The Standard Model
Quantum Chromodynamics (QCD)

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Preface

QCD is the final symmetry of the Standard Model. Quarks are subject to the forces of a non Abelian gauge field theory *Quantum Chromodynamics*, based on the gauge group $SU(3)$. The full Standard Model gauge group is then $SU(3) \times SU(2) \times U(1)_Y$. Leptons, Higgs, and electroweak gauge fields are colour $SU(3)$ singlets. The vector particles corresponding to the $SU(3)$ gauge field are referred to as gluons and the quantum number carried by the quarks is colour. Hadrons (baryons and mesons) can be regarded as composite bound states of quarks and gluons. However, neither quarks nor gluons have ever been observed as free particles; they appear to be permanently confined in hadrons that transform as singlets with respect to $SU(3)$. This is due to non-perturbative effects that can be calculated roughly but the physical mechanism for which is still not fully understood. Nevertheless, a careful analysis of QCD as a renormalizable field theory shows that the strong coupling constant runs with the energy scale of a process. It is very strong for scales < 1GeV, resulting in the binding of the quarks and gluons into composite states. However, it grows weaker with increasing scales (asymptotic freedom), and for processes involving energies much higher than 1GeV, perturbation theory in terms of the quark and gluon states is is applicable, which allows detailed comparison with experiment.

1 Symmetries and Interactions

Quark fields form a complex 3-dimensional fundamental representation of the QCD gauge group, $SU(3)$ 'colour'; anti-quarks are in the conjugate representation, of course. Real gluon gauge fields $A_{\mu a}$, $a \in \{1, \ldots 8\}$, are linked to the generators iT_a of $SU(3)$. In the fundamental representation $T_a \to \frac{1}{2}\lambda_a$, where λ_a are the 3×3 Gell-Mann matrices, a generalisation of the Pauli matrices to

$SU(3)$. The corresponding field strength is

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - g_s\, f_{abc} A_{\mu b} A_{\nu c}\,, \tag{1}$$

where g_s is the QCD gauge coupling and f_{abc} are the totally antisymmetric structure constants of $SU(3)$, with $[\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b] = i f_{abc} \frac{1}{2}\lambda_c$. The most general renormalizable $SU(3)$ gauge-invariant Lagrangian density is then (up to a term $\propto \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu a} F_{\rho\lambda a}$ which contains no local physics, and the coefficient of which is known to be $< 10^{-9}$, and is usually assumed to be zero)

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4} F^{\mu\nu}{}_a F_{\mu\nu a} + \sum_{f=1}^{n_f} \overline{q}_f (i\gamma^\mu D_\mu - m_f) q_f\,, \tag{2}$$

where the covariant derivative is defined by

$$D_\mu q_f = \partial_\mu q_f + i g_s A_{\mu a} \tfrac{1}{2}\lambda_a q_f\,, \tag{3}$$

and colour indices $i \in \{1,2,3\}$, as well as spinor indices, for the quark fields are suppressed. The sum over f is for the different quark flavours, so that $q_f \in \{u, d, s, \ldots\}$, which from the point of view of QCD are distinguished only by their differing masses m_f. The Lagrangian is invariant under the local $SU(3)$ gauge symmetry.

In fact, quark mass terms are not really part of the QCD Lagrangian since they arise from the electroweak sector, through coupling to the Higgs field. In practice m_u, m_d and m_s are all $\ll 1$GeV and so lead to corrections smaller than nonperturbative QCD corrections, and are usually set to zero. For the other quarks, if we are at scale Q, then if Q is below m_q we may ignore the quark in a process up to corrections of $\mathcal{O}(Q^2/m_q^2)$ while if Q is above m_q we may treat the quark as massless up to corrections of $\mathcal{O}(m_q^2/Q^2)$. Hence it is common to turn the quark on at mass scale $Q^2 = m_q^2$, and treat it as massless above this scale but ignore it below this scale. Hence at any scale we have n_f "active" quark flavours, e.g. at $Q^2 = M_Z^2$ $n_f = 5$ because we ignore the top quark.

To write down Feynman rules for QCD, it is necessary to add extra gauge fixing terms in order to set up a perturbative expansion, starting from a zeroth order free field theory of quarks and gluons. The Feynman rules involve quark and gluon propagators, three and four gluon vertices, which are proportional to g_s and g_s^2 respectively, and also an $\mathrm{O}(g_s)$ vertex when a gluon couples to a quark. Ghost propagators couple to other lines in a Feynman graph through a single gluon vertex $\propto g_s$. The perturbative expansion in g_s defines a renormalizable quantum field theory, so that no new parameters, beyond those present in the initial classical Lagrangian and gauge fixing terms, need be introduced. In lattice field theory no gauge fixing is necessary since a propagator does not need to be defined.

1.1 Global Symmetries

The classical Lagrangian density eqn.(2) possesses a number of global symmetries also. If we write

$$q(x) = \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ q_{n_f} \end{pmatrix} \tag{4}$$

then, even if $m_f \neq 0$,

$$q \to e^{i\xi} q \tag{5}$$

is a $U(1)_V$ global symmetry. The V is for 'vector-like'. The corresponding conserved charge — baryon number — counts the number of quarks minus anti-quarks. This is observed to be a good symmetry to the extent that we do not observe baryon number violation in any experiments, e.g. proton decay has not been observed despite much effort. However, the obvious baryon—anti-baryon asymmetry that now exists was presumably due to baryon-number violating processes, in the early universe, that cannot be accommodated within the Standard Model.

If all m_f are equal (which is a fair approximation if one restricts to flavours u, d, s and a good approximation for just u, d), then

$$q \to e^{i\xi_b \theta_b} q \tag{6}$$

is an $SU(n_f)_V$ flavour symmetry. Here, ξ_b's are global parameters and θ_b the fundamental representation of the generators of $SU(n_f)_V$, $b \in \{1, \ldots, n_f^2 - 1\}$. In the case $n_f = 2$, this symmetry is called *isospin*, and means that one can obtain a neutron from a proton just by swapping up and down quarks. Isospin is only violated at a very small level.

2 Renormalization and Running Coupling

QCD is a renormalizable quantum field theory and if quarks masses are neglected depends upon only one parameter, the gauge coupling g_s. Because g_s is much larger than the electroweak couplings g and g' low order calculations are far less accurate than for electroweak physics and loop corrections are essential. Also, because g_s is large the running of the coupling in QCD is an essential part of the structure of the theory, whereas in the electroweak sector it is present, but leads only to small corrections.

For simplicity, we hereafter neglect the quark masses m_f in discussing the renormalization of QCD, although the treatment can be extended to include them. In general, setting mass terms to zero may generate additional infra red divergences in Feynman amplitudes. But with appropriate prescriptions, and due caveats to be made clearer later, these can be avoided and the massless limit of perturbative QCD exists. Because of short distance ultra-violet divergences, it is necessary to introduce some regularisation for the loop integrals which appear in the perturbative expansion of physical amplitudes. Without specifying any details we suppose there is a cut off, M, which renders Feynman integrals finite and preserves Lorentz invariance, unitarity, etc. for energy scales $\ll M$. Any regularisation introduces a mass scale like M, even if the original theory has no mass parameters, such as QCD in the massless limit. Even though this scale M is ultimately taken to ∞, and all physical amplitudes are independent of M in this limit, the removal of the divergences requires the introduction of an additional finite scale μ, and this results in scale dependence.

(For QCD, dimensional regularisation is almost universally used, since this preserves gauge invariance, which a simple ultraviolet cut-off does not. This means we work in $4-2\epsilon$ dimensions, and remove divergences like $1/\epsilon$ when we let $\epsilon \to 0$. In this case $\int d^{4-2\epsilon}(\partial \cdot A)^2$ has dimension zero because the action is dimensionless, and such terms appear in the kinetic part of the action. A spatial derivative has mass dimension -1, so the field A must have dimension $1-\epsilon$. There are also terms of the type $\int d^{4-2\epsilon} g_s^2 A^4$ which must also have dimension 0. This means g_s has dimension ϵ. In order to give the dimensionless g_s a mass dimension we must write it as $\mu^\epsilon g_s$, where g_s is the usual coupling, and μ is some arbitrary mass scale. In the process of renormalization μ does not disappear from physical amplitudes when $\epsilon \to 0$.)

2.1 Renormalisation

Let us now consider some physical amplitude F, which we take to be characterised by a set of momenta p_i, and which has a perturbative expansion, so that we may write $F(g_0, M; p_i)$, where we display explicitly the necessary dependence on the cut off M and have relabelled the coupling appearing in the Lagrangian $g_s \to g_0$. It is important to realize now that this so-called *bare* coupling g_0 is not a physical quantity, but simply a parameter appearing in the *bare* Lagrangian. As such we know nothing about it directly, and it turns out to be divergent as $M \to \infty$ if we wish to have finite amplitudes. The same would be true of the bare masses m_0. They are not the real physical masses, but just a parameter which appears in a calculation of some physical quantity which may define the real physical mass.

