pi} m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 G_F^2 F_\pi^2 \cos^2\theta_C \ . \tag{56}$$

The ratio of the rates

$$R_0 = \frac{\Gamma(\pi^- \to e^- + \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \overline{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2} \frac{(m_\pi^2 - m_e^2)^2}{(m_\pi^2 - m_\mu^2)^2} \ , \tag{57}$$

is independent of F_π. Inserting the appropriate masses we find $R_0 = 1.28 \times 10^{-4}$, which should be compared with the experimental result $R_{\text{expt.}} = 1.23 \times 10^{-4}$. While there is a reasonably good comparison between theory and experiment, it can be improved considerably by including appropriate quantum corrections (i.e. loop corrections due to virtual photons). There are many other decay processes that can be estimated, with results that support the theory.

2.2 Non-Leptonic Processes

A crucial simplification in the semi-leptonic processes is that there is no more than one hadron in initial or final states. This allowed a factorized form for the matrix element, with the hadronic piece simply expressed in terms of the dominant quark content of one hadron. For weak interactions involving only hadrons, the whole process is complicated by the (QCD) interactions that bind quarks in a hadron, which are very strong at low energy. Consider for example, the weak decay of the Σ baryon (consisting of dds)

$$\Sigma^- \to n\pi^- \quad , \quad s \to u + d + \overline{u} \ . \tag{58}$$

If one could neglect the (QCD) interactions between the quarks in the final n and π^-, one could write a factorized form

$$\mathcal{M} \sim \langle \pi^- | \overline{d} \gamma_\alpha (1-\gamma_5) u | 0 \rangle \cos\theta_C \langle n | \overline{u} \gamma^\alpha (1-\gamma_5) s | \Sigma \rangle \sin\theta_C \ . \tag{59}$$

However, the quarks in the n interact strongly with the quarks in the π in the decay region, so this not a good approximation. Moreover, the second overlap would depend upon the full Fock space structure of the n and Σ. For these reasons one needs non-perturbative methods that take into account the forces that bind quarks in hadrons.

2.3 CP Violation

We now return to the full electroweak theory and consider its application to CP violation. CP violation is interesting on general grounds since it is one of the conditions needed to produce a baryon—anti-baryon asymmetry in the universe. C violation alone is not able to generate a particle-antiparticle number asymmetry because it is possible that the C violation will lead to the decays of particles having a different distribution to that of antiparticles, but the same total rate. As we have seen the CKM matrix for 3 families is necessarily complex and this results in the charged current weak interaction being CP-non-invariant.

The weak decay of K^0 mesons (kaons) exhibits most clearly this CP-violation. The K^0 and its anti-particle $\bar{K}^0$ are pseudoscalar mesons with dominant quark structure $\bar{s}d$ and $\bar{d}s$, having strangeness 1 and -1 respectively ('strangeness' is defined to be number of s anti-quarks minus number of s quarks). Strangeness is conserved by the (QCD) interactions that bind quarks in the meson, and the kaon is the lightest particle with strangeness. In the absence of any other interactions the hadron state $|K^0\rangle$ is therefore stable and distinguished from $|\overline{K}^0\rangle$ only by strangeness. $|K^0\rangle$ and $|\overline{K}^0\rangle$ are the strong interaction eigenstates, and are hence the states produced in QCD interactions. With suitable phase convention for $|\overline{K}^0\rangle$, under combined charge conjugation and parity the $K^0, \bar{K}^0$ states (at rest) may be chosen so as to transform as

$$\hat{C}\hat{P}|K^0\rangle = |\bar{K}^0\rangle \,, \qquad \hat{C}\hat{P}|\bar{K}^0\rangle = |K^0\rangle \,, \tag{60}$$

so that we may define $CP = +1$ and -1 eigenstates by

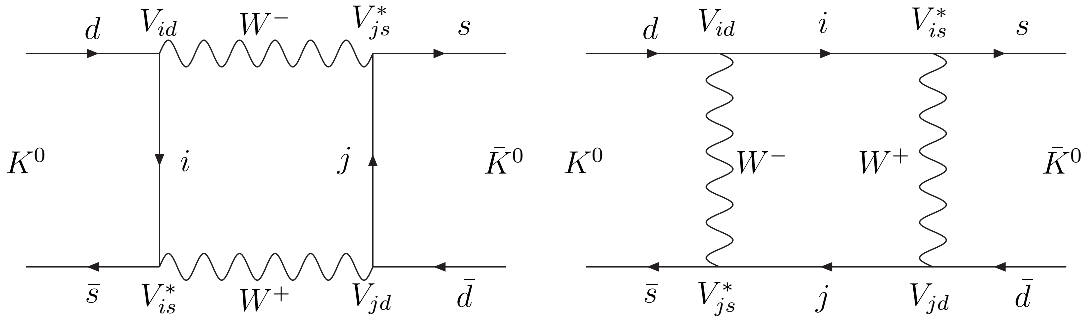
$$|{K_1}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle + |\bar{K}^0\rangle\Big) \,, \qquad |{K_2}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle - |\bar{K}^0\rangle\Big) \,. \tag{61}$$

CP-violating, strangeness violating, weak interactions lift the degeneracy between these states and allow for decays to pions. We can write an effective hermitian mass matrix for kaons

$$M = \begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\bar{K}^0\rangle \\ \langle \bar{K}^0|H'|K^0\rangle & \langle \bar{K}^0|H'|\bar{K}^0\rangle \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \,, \tag{62}$$

where H' is an effective Hamiltonian arising from weak processes that mix $|K^0\rangle$ and $|\overline{K}^0\rangle$, and weak and QCD processes that are diagonal. The eigenvalues of this mass matrix are the mass eigenstates, i.e. the states which propagate independently.

