1. $(\gamma^5)^2 = - \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3$

$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ if $\nu \neq \mu$

$= 1 \qquad \nu = \mu = 0$

$= -1 \qquad \nu = \mu = 1, 2, 3$

$\rightarrow (\gamma^5)^2 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^0\gamma^1\gamma^2$

$= -\gamma^0\gamma^1\gamma^2\gamma^0\gamma^1\gamma^2$

$= -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^0\gamma^1$

$= \gamma^0\gamma^1\gamma^0\gamma^1$

$= -\gamma^0\gamma^1\gamma^1\gamma^0 = \gamma^{0\,2} = \underline{\underline{1}}$

For plane wave solutions $\psi = u(p)\, e^{-ip\cdot x}$

$(p\cdot\gamma - m) = 0$

$\therefore (E\gamma^0 - \underline{p}\cdot\underline{\gamma} - m)\, u = 0$

let $u = \begin{pmatrix}\phi \\ \chi\end{pmatrix}$ where ϕ and χ have 2 components

$\therefore \left(E\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} - \underline{p}\cdot\begin{pmatrix}0 & \underline{\sigma} \\ -\underline{\sigma} & 0\end{pmatrix} - m\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}\phi \\ \chi\end{pmatrix} = 0$

$\therefore \begin{pmatrix}-m & E - \underline{\sigma}\cdot\underline{p} \\ E + \underline{\sigma}\cdot\underline{p} & -m\end{pmatrix}\begin{pmatrix}\phi \\ \chi\end{pmatrix} = 0$

$\therefore (E - \underline{\sigma}\cdot\underline{p})\,\chi = \phi m$

$(E + \underline{\sigma}\cdot\underline{p})\,\phi = \chi m$

$(E - \underline{\sigma}\cdot\underline{p})(E + \underline{\sigma}\cdot\underline{p})\,\phi = m^2\phi$

$\rightarrow (E^2 - \underline{p}^2)\,\phi = m^2\phi$ ditto for χ

$u = \begin{pmatrix}\phi \\ \dfrac{E + \underline{\sigma}\cdot\underline{p}}{m}\,\phi\end{pmatrix}$

$m = 0$

$$E\phi = -\underline{\sigma}\cdot\underline{p}\,\phi$$

$$E\chi = \underline{\sigma}\cdot\underline{p}\,\chi$$

but $\quad \underline{p} = \hat{\underline{p}}\,E. \qquad \hat{\underline{p}}$ direction of $\underline{p}$

$\therefore \quad \underline{\sigma}\cdot\hat{\underline{p}}\,\chi = +\chi \qquad$ (positive helicity)

$\qquad \underline{\sigma}\cdot\hat{\underline{p}}\,\phi = -\phi \qquad$ (negative helicity).

$\therefore$ solution decouple into top two components (+ve helicity) or bottom two (−ve helicity).

$$\gamma^5 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} \sigma_1\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_2\sigma_3 \end{pmatrix}$$

$$= i\begin{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \frac{1}{2}\left(1 - \gamma^5\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\frac{1}{2}\left(1 + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore \quad \frac{1}{2}(1+\gamma^5) = P_R$ picks out χ i.e. +ve helicity

$\qquad \frac{1}{2}(1-\gamma^5) = P_L$ picks out ϕ i.e. −ve helicity

As we would expect.

From the general formula for $2 \to 2$ scattering we have, neglecting masses,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} |M|^2$$

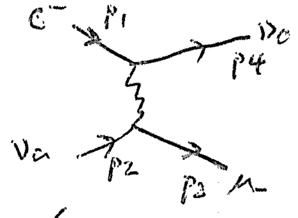
where $|M|^2$ is the appropriate spin-averaged matrix element squared. For neutrino-electron scattering we average over initial electron states and sum over final muon states (neutrinos have a definite helicity). We can, for convenience, sum over both helicity states of the neutrino since the $(1-\gamma_5)$ factors automatically guarantee that right-handed neutrinos make no contribution.

For ~~our~~ example we get.

$$|M|^2 = \left(\frac{G_F^2}{2}\right) \mathrm{Tr}\left(\not{p}_3 \gamma_\alpha (1-\gamma_5) \not{p}_2 \gamma_\nu (1-\gamma_5)\right] \tfrac{1}{2} \mathrm{Tr}\left(\not{p}_4 \gamma^\alpha (1-\gamma_5) \not{p}_1 \gamma^\nu (1-\gamma_5)\right)$$

where e^- has p_1, ν_μ has p_2, μ^- has p_3 and ν_e has p_4.

Therefore defining



$\therefore q = p_1 - p_4$
$t = (p_1 - p_4)^2$
$u = (p_1 - p_3)^2$
$s = (p_1 + p_2)^2$.

$$|M|^2 = \left(\frac{G_F^2}{2}\right) N_{\alpha\nu} E^{\alpha\nu}$$

the $\nu_\mu \to \mu^-$ tensor is $N_{\alpha\nu}$ and is given by

$$N_{\alpha\nu} = \mathrm{Tr}\left[\not{p}_3 \gamma_\alpha (1-\gamma_5) \not{p}_2 \gamma_\nu (1-\gamma_5)\right]$$

and the $e^- \to \nu_e$ tensor $E^{\alpha\nu}$ is

$$E^{\alpha\nu} = \frac{1}{2} \mathrm{Tr}\left[\not{p}_4 \gamma^\alpha (1-\gamma_5) \not{p}_1 \gamma^\nu (1-\gamma_5)\right]$$

and has a factor of $\frac{1}{2}$ for spin averaging.

By commuting the $(1-\gamma_5)$ factor through two γ matrices and using $(1-\gamma_5)^2 = 2(1-\gamma_5)$

$$N_{\alpha\nu} = 2\mathrm{Tr}\left[\not{p}_3 \gamma_\alpha (1-\gamma_5) \not{p}_2 \gamma_\nu\right]$$

$$= 2\mathrm{Tr}\left(\not{p}_3 \gamma_\alpha \not{p}_2 \gamma_\nu\right) - 2\mathrm{Tr}\left(\gamma_5 \not{p}_3 \gamma_\alpha \not{p}_2 \gamma_\nu\right)$$

The first is a standard result.

$$\mathrm{Tr}(\not p_3 \gamma_\mu \not p_2 \gamma_\nu) = 4[p_{3\mu} p_{2\nu} + p_{2\mu} p_{3\nu} - \underbrace{p_2 \cdot p_3}_{= t/2} g_{\mu\nu}]$$

The second trace can be evaluated using

$$\mathrm{Tr}(\gamma_5 \not a \not b \not c \not d) = 4 i \, \epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta$$

We will also need to contract two ϵ tensors in the contraction of the two traces. We can use

$$\epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu\nu\gamma\delta} = -2(\delta^\gamma_\alpha \delta^\delta_\beta - \delta