The fundamental requirement of renormalizability, which may be proven order by order in the perturbative expansion, asserts that in general we can introduce a suitable 'wavefunction renormalisation' $Z(M)$ which is independent of the momenta, such that in the limit $M \to \infty$

$$ZF(g_0, M; p_i) \longrightarrow f(g, \mu; p_i) \quad \text{as} \quad M \to \infty\,, \tag{7}$$

where $f(g, \mu; p_i)$ is finite and obeys the general axioms of quantum field theory. μ is some arbitrary scale, known as the renormalization scale, which appears in the process of renormalization (as we will soon see), and g is a *renormalized* coupling which is related to the physical quantity $f(g, \mu; p_i)$ and whose value may be determined by measuring $f(g, \mu; p_i)$ for some p_i and making a choice of μ

(e.g. $\mu \sim p_i$). The statement (7) is valid order by order in a perturbative expansion in the finite renormalised coupling g, which must be process-independent and therefore cannot depend on p_i. In general

$$g_0\Big(g, \frac{M}{\mu}\Big) = g + \mathrm{O}(g^3)\,, \qquad Z\Big(g, \frac{M}{\mu}\Big) = 1 + \mathrm{O}(g^2)\,, \tag{8}$$

are also given as an expansion in g. As we see the appearance of μ is tied up with the precise definition of g.

In QCD, an amplitude would usually consist of a tree-level part plus loop corrections which diverge logarithmically as $M \to \infty$. Consider, for example, the typical form for an amplitude in QCD to one-loop level which does not require wavefunction renormalisation and depends on external momentum p,

$$F(g_0, M; p) = g_0^2 - 2g_0^4\Big(\tilde{a} + b\ln\left(\frac{M}{p}\right)\Big) = g_0^2 - 2g_0^4\left(\tilde{a} + b\ln\left(\frac{M}{\mu}\right) + b\ln\left(\frac{\mu}{p}\right)\right). \tag{9}$$

By defining g via

$$g_0(g, M/\mu) \equiv g + g^3\left(a + b\ln\left(\frac{M}{\mu}\right)\right) + O(g^5)\ , \tag{10}$$

we can obtain the renormalized amplitude

$$f(g, \mu; p) = g^2 - 2g^4\left(b\ln\left(\frac{\mu}{p}\right) + \tilde{a} - a\right) + O(g^6). \tag{11}$$

The coefficient b of the divergent loop diagrams is fixed by the basic structure of the theory (gauge group, number of flavour etc.), while the coefficient a of the finite part of g_0 can be changed arbitrarily by changing the precise definition of g (the renormalisation scheme). Similar considerations usually also apply for wavefunction and mass renormalization.

In order to introduce a finite renormalised coupling g that is independent of external momenta, it was necessary to introduce the arbitrary scale μ, on which g will now depend. Once $g_0(g, M/\mu)$ is precisely specified in terms of g in this way, g can be determined by an experiment measuring any one amplitude f. (In dimensional regularisation, a common renormalisation scheme for handling finite parts like a is termed minimal subtraction (MS). In that prescription, only the poles in ϵ are subtracted to define the finite physical amplitude in the limit $d \to 4$. An alternative scheme called modified minimal subtraction $\overline{\mathrm{MS}}$ removes the poles plus common finite constants $\ln(4\pi) - \gamma_E$.)

Differentiating eqn.(10) with respect to μ at fixed g_0, and defining

$$\beta(g) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g\Big|_{g_0}\,, \tag{12}$$

we find

$$0 = \beta(g)\Big(1 + 3g^2\big(a + b\ln\frac{M}{\mu})\Big) - g^3 b + O(g^5) \tag{13}$$

giving

$$\beta(g) = g^3 b + O(g^5)\,. \tag{14}$$

This could also be obtained by differentiating eqn.(11) with respect to μ. As mentioned b is determined by the short distance divergences while a depends on the precise definition of g. In general, beyond lowest order $\beta(g)$ is not unique, but depends on the choice of renormalisation scheme. Different schemes correspond to couplings which are related by a reparameterisation, $g \to g'(g) = g + \mathrm{O}(g^3)$. It is important to use the same scheme for calculations of different processes, such as consistently using dimensional regularisation with minimal subtraction, or to take account of the appropriate redefinition when comparing calculations according to differing schemes.

2.2 Running Coupling

Although $f(g(\mu), \mu; p_i)$ is formally independent of μ, up to a possible overall scaling, the coupling itself depends on the choice for the renormalization scale, and the most appropriate choice would seem to be that $\mu \sim p_i$. For example, if f is a dimensionless quantity with only one physical scale p, then we have $f(g(\mu), p/\mu)$. We could then choose $\mu \gg p$ or $\mu \ll p$, but this would introduce a dependence on the very large or small ratio p/μ and any expansion in $g(\mu)$ might not be reliable. If $\mu \approx p$ then there is no obvious source of a large number, and we can examine the expression in terms of $g(\mu)$ with more confidence. Hence, the main features of the scale dependence of the theory depends on the dependence of $g(\mu)$ on μ and hence on the qualitative form of $\beta(g)$.

In any renormalizable quantum field theory it is straightforward to calculate the β-function in perturbation theory to one or two, or sometimes more, loops. For a non-Abelian gauge theory, with a simple gauge group so that there is a single gauge coupling g, the corresponding β-function may be written as

$$\beta(g) = -\beta_0 \,\frac{g^3}{16\pi^2} + \mathrm{O}(g^5)\,. \tag{15}$$

We suppose that, as in QCD, the gauge field is only coupled to fermion fields ψ_f through covariant derivatives $D_\mu \psi_f = \partial_\mu \psi_f + igA_{\mu a} T_{fa} \psi_f$, where T_{fa} are matrix generators of the Lie algebra of the gauge group for the irreducible representation defined by ψ_f, $[T_{fa}, T_{fb}] = i f_{abc} T_{fc}$. In this case the general formula for β_0 (this assumes that the gauge field coupling does not distinguish between left and right handed fermions, there is no γ_5 involved) is

$$\beta_0 = \tfrac{11}{3}\, C_A - \tfrac{4}{3} \sum_f T_f\,, \tag{16}$$

where C_A, T_f are group theory factors defined by

$$f_{acd} f_{bcd} = C_A \delta_{ab}\,, \qquad \mathrm{tr}(T_{fa} T_{fb}) = T_f \delta_{ab}\,. \tag{17}$$

For gauge group $SU(N)$, $C_A = N$, while if the fermions are in the fundamental representation, as are the quark fields for $SU(3)_{\text{colour}}$, then $T_f = \frac{1}{2}$. The origin of each contribution to β_0 can be justified as follows. Just as dipoles tend to screen an electric field in electromagnetism, so pair production of e^+, e^- pairs in the vacuum will tend to screen the electric field around a test charge, and hence the apparent value of the test charge. The farther from the test charge (the lower the momentum scale μ) the more dipoles will screen. This phenomena also occurs in non-Abelian gauge theories, where pair production of fermions screens the non-Abelian gauge field, and is the origin of the second term in eqn.(16). However, unlike in Abelian QED, there is also pair production of gauge field particles due to their self-interaction in a non-Abelian theory. Because of the opposite statistics for bosons and fermions, this contribution to β_0 has opposite sign. The first term in eqn.(16) therefore signifies the anti-screening that takes place due to gauge particle fluctuations.

For QCD, the formula therefore becomes

$$\beta_0 = 11 - \tfrac{2}{3} n_f \,, \tag{18}$$

where n_f is the number of quark flavours which contribute to the β-function. In determining the running coupling $g(\mu)$, those quarks with masses $\gtrsim \mu$ should not contribute to the β-function, i.e. their pair production in vacuum can be neglected over sufficiently large distance scales. The maximum value of n_f is 6 so $\beta_0 > 0$, and we have asymptotic freedom.