The mixing proceeds through second order charged current interactions and some of the underlying quark weak processes that contribute to the mixing part of H' are



where i and j run over the u, c and t quarks. We can investigate the consequences of C, P, and T on this mass matrix. We first look at the diagonal elements and assume CPT invariance.

First we insert $T^\dagger T$ into the matrix element $\langle K^0|H'|K^0\rangle$, and use the anti-unitary property of the T operator.

$$\begin{aligned}\langle K^0|T^\dagger T H'|K^0\rangle &= \langle K^0|H'|K^0\rangle^* \\ &= \langle K^0|H'^\dagger|K^0\rangle \\ &= \langle K^0|H'|K^0\rangle\end{aligned}$$

using the fact that $H'^\dagger = H'$. Additionally inserting $P^\dagger C^\dagger CP = 1$ we obtain

$$\begin{aligned}\langle K^0|T^\dagger P^\dagger C^\dagger CPTH'|K^0\rangle &= \langle K^0|T^\dagger P^\dagger C^\dagger H'CPT|K^0\rangle \\ &= \langle \bar{K}^0|H'|\bar{K}^0\rangle \\ &= \langle K^0|H'|K^0\rangle,\end{aligned}$$

where for the particle at rest $T|K^0\rangle = |K^0\rangle$ up to a phase (which vanishes in the matrix element) and $CP|K^0\rangle = |\bar{K}^0\rangle$. Thus, assuming CPT is obeyed $M_{11} = M_{22}$.

In a similar fashion, inserting $P^\dagger C^\dagger CP$ into $\langle K^0|H'|\bar{K}^0\rangle$ and assuming CP invariance we obtain

$$\langle K^0|H'|\bar{K}^0\rangle = \langle \bar{K}^0|H'|K^0\rangle, \tag{63}$$

and hence $M_{12} = M_{21}$. If this were true then the two eigenvalues of the mass matrix would be $M_{11} \pm M_{12}$ with eigenstates $|K_1{}^0\rangle$ and $|K_2{}^0\rangle$ respectively. However, CP violation means that $M_{12} \neq M_{21}$ precisely. This results in the mass eigenstates

$$|K_S{}^0(t)\rangle = \frac{e^{-im_S t - \Gamma_S t}}{(1+|\epsilon_1|^2)^{\frac{1}{2}}}\Big(|K_1{}^0\rangle + \epsilon_1 |K_2{}^0\rangle\Big), \quad |K_L{}^0(t)\rangle = \frac{e^{-im_L t - \Gamma_L t}}{(1+|\epsilon_2|^2)^{\frac{1}{2}}}\Big(|K_2{}^0\rangle + \epsilon_2 |K_1{}^0\rangle\Big), \tag{64}$$

where m_S and m_L are the mass eigenvalues and Γ_S and Γ_L the widths (i.e. the inverse of the decay rates). If we assume CPT invariance then

$$\epsilon_1 = \epsilon_2 = \frac{\sqrt{M_{12}} - \sqrt{M_{21}}}{\sqrt{M_{12}} + \sqrt{M_{21}}} \,. \tag{65}$$

The amount of CP violation can be measured experimentally through the decays of the kaons, which predominantly decay in hadronic channels to either two or three pions. Under C, $\pi^+ \leftrightarrow \pi^-$ while $\pi^0 \to \pi^0$ and, in the centre of mass frame for two pions, P interchanges the two particles and hence $\hat{C}\hat{P}|\pi^+\pi^-\rangle = |\pi^+\pi^-\rangle$, $\hat{C}\hat{P}|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle$. In the limit of CP conservation $\epsilon_1 = \epsilon_2 = 0$, the decay $K_S^0 \to \pi\pi$ is allowed whereas the decay $K_L^0 \to \pi\pi$ is forbidden. The state K_L^0 must then decay to at least three pions (where a $CP = -1$ combination is possible), but will have a longer lifetime as a result of the reduced phase space available in the final state (hence the $L =$ Long subscript). Experimentally, one can separate out the K_L^0 states from the K_S^0 by waiting long enough. However, a small fraction of these K_L^0 states are observed to decay to $\pi\pi$, indicating small CP violation. If we define

$$\eta_{+-} = \frac{\langle \pi^+\pi^- | H_W | {K_L}^0 \rangle}{\langle \pi^+\pi^- | H_W | {K_S}^0 \rangle} \,, \qquad \eta_{00} = \frac{\langle \pi^0\pi^0 | H_W | {K_L}^0 \rangle}{\langle \pi^0\pi^0 | H_W | {K_S}^0 \rangle} \,, \tag{66}$$

where H_W is the weak interaction Hamiltonian that causes the decay, then experimentally $|\eta_{+-}| \approx |\eta_{00}| \approx 2.28 \times 10^{-3}$. Most of this is due to the lack of CP invariance in the mass eigenstate $|K_L^0\rangle$, from CKM matrix elements in the mass matrix. It can also be shown that the Standard Model predicts an even smaller CP violation in the decay amplitude itself, due to H_W. In the absence of this one would have $\eta_{+-} = \eta_{00} = \epsilon_2$, but experimentally it is found that $|\eta_{+-}/\eta_{00}|^2 - 1 \approx 2 \times 10^{-2}$.

Similar effects can also be seen in the B^0 meson sector. In fact the theoretical calculations are cleaner in this situation, but the experiments are harder, but CP-violation is clearly confirmed.

Because it is already very small in the Standard Model, and many models of new physics beyond the Standard Model predict significant CP violation, its study is one of the main ways in which we may obtain some signs of this new physics. Indeed, according to current understanding of the processes that lead to matter—anti-matter asymmetry in the early universe, the Standard Model alone does not seem to have strong-enough CP violation to account for the cosmological observations.