Using the evolution equation we can describe the scale-dependence of the coupling quantitatively. It is conventional to define for the QCD coupling an analogue of the QED fine structure constant,

$$\alpha_s = \frac{g_s^2}{4\pi}, \tag{19}$$

since it is this quantity which actually appears naturally in perturbative expansions of physical quantities. Using this definition of the strong coupling constant we can write

$$\frac{d\,\alpha_s(\mu^2)}{d\,\ln\mu^2} = -\frac{\beta_0}{4\pi}\alpha_s^2(\mu^2) + \mathrm{O}(\alpha_s^3). \tag{20}$$

Ignoring the $\mathrm{O}(\alpha_s^3)$ corrections, i.e. working at lowest order this may easily be solved, i.e.

$$-\int_{\mu_0^2}^{\mu^2} d\,\ln\tilde{\mu}^2 = \frac{4\pi}{\beta_0}\int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\,\tilde{\alpha}_s}{\tilde{\alpha}_s^2}, \tag{21}$$

where μ_0 is some fixed scale. Hence,

$$-\ln(\mu^2/\mu_0^2) = \frac{4\pi}{\beta_0}\Big[\frac{1}{\alpha_s(\mu_0^2)} - \frac{1}{\alpha_s(\mu^2)}\Big]. \tag{22}$$

This leads to

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln(\mu^2/\mu_0^2) + \frac{4\pi}{\beta_0\alpha_s(\mu_0^2)}}. \tag{23}$$

From this expression we can indeed see that $\alpha_s(\mu^2)$ decreases as μ^2 increases, and that $\alpha_s(\mu^2) \to 0$ as $\mu^2 \to \infty$. However, the definition relies on an arbitrary boundary condition for the coupling at some fixed scale μ_0^2.

It is simpler, and more illustrate to rewrite the solution for $\alpha_s(\mu^2)$ slightly. Eqn.(23) may be expressed as

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln(\mu^2) - (\ln(\mu_0^2) - \frac{4\pi}{\beta_0\alpha_s(\mu_0^2)})}. \tag{24}$$

Defining a scale Λ_{QCD} by

$$\ln(\mu_0^2) - \frac{4\pi}{\beta_0\alpha_s(\mu_0^2)} = \ln(\Lambda_{QCD}^2), \tag{25}$$

i.e. Λ_{QCD} is the value of μ_0^2 for $\alpha_s(\mu_0^2) \to \infty$, results in the solution

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda_{QCD}^2)}. \tag{26}$$

Λ_{QCD}, which may be regarded as a constant of integration in the solution of the evolution equation, provides a basic QCD mass scale even in the absence of any quark masses. Hence, in QCD we have replaced a dimensionless coupling constant as the free parameter with a mass scale. This is often known as dimensional transmutation.

We now see explicitly from eqn.(26) that the coupling falls logarithmically with increasing μ^2. However it becomes large as $\mu^2 \to \Lambda_{QCD}^2$ from above. Since $4\pi/\beta_0 \approx 1.5$ perturbation theory loses any real meaning when $\ln(\mu^2/\Lambda_{QCD}^2)$ becomes ~ 2 or less. Since the mass of hadrons is typically of order 1GeV it is thus consistent that measurements of α_s lead to $\Lambda_{QCD} \approx 0.3$GeV. Thus, over distance scales corresponding to energies of $\sim$ 1GeV the coupling is strong, and attempts to pull quarks and gluons with these energies further apart decreases μ^2 and increases the coupling even more. In very general terms this is the mechanism for confinement, though the details and proof are a nonperturbative problem. Alternatively, at very short distances, corresponding to $\mu^2 \gg 1GeV^2$ the coupling is relatively weak, e.g. $\alpha_s(M_Z^2) = 0.12$ and interactions involving quarks and gluons can be treated using perturbation theory.

Hence, the result (26) shows clearly how $\alpha_s(\mu^2)$ behaves. The contributions from $\mathcal{O}(\alpha_s^3)$ in eqn.(20) lead to corrections of $\mathrm{O}((\ln\mu^2/\Lambda^2)^{-2})$. These become important if μ^2 is as low as only a few GeV^2. Beyond leading order strictly speaking Λ_{QCD} is a renormalization scheme dependent quantity. This is because a change of scheme leads to different $\mathcal{O}(\alpha_s^3)$ contributions in eqn.(20), and a change in these contributions is equivalent to a rescaling of Λ_{QCD}. Hence, the precise value of Λ_{QCD} has significance only in the context of a particular scheme.

For some theories, e.g. QED and scalar field theory ($\lambda\phi^4$), the β-function has the opposite sign:

$$\frac{d\,\alpha(\mu^2)}{d\,\ln\mu^2} = \frac{1}{3\pi}\alpha^2(\mu^2) \qquad \frac{d\,\lambda(\mu^2)}{d\,\ln\mu^2} = \frac{3}{32\pi}\lambda^2(\mu^2). \tag{27}$$

This means the renormalization group equation may be written (at lowest order) as

$$\frac{d\,\lambda(\mu^2)}{d\,\ln\mu^2} = \frac{\beta_0}{4\pi}\lambda^2(\mu^2). \tag{28}$$

We may solve in the same manner as before, obtaining

$$\frac{1}{\lambda(\mu^2)} = \frac{1}{\lambda(\mu_0^2)} - \frac{\beta_0}{4\pi}\ln(\mu^2/\mu_0^2), \tag{29}$$

which may be rearranged to give

$$\lambda(\mu^2) = \frac{\lambda(\mu_0^2)}{1 - \lambda(\mu_0^2)\frac{\beta_0}{4\pi}\ln(\mu^2/\mu_0^2)}. \tag{30}$$

This makes it clear that $\lambda(\mu^2)$ grows as μ^2 increases above μ_0^2. However, we also see that

$$\lambda(\mu^2) \to \infty \quad \text{as} \quad \ln(\mu^2/\mu_0^2) \to \frac{4\pi}{\beta_0\lambda(\mu_0^2)}. \tag{31}$$

This is known as the Landau singularity. This implies that perturbation theory breaks down in a theory with this type of β-function at some very high scales. One can verify that higher-order corrections do not alter this picture, simply the precise position of the singularity. Additionally, strong coupling expansions and lattice studies, prove that the feature is not simply a breakdown of perturbation theory, but a fundamental inconsistency in the quantum field theory - if the coupling is finite at some low scale it genuinely becomes infinite at some high scale, the value indeed approximately equal to that in eqn.(31). This problem is known as triviality because it means the theory is inconsistent unless the coupling is zero. It is true for both QED and for the Higgs coupling in the Standard Model.

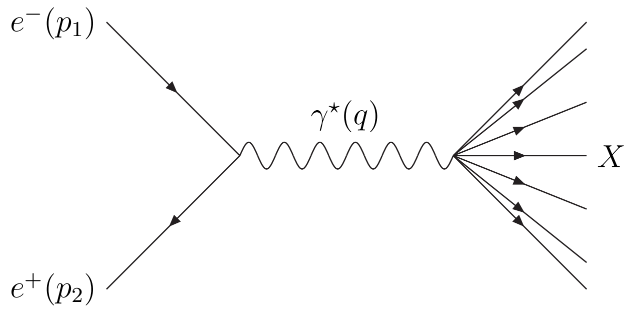
Because the in QED the coupling is very small the running is very slow - $\alpha(m_e^2) = 1/137$ and this increases to $\alpha(m_Z^2) = 1/128$. Hence over the region of experimental physics so far probed the running of α leads to corrections of a few percent. The triviality problem could be solved by the addition of new physics before the scale at which the coupling diverges (e.g. the problem would be avoided if the $U(1)$ group of QED came from some larger non-Abelian simple group, i.e. there was a unification of couplings), and in QED this scale is many orders of magnitude away.

However, in the Higgs sector of the Standard Model $m_H = \sqrt{2\lambda(m_H^2)}v$, where $v = 246$GeV. Hence, a larger value of the Higgs mass leads to a larger scalar coupling, and from eqn.(31) a lower scale at which the divergence of the coupling appears, and below which new physics must appear to avoid this problem. So the higher the Standard Model Higgs mass the stronger the constraint of when some new physics beyond the Standard Model must appear. Even before the discovery of the Higgs boson our failure to yet see such new physics puts a weak upper bound on the Higgs mass of about 500GeV.

3 $e^-e^+ \to$ Hadrons

In many ways the cleanest application of asymptotic freedom in QCD is to the total cross section for $e^-(p_1) + e^+(p_2) \to$ hadrons. To lowest order in the electromagnetic coupling e, the e^-e^+ annihilate to produce a virtual photon, with momentum $q = p_1 + p_2$, which then decays to form physical hadron states labelled by X, i.e.

$$e^-(p_1) + e^+(p_2) \to \gamma^\star(p_1 + p_2 = q) \to X \quad \text{(hadrons)}. \tag{32}$$



For a final hadronic state X the amplitude is

$$\mathcal{M}_X = e^2 \frac{1}{q^2} \langle X | J_{\rm h}^\mu | 0 \rangle \, \overline{v}(p_2) \gamma_\mu u(p_1) \,, \tag{33}$$

where $J_{\rm h}^\mu$ is the hadronic contribution to the electromagnetic current which may be expressed in term of quark fields by

$$J_{\rm h}^\mu = \overline{q} \gamma^\mu Q q \,, \tag{34}$$

for Q the diagonal matrix of quark charges. This results in the expression for the total cross-section,

$$\sigma_{\mathrm{tot},\, e^-e^+ \to \mathrm{hadrons}} = \frac{1}{F} \sum_X \frac{1}{4} \sum_{\mathrm{spins}} (2\pi)^4 \delta^4(q - p_X) \, |\mathcal{M}_X|^2 \,, \tag{35}$$

where F is the flux factor $F = 4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2} = 2q^2$, where we assume we can let $m_e = 0$ since $q^2 \gg m_e^2$. This matrix element is similar to many we have seen previously, and is easily shown to factorize into the form

$$\sigma_{\mathrm{tot},\, e^-e^+ \to \mathrm{hadrons}} = \frac{1}{F} \frac{e^4}{q^4} L_{\mu\nu} \sum_X (2\pi)^4 \delta^4(q - p_X) \, \langle 0 | J_{\rm h}^\mu | X \rangle \langle X | J_{\rm h}^\nu | 0 \rangle \,, \tag{36}$$

where $L_{\mu\nu}$ is the standard leptonic tensor

$$L_{\mu\nu} = 4(p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu} - g_{\mu\nu} p_1 \cdot p_2). \tag{37}$$

In principle this seems to be a difficult problem requiring nonperturbative QCD. However, wee can simplify the problem significantly by postulating,

$$\sum_{X = \mathrm{hadrons}} |X\rangle\langle X| = \sum_{X = q, \overline{q}, g \text{ states}} |X\rangle\langle X| \,, \tag{38}$$

at least in application to high energy processes. The justification for this is is essentially beyond the scope of perturbation theory. However, a physical 'hand-wave' picture for the above analysis emerges if one considers the time and distance scales involved in the decay of the high energy virtual photon intermediate state, and noting that $q^2 \gg \Lambda^2_{QCD}$. The photon with virtuality q^2 can fluctuate into a set of partons (e.g. at lowest order just a $q\bar{q}$ pair), and from the uncertainty principle this happens on a timescale $\mathcal{O}(1/\sqrt{q^2})$. This is then a perturbative process and any QCD corrections to this process, from gluon exchange between the quarks for example, are suppressed by the small value of α_s at large momentum scales. At later times, as the quarks separate, the QCD coupling becomes stronger, so that large numbers of quarks and gluons are created and exchanged, as the original quarks 'hadronize'. The final asymptotic state is then usually a complicated mess of hadrons. However, this latter process occurs over a timescale $\mathcal{O}(1/\Lambda_{QCD})$. So there is very little interference between the formation of quarks and gluons from the photon and the subsequent hadronization. This implies that

$$\sigma_{\mathrm{tot},\, e^-e^+\to \mathrm{hadrons}} = \sigma_{\mathrm{tot},\, e^-e^+\to q,\bar{q},g\ldots} + \mathcal{O}\bigg(\Big(\frac{\Lambda^2_{QCD}}{q^2}\Big)^n\bigg). \tag{39}$$

A more rigorous argument shows that for total cross-sections the nonperturbative hadronization correction is $\mathcal{O}\Big(\big(\frac{\Lambda^2_{QCD}}{q^2}\big)\Big)$, but the power depends on the type of process being investigated and can be $\mathcal{O}\Big(\big(\frac{\Lambda^2_{QCD}}{q^2}\big)^{\frac{1}{2}}\Big)$ if one looks at some details of the final state rather than just total cross-sections.

The calculation of $\sigma_{\mathrm{tot},\, e^-e^+\to \mathrm{hadrons}}$ at leading order (LO) is therefore the same as $\sigma_{\mathrm{tot},\, e^-e^+\to \sum q\bar{q}}$, where the sum is over all possible $q\bar{q}$ pairs. This cross-section is identical in form to the cross-section for $e^-e^+ \to \mu^-\mu^+$, and ignoring particle masses the differential cross section is

$$\frac{\mathrm{d}\sigma_{e^-e^+\to q\bar{q}}}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\, Q_q^2\, (1+\cos^2\theta)\,, \tag{40}$$

where $\alpha = e^2/4\pi$ and Q_q is the fractional quark charge, i.e. $Q_q = 2/3$ for up-type quarks and $1/3$ for down-type quarks. It is easy to integrate this to find the total cross section

$$\sigma_{\mathrm{tot},\, e^-e^+\to q\bar{q}} = \frac{4\pi\alpha^2}{3q^2}\, Q_q^2\,, \tag{41}$$

which would be valid for $\sqrt{q^2} \gg m_q$.

Therefore at LO

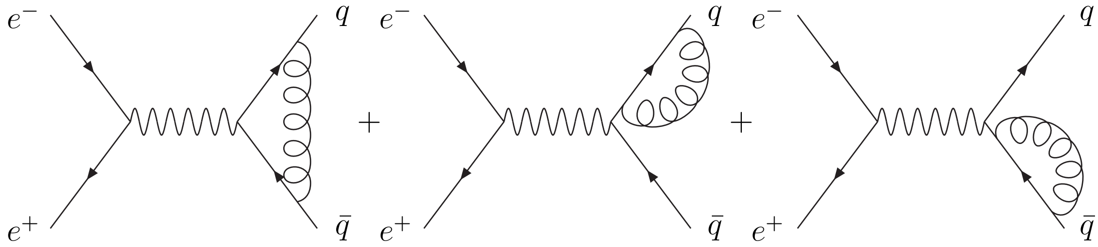
$$\sigma_{\mathrm{tot},\, e^-e^+\to \mathrm{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\, 3\sum_f Q_f^2\,, \tag{42}$$

Where the 3 comes from the number of quark colours and the sum is over the number of active quark flavours. The measurement of this cross-section was an early piece of evidence for quarks and the fact that they had 3 colours. We also see that the cross-section jumps at each quark threshold, though in practice the mass corrections smooth this jump. However the abrupt change from $\sum_f Q_f^2 = 6/9$ below the charm threshold to $\sum_f Q_f^2 = 10/9$ is clearly seen in data.

Working beyond LO we can express the cross-section as

$$\sigma_{\text{tot},\, e^-e^+\to\text{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\, 3 \sum_f Q_f^2 K(\alpha_s(\mu^2), q^2/\mu^2)\,. \tag{43}$$

At LO $K(\alpha_s(\mu^2), q^2/\mu^2) = 1$. At NLO, i.e. at first order in α_s we have corrections to $\sigma_{\text{tot},\, e^-e^+\to q\bar{q}}$ of the form

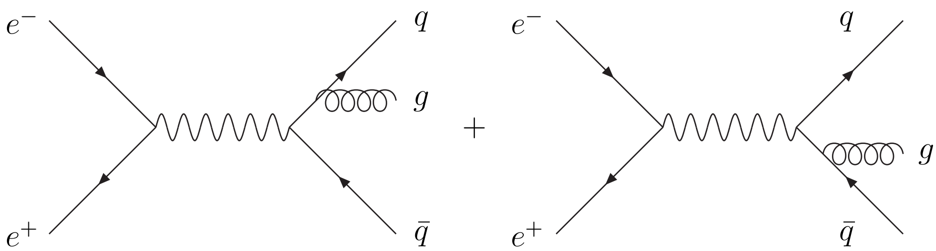


The sum of these diagrams turns out to be ultraviolet finite, and hence independent of any ultraviolet regularization. However, there are infrared divergences coming from the region of integration where the loop momenta $\to 0$, and present because of the masslessness of the quarks and gluon. These can be regularized using dimensional regularization, i.e evaluating loop integrals in $4 + 2\epsilon$ dimensions (it would be $4 - 2\epsilon$ for ultraviolet regularization). The NLO correction to the cross-section for quark-antiquark production is given by the interference of these diagrams with the LO amplitude, and is equal to

$$\sigma^{NLO}_{\text{tot},\, e^-e^+\to q\bar{q}} = \sigma_0 3 \sum_f Q_f^2 \frac{C_F \alpha_s(\mu^2)}{2\pi} H(\epsilon)\Big[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\Big], \tag{44}$$

where $\sigma_0 = \frac{4\pi\alpha^2}{3q^2}$, $C_F = 4/3$ and $H(\epsilon) = 1 + \mathcal{O}(\epsilon)$. Hence this NLO contribution is not well-defined.

However, there are contributions of the same order due to the process $e^-e^+ \to q\bar{q}g$. These come form the matrix elements below.



This can be calculated giving a contribution

$$\sigma_{\mathrm{tot},\,e^-e^+\to q\bar{q}g} = \sigma_0 3\sum_f Q_f^2 \frac{C_F\alpha_s(\mu^2)}{2\pi} H(\epsilon)\Big[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}+\mathcal{O}(\epsilon)\Big]. \tag{45}$$

Hence, whilst the cross-sections for specific final states of quarks and gluons are infrared divergent, it is easy to see that the total NLO contribution to the cross-section obtained by adding (44) and (45) is

$$\sigma_{\mathrm{tot},\,e^-e^+\to hadrons} = \sigma_0 3\sum_f Q_f^2 \frac{\alpha_s(\mu^2)}{\pi}. \tag{46}$$

So up to NLO

$$K(\alpha_s(\mu^2), q^2/\mu^2) = 1 + \frac{\alpha_s(\mu^2)}{\pi}. \tag{47}$$

Comparison with the experimental measurements of the cross-section confirm the variation of $K(\alpha_s(\mu^2), q^2/\mu^2)$ with scale and determine that $\alpha_s(M_Z^2) \approx 0.12$. However, this relies on assuming that the suitable choice for the renormalization scale μ is $\mu^2 \approx q^2$. Since q^2 is the only scale in the problem this seems sensible, but in principle we can choose $\mu^2 = cq^2$ where c is any constant, and could be either large or small. These choices could result in $\alpha_s(\mu^2)$ being either large or small.

In order to at least partially remove this ambiguity, as well as obtain intrinsically higher accuracy, it is necessary to go to next order in $\alpha_s(\mu^2)$. The NNLO calculation for the total cross-section has been calculated, and results in

$$K(\alpha_s(\mu^2), q^2/\mu^2) = 1+\frac{\alpha_s(\mu^2)}{\pi}+\frac{\alpha_s^2(\mu^2)}{\pi^2}\Big(1.99-0.11n_f-\pi\frac{\beta_0}{4\pi}\ln(q^2/\mu^2)\Big). \tag{48}$$

The $\ln(q^2/\mu^2)$ originates from the running of the coupling, and indicates that, indeed, μ^2 should be chosen approximately equal to q^2, otherwise the NNLO correction contains a large logarithm. However, the whole expression is now much more insensitive to the choice of μ^2 than at NLO, the variation of the explicit $\ln(q^2/\mu^2)$ term compensating for that of $\alpha_s(\mu^2)$, and hence more precise. An all-orders calculation would remove the dependence on the arbitrary scale μ^2 completely. In practice $\sigma_{\mathrm{tot},\,e^-e^+\to hadrons}$ is calculated to $\mathcal{O}(\alpha_s^3)$, but this is only the case for a very few quantities.

4 Hadrons in Initial States - Deep Inelastic Scattering

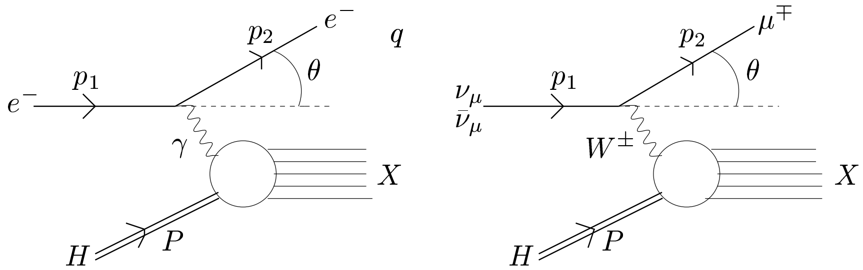
In order to obtain very high energies more easily many particle colliders have hadrons, in particular protons and antiprotons, in the initial state. Thus in order to understand any of the results of these experiments one needs to understand how the incoming hadron is made up from the constituent quarks and gluons, the interactions of which we then know how to calculate using perturbation theory as long as there is a large scale in the process so that perturbation theory is applicable. We can use use deep inelastic scattering (DIS) experiments to probe the structure of hadrons and the fundamental interactions of quarks, gluons, and leptons. In fact, DIS was the first method to directly detect quarks in hadrons. In DIS an elementary particle transfers large energy-momentum to a hadron, which then breaks up inelastically. Essentially it knocks a quark out of the target hadron, which then hadronizes. Similar to the discussion of e^+e^- annihilation, the assumption is that for high energy momentum transfers, corrections to these basic processes from gluon exchange between quarks can be treated in QCD perturbation theory. The hadronization process, where perturbation theory cannot be used, takes place over much longer timescales and larger distance scales than the initial point-like electroweak scattering. However, unlike the previous case we have to consider the nonperturbative initial state.

4.1 Kinematics

We consider DIS of electrons (or neutrinos) as a typical examples. On a hadron H, of mass M, we have in the first case

$$e(p) + H(P) \longrightarrow e(p') + X \,, \tag{49}$$

where X is an arbitrary final state. To lowest order in e, the electron couples to the hadron through a virtual photon,



Deep inelastic electron, neutrino scattering on a hadron

where we also show the corresponding diagram for neutrino interaction via a W. Concentrating first on the electromagnetic scattering process, the lowest order QED amplitude for this is

$$i\mathcal{M} = (ie)^2 \overline{u}(p')\gamma^\mu u(p)\, i\frac{-g_{\mu\nu}}{q^2} \langle X|J_{\rm h}^\nu|H,P\rangle \,, \quad q = p - p' \,. \tag{50}$$

In the hadron rest frame $P = (M, \mathbf{0})$, $p = (E, \mathbf{p})$ and $p' = (E', \mathbf{p}')$. The basic relativistic invariant variables are

$$\nu \equiv P{\cdot}q = M(E - E') \,, \qquad Q^2 \equiv -q^2 = 2p{\cdot}p' = 2EE'(1 - \cos\theta) \,, \tag{51}$$

where we have neglected the electron mass, so that $E = |\mathbf{p}|$, $E' = |\mathbf{p}'|$, and θ is the electron scattering angle. Clearly $Q^2 \geq 0$ and also

$$M_X^2 = (P + q)^2 \geq M^2 \quad \Rightarrow \quad Q^2 \leq 2\nu \,. \tag{52}$$

The standard expression for the differential cross section gives

$$\mathrm{d}\sigma = \frac{1}{F}\,\frac{\mathrm{d}^3 p'}{(2\pi)^3 2p'^0}\,\sum_X (2\pi)^4\delta^4(q+P-p_X)\,\tfrac{1}{2}\sum_{e\ \mathrm{spins}}|\mathcal{M}|^2\,, \tag{53}$$

where F is the flux factor, $F = 4EM$, in the hadron rest frame. From (50)

$$\sum_{e\ \mathrm{spins}}|\mathcal{M}|^2 = \frac{e^4}{(q^2)^2}\,L_{\mu\nu}\,\langle H,P|J_{\mathrm{h}}^{\mu}|X\rangle\,\langle X|J_{\mathrm{h}}^{\nu}|H,P\rangle\,, \tag{54}$$

where, setting $m_e = 0$,

$$L_{\nu\mu} = 4(p_\mu p'_\nu + p_\mu p'_\nu - g_{\mu\nu}\,p{\cdot}p')\,. \tag{55}$$

If we define

$$W_H^{\mu\nu}(q,P) = \frac{1}{4\pi}\sum_X\,(2\pi)^4\delta^4(q+P-p_X)\,\langle H,P|J_{\mathrm{h}}^{\mu}|X\rangle\,\langle X|J_{\mathrm{h}}^{\nu}|H,P\rangle\,, \tag{56}$$

where we implicitly average over the hadron spin in this definition, then the cross section formula (53) becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^3p'} = \frac{e^4}{8(2\pi)^2EME'}\,\frac{1}{(Q^2)^2}\,L_{\mu\nu}W_H^{\mu\nu}(q,P)\,. \tag{57}$$

By virtue of conservation of the electromagnetic current $(p_X-P)_\mu\langle X|J_{\mathrm{h}}^{\mu}|H,P\rangle = 0$, we have $q_\mu W_H^{\mu\nu}(q,P) = q_\nu W_H^{\mu\nu}(q,P) = 0$. The most general Lorentz covariant form compatible with this is

$$W_H^{\mu\nu}(q,P) = \Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big)W_1 + \Big(P^\mu - \frac{P{\cdot}q}{q^2}\,q^\mu\Big)\Big(P^\nu - \frac{P{\cdot}q}{q^2}\,q^\nu\Big)W_2\,, \tag{58}$$

where $W_{1,2}$ are Lorentz scalar functions characteristic of the hadron H, which depend on the two variables Q^2 and ν. In writing (58) we have neglected a possible term involving the ϵ-tensor but this can be excluded by using parity invariance. To calculate the contraction in (57) we may use the fact that current conservation also leads to $L_{\mu\nu}q^\nu = L_{\mu\nu}q^\mu = 0$ so that from (55) and (58) we have,

$$\begin{aligned} L_{\mu\nu}W_H^{\mu\nu}(q,P) &= 8p'{\cdot}p\,W_1 + 4(2p{\cdot}P\,p'{\cdot}P - M^2p'{\cdot}p)W_2 \\ &= 4Q^2\,W_1 + 2M^2(4EE'-Q^2)\,W_2 \end{aligned} \tag{59}$$

where in the second line we have used $p{\cdot}p' = -\frac{1}{2}q^2$, if $m_e = 0$, together with $p{\cdot}P = ME$, $p'{\cdot}P = ME'$. In the limit of large momentum transfer $Q^2 \sim \mathrm{O}(\nu) \to \infty$, define dimensionless variables x, y

$$x = \frac{Q^2}{2\nu}\,, \qquad y = \frac{\nu}{P{\cdot}p} = \frac{\nu}{ME} = 1 - \frac{E'}{E}\,, \tag{60}$$

which stay fixed. It is easy to see that

$$0 \le x \le 1\,, \qquad 0 \le y \le 1\,. \tag{61}$$

Then from eqn.(59)

$$L_{\mu\nu}W_H^{\mu\nu}(q,P) = 8EM\Big(xy\,W_1 + \frac{1-y}{y}\,\nu W_2\Big)\left[1 + O\left(\frac{M^2}{Q^2}\right)\right]\,. \tag{62}$$

Since

$$\mathrm{d}^3p' \to 2\pi\,E'^2\mathrm{d}(\cos\theta)\,\mathrm{d}E' = \pi E'\,\mathrm{d}Q^2\,\mathrm{d}y = 2\pi E'\nu\,\mathrm{d}x\,\mathrm{d}y\,, \tag{63}$$

we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} = \frac{4\pi\alpha^2}{Q^4}\,2ME\Big((1-y)F_2(x,Q^2) + xy^2F_1(x,Q^2)\Big)\left[1 + O\left(\frac{M^2}{Q^2}\right)\right]\,, \tag{64}$$

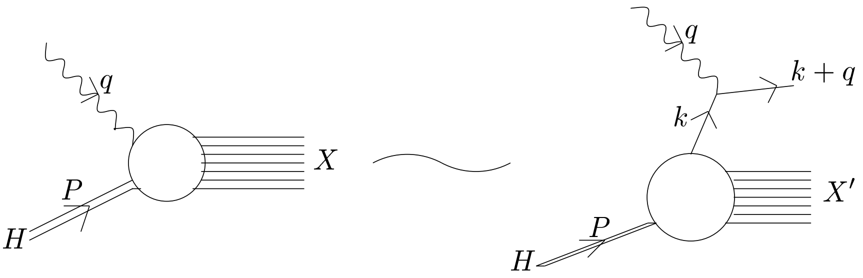
where $\alpha = e^2/4\pi$ and

$$F_2(x,Q^2) = \nu W_2\,, \qquad F_1(x,Q^2) = W_1\,, \tag{65}$$

are dimensionless, frame invariant "structure functions". Clearly comparison of cross section measurements with (64) allows $F_{1,2}$ to be disentangled.

4.2 Factorization

$W_H^{\mu\nu}(q,P)$ could be evaluated exactly if one knew the wavefunctions of $|H\rangle$ and $|X\rangle$ in terms of quark and gluon Fock states. In practice this is a difficult non-perturbative problem. We could apply the same assumption to the final state as in the previous section, i.e. that we sum over final state quarks and gluons rather than hadrons. However, we now have the added complication that the target hadron momentum P satisfies $P^2 = M^2$ which is fixed and the hadron wave function depends on low energy scales. It is necessary to introduce a further factorisation assumption, which can be derived to all orders in the perturbation expansion, in order to justify using the ideas of asymptotic freedom. We use similar physical reasoning to the previous section and assume that is the large momentum transfer from the virtual photon takes place to a single quark which has fluctuated out of the proton over the short time (and distance) scale $\mathcal{O}(1/Q)$, and we can neglect the QCD interactions between hadron constituents due to asymptotic freedom. On a larger time (and distance) scale $\mathcal{O}(1/\Lambda_{QCD})$ the struck quark and the remaining quarks and gluons interact strongly via QCD forces in order to hadronize in the final state, and this processes is largely independent of the former so-called hard process. One can prove in DIS scattering that this factorization holds up to corrections $\mathcal{O}(\Lambda_{QCD}^2/Q^2)$. Within this framework the leading term in the deep inelastic limit is then given in (56) by letting $|X\rangle \to |q_f,\tilde{k}\rangle|X'\rangle$, as illustrated below, where $|q_f,\tilde{k}\rangle$ denotes an on-shell 'parton', either a single quark or anti-quark state with flavour index f and 4-momentum $\tilde{k}$, and $|X'\rangle$ denotes the remnant of the scattered proton.



Parton model for deep inelastic scattering

In order to analyse the behaviour of $W_H^{\mu\nu}(q,P)$ defined in (56) in the deep inelastic limit, $Q^2, \nu \to \infty$ with $x = Q^2/2\nu = \mathrm{O}(1)$, it is very convenient to introduce an alternative basis for 4-vectors which give what are termed light cone variables. For an arbitrary 4-vector V, we define

$$V^{\pm} = V^0 \pm V^3\,, \qquad \mathbf{V}_\perp = (V^1, V^2)\,, \tag{66}$$

and the Lorentz invariant scalar product for two 4-vectors V and U is

$$V{\cdot}U = \tfrac{1}{2}(V^+U^- + V^-U^+) - \mathbf{V}_\perp{\cdot}\mathbf{U}_\perp\,. \tag{67}$$

To discuss $W_H^{\mu\nu}(q,P)$ and $W_H^{\pm\mu\nu}(q,P)$ we choose a frame such that

$$\mathbf{P}_\perp = \mathbf{q}_\perp = \mathbf{0}\,, \tag{68}$$

e.g. we can remain in the hadron rest frame and define $P = (M, \mathbf{0})$ and

$$\begin{aligned} q &= (q^0, 0, 0, q^3) \\ &= (\nu/M, 0, 0, -\sqrt{\nu^2/M^2 + Q^2}) \\ &\approx (\nu/M, 0, 0, -\nu/M - Mx), \end{aligned} \tag{69}$$

where we have simply applied the definitions of the relativistic invariants in eqn.(51) and eqn.(60). In light cone coordinates

$$Q^2 = -q^+q^-\,, \qquad \nu = \tfrac{1}{2}(q^+P^- + q^-P^+)\,. \tag{70}$$

The deep inelastic limit is $q^- \to \infty$ with $q^+ = \mathrm{O}(P^+)$ so that

$$x \to -\frac{q^+}{P^+}\ \text{finite}\,, \qquad \nu \to \tfrac{1}{2}q^-P^+\,. \tag{71}$$

From the above figure we can see that when writing the final state as $|X\rangle \to |q_f, \tilde{k}\rangle|X'\rangle$, in more detail we mean that we may now make the replacement

$$\sum_X \to \sum_f \sum_{X'} \frac{1}{(2\pi)^3} \int \mathrm{d}^4\tilde{k}\, \theta(\tilde{k}^0)\delta(\tilde{k}^2) \sum_{q\ \text{spins}} \tag{72}$$

and in fact $\tilde{k} = k+q$, and this must correspond to an on-shell quark or antiquark. k is the momentum of a quark (or antiquark) inside the proton, and thus should have a very small probability of having any momentum components greater than $\mathcal{O}(\Lambda_{QCD})$. Thus, $(k+q)^0 \approx q^0 > 0$ so $\theta((k+q)^0) = 1$, i.e. the quark satisfies the condition of positive energy. Also using (71), in the DIS limit

$$(k+q)^2 \to q^-(k^+ + q^+) = q^- P^+ \Big(\frac{k^+}{P^+} - x \Big) , \tag{73}$$

and hence

$$\delta((k+q)^2) \sim \frac{1}{q^- P^+} \, \delta\Big(\frac{k^+}{P^+} - x \Big) . \tag{74}$$

Hence, x, which is a relativistic invariant depending on the momentum of the proton target and virtual photon, has the physical interpretation of the fraction of the + component of the hadron momentum carried by the struck quark.

It may then be shown that if we define the parton density functions

$$\begin{aligned} \frac{1}{2P} \int \mathrm{d}^4 k \, \delta\Big(\frac{k}{P} - x \Big) \, \mathrm{tr}\Big(\gamma \Gamma_{H,f}(P,k) \Big) &= q_f(x) , \\ \frac{1}{2P} \int \mathrm{d}^4 k \, \delta\Big(\frac{k}{P} - x \Big) \, \mathrm{tr}\Big(\gamma \overline{\Gamma}_{H,f}(P,k) \Big) &= \overline{q}_f(x) , \end{aligned} \tag{75}$$

where

$$\begin{aligned} \Gamma_{H,f}(P,k)_{\beta\alpha} &= \sum_{X'} \delta^4(P - k - p_{X'}) \, \langle H, P | \overline{q}_{f\alpha} | X' \rangle \, \langle X' | q_{f\beta} | H, P \rangle , \\ \overline{\Gamma}_{H,f}(P,k)_{\beta\alpha} &= \sum_{X'} \delta^4(P - k - p_{X'}) \, \langle H, P | q_{f\beta} | X' \rangle \, \langle X' | \overline{q}_{f\alpha} | H, P \rangle , \end{aligned} \tag{76}$$

and the average over the hadron spins is implicit, we find

$$F_1(x, Q^2) \to \tfrac{1}{2} \sum_f Q_f^2 \Big(q_f(x) + \overline{q}_f(x) \Big) . \tag{77}$$

This demonstrates that F_1 depends only on the dimensionless variable $x = Q^2/2\nu$ in the deep inelastic limit, which is known as Bjorken scaling. The experimental observation of this scaling was the first direct evidence for point-like constituents in hadrons. The quark distribution functions $q_f(x)$, $\overline{q}_f(x)$ defined by (75) for $x \geq 0$ are an intrinsic non-perturbative property of the hadron H. They may be interpreted as momentum distributions for quarks and anti-quarks inside the hadron and in principle (thought not yet in practice) they can be computed from a non-perturbative analysis in QCD. These distribution functions must simply be determined experimentally at present from (mainly) DIS experiments. We also find that

$$F_2(x, Q^2) = 2x F_1(x, Q^2) = x \sum_f Q_f^2 \Big(q_f(x) + \overline{q}_f(x) \Big) . \tag{78}$$

The form of the relation between F_1 and F_2 is a consequence of the spin $1/2$ nature of the struck quark.

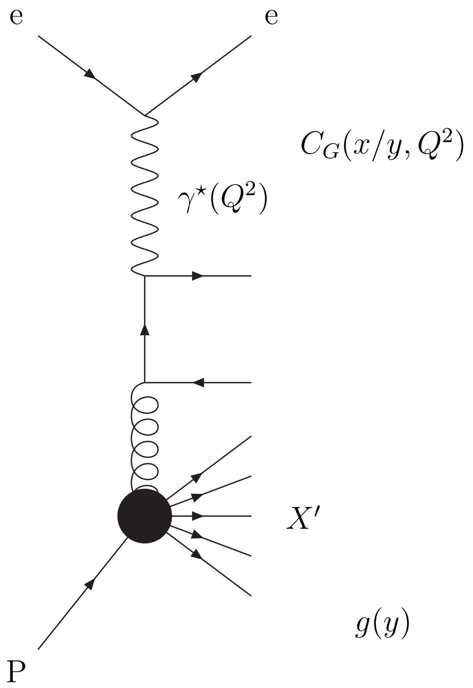
Applying these results to deep inelastic scattering on a proton target say, the proton wavefunction is dominated by $\sim uud + \cdots$ where the dots indicate

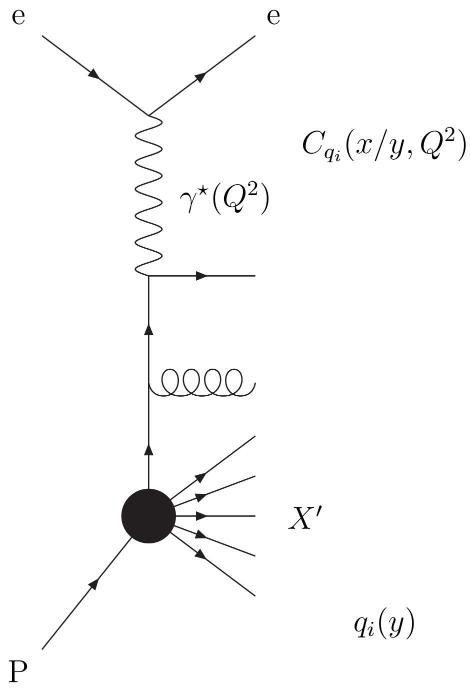
uud plus further quarks (including heavy flavours). With an evident notation $q_u(x) = u(x)$, $\overline{q}_u(x) = \overline{u}(x)$ etc,

$$F_{2,\text{proton}}(x,Q^2) \sim x\Big(\tfrac{4}{9}(u(x)+\overline{u}(x)) + \tfrac{1}{9}(d(x)+\overline{d}(x)) + \text{ heavy flavours}\Big)\,. \quad (79)$$

4.3 QCD Corrections

We have assumed that the quark interacts with the virtual γ for large Q^2 with a point-like coupling, not including any corrections due to QCD. However, there are calculable perturbative QCD corrections to Bjorken scaling. To simplify the discussion, we examine a generic structure function $F(x,Q^2)$, such as might be measured in deep inelastic scattering. The dominant contributions for $Q^2 \to \infty$ arise from the elementary particles of perturbative QCD, quarks and gluons, but QCD corrections are no longer ignored and $F(x,Q^2)$ cannot any more be represented in terms of solely point-like couplings to the quarks. Hence, we recognise that we can now create a number of quarks, antiquarks and gluons in the final state via a hard QCD perturbative process. The point-like vertex is now also replaced by a "coefficient function" $C_i(q,k)$ representing this hard scattering process, where $i = q_f, \overline{q}_f, G$ for an incoming quark, antiquark or gluon with 4-momentum k coupling to a current J carrying 4-momentum q, $q^2 = -Q^2$, and which includes all (perturbative) QCD corrections. Some of the leading α_s corrections to the lowest order diagram are illustrated below





In the relevant limit $Q^2 = -q^2 \to \infty$, $x = Q^2/2\nu$ ($\nu = P{\cdot}q$) fixed, $F(x,Q^2)$ is assumed to have the form of a sum over contributions for different $i = q_f, \overline{q}_f, G$, Taking into account these considerations the expression for the structure function reduces to a single variable integral

$$F(x, Q^2) \sim \sum_{i=q_f,\overline{q}_f,G} \int_x^1 \frac{\mathrm{d}y}{y}\, C_i\Big(\frac{x}{y}, \frac{Q^2}{\mu_F^2}; \alpha_s\Big)\, f_i(y, \mu_F^2)\,, \tag{80}$$

where

$$f_i(y, \mu^2) = \Big(q_f(y, \mu^2), \overline{q}_f(y, \mu^2), G(y, \mu^2)\Big)\,, \quad i = q_f, \overline{q}_f, G\,, \tag{81}$$

and we now integrate over the possible values of the momentum fraction y.

The definition of the parton distributions is the same as in the previous argument except for three points.

1. Now we also have a nonperturbative contribution corresponding to the possibility of scattering off a gluon in the hadron.
2. The momentum fraction of the parton leaving the hadron is denoted by y, where $y \geq x$ since some of the original momentum may be lost by branching to other particles before the scattering with the photon which defines the variable x.
3. The infrared singularities in the coefficient functions which have been regularized by μ_F must be absorbed into the nonperturbative definition of $\Gamma(P, k)$ rendering it μ_F dependent when we include QCD corrections. This is natural because the singularities come from the infrared limit of the integral over k where the coupling is strong and really we should be using nonperturbative physics. The divergences are determined entirely in terms of the incoming parton, and are independent of the particular scattering process as long as it is one which sums over final states, as in $e^+e^- \to$ hadrons (though we can be less inclusive and define final state jets as in this previous case).

It is important to recognise that $F(x, Q^2)$ as a potentially measurable physical quantity must be independent of μ_F. In general for vectors A_i, B_i

$$\mu_F \frac{\mathrm{d}}{\mathrm{d}\mu_F}\Big(A_i B_i\Big) = 0 \quad \Rightarrow \quad \mu_F \frac{\mathrm{d}}{\mathrm{d}\mu_F} A_i = -A_j P_{ji}\,, \quad \mu_F \frac{\mathrm{d}}{\mathrm{d}\mu_F} B_i = P_{ij} B_j\,. \tag{82}$$

The integral convolution in (80) can be regarded similarly as a form of matrix multiplication for two μ_F-dependent factors. The analogous version of the equations for A, B in (82) become integral relations

$$\mu_F \frac{\mathrm{d}}{\mathrm{d}\mu_F} C_i\Big(x, \frac{Q^2}{\mu_F^2}; \alpha_s\Big) = -\sum_{j=q_f,\overline{q}_f,G} \int_x^1 \frac{\mathrm{d}y}{y}\, C_j\Big(y, \frac{Q^2}{\mu_F^2}; \alpha_s\Big)\, P_{ji}\Big(\frac{x}{y}; \alpha_s\Big)\,, \tag{83}$$

$$\mu_F \frac{\mathrm{d}}{\mathrm{d}\mu_F} f_i(y, \mu_F^2) = \sum_{j=q_f,\overline{q}_f,G} \int_y^1 \frac{\mathrm{d}z}{z}\, P_{ij}\Big(\frac{y}{z}; \alpha_s\Big)\, f_j(z, \mu_F^2)\,, \tag{84}$$

where the $P_{ij}(y; \alpha_s)$ are determined by the form of the infrared divergences regularized by μ_f and absorbed into the nonperturbative definition of the partons. As such they are independent of Q^2, the particular current J and the hadron H, and may be determined as an expansion in α_s from (83). In general all components of $P_{ij}(y; \alpha_s)$ are non zero.

The equations (83,84), are referred to as the Altarelli-Parisi equations, and the perturbatively calculable $P_{ij}(y; \alpha_s)$ are known as splitting functions. In these equations we should take $\alpha_s \to \alpha_s(\mu^2)$ the running coupling, which is explicitly given by (26) to lowest order. It is important to note that $\alpha_s(\mu^2)$ is a function of the renormalization scale μ not the factorization scale μ_F since its running is determined by the renormalization of the ultraviolet divergences in the theory and is nothing to do with the infrared regularization which introduces μ_F.

Since μ and μ_F are arbitrary we may choose their values independently. However, it is natural, and very common to set $\mu^2 = \mu_F^2 = Q^2$ so that (80) becomes

$$F(x, Q^2) \sim \sum_{i=q_f, \overline{q}_f, G} \int_x^1 \frac{\mathrm{d}y}{y} \, C_i\Big(\frac{x}{y}, 1; \alpha_s(Q^2)\Big) \, f_i(y, Q^2) \,, \tag{85}$$

where from (84)

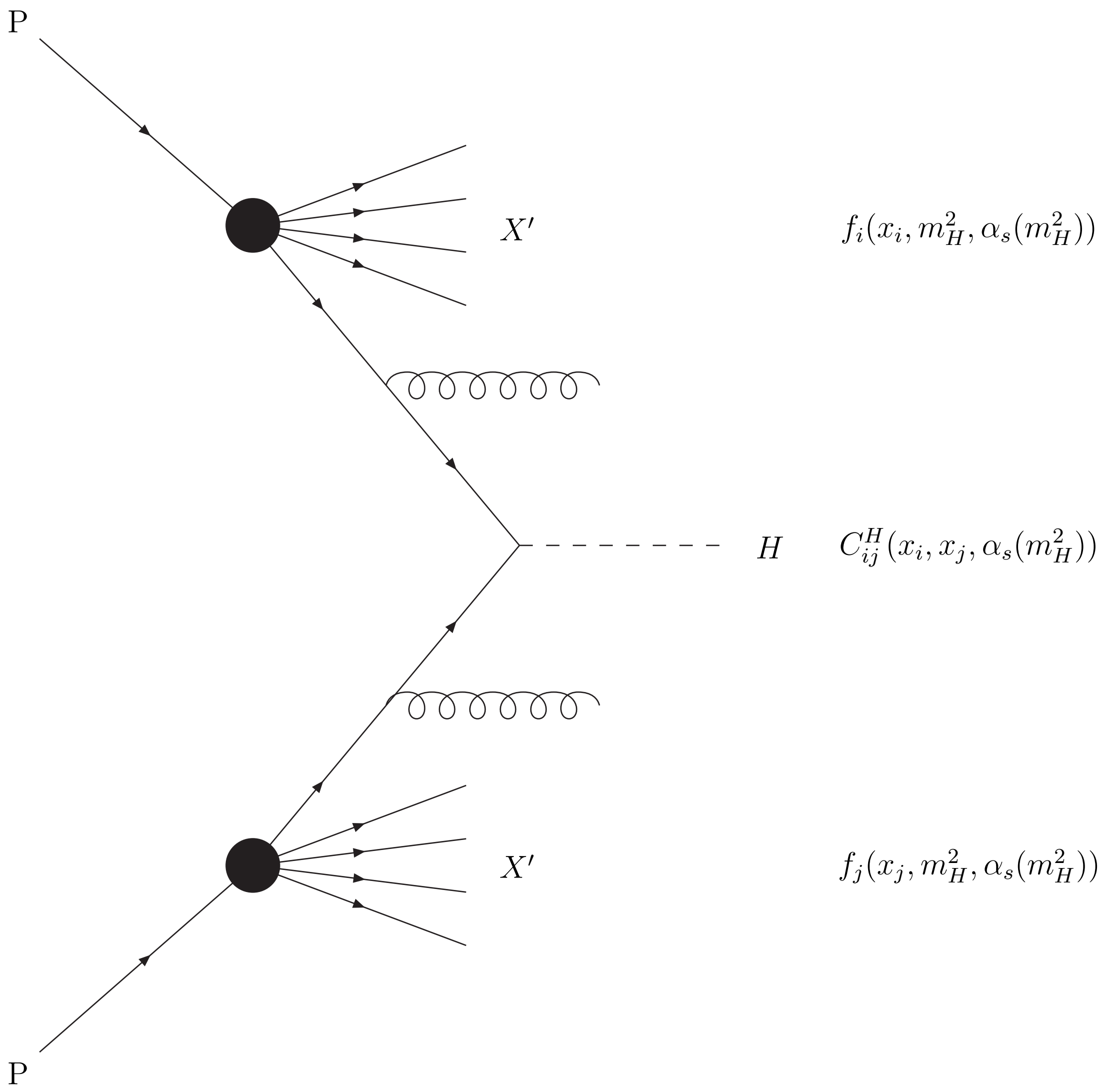
$$Q \frac{\mathrm{d}}{\mathrm{d}Q} f_i(y, Q^2) = \sum_{j=q_f, \overline{q}_f, G} \int_y^1 \frac{\mathrm{d}z}{z} \, P_{ij}\Big(\frac{y}{z}; \alpha_s(Q^2)\Big) \, f_j(z, Q^2) \,. \tag{86}$$

The results (85) and (86) then provide the justification for the claim that asymptotic freedom allows the Q^2 *evolution* of $F(x, Q^2)$ to be calculated perturbatively in the deep inelastic limit. Hence, once we have measured the parton distributions at some low scale Q_0^2 we can calculate their evolution to higher scales perturbatively. Comparison of theory and data on structure functions and their scaling violations works extremely well, and is one of the best tests of QCD.

We can apply the same sort of reasoning as above to hadron-hadron collisions. The coefficient functions $C_i(x, \alpha_s(\mu^2))$ describing a particular hard scattering process involving incoming partons are process dependent but are calculable as a power-series in the strong coupling constant $\alpha_s(\mu^2)$.

$$C^P(x, \alpha_s(\mu^2)) = \sum_k C^{P,k}(x) \alpha_s^k(\mu^2).$$

The scale of the coupling will be set by the hard scale q^2 in the particular process, e.g. if one produces a particle with large mass m in the final state then $q^2 = m^2$. If there is no hard scale in the perturbative scattering process, e.g. if we simply have proton-proton scattering to hadrons with no identified hard final state, perturbation theory cannot be reliably used. Since the parton distributions $f_i(x, q^2)$ are process-independent, i.e. *universal*, once they have been measured at one experiment, one can predict many other scattering processes. Consider for example the diagram for proton-proton scattering to form hadrons plus a Higgs boson, a contribution to which is shown below.



The definition of the parton distributions is exactly the same for this diagram as it is in Deep Inelastic Scattering. Hence, once we calculate $C_{ij}^H(x_i, x_j, \alpha_s(m_H^2))$ we can calculate the cross-section for Higgs production at a proton-proton collider, i.e. the Large Hadron Collider (LHC). This is given simply by

$$\sigma_H(x_1, x_2, m_H^2) = \sum_{i,j=q_f,\overline{q}_f,G} \int_{x_1}^1 \int_{x_2}^1 \frac{\mathrm{d}y_1}{y_1} \frac{\mathrm{d}y_2}{y_2} \, C_{ij}^H\Big(\frac{x_1}{y_1}, \frac{x_2}{y_2} \frac{m_H^2}{\mu^2}; \alpha_s\Big) \, f_i(y_1, \mu^2) f_j(y_2, \mu^2) \,, \tag{87}$$

This general procedure can be applied to any process, so although parton distributions are essentially nonperturbative their determination in a small number of experiments then leads to huge predictive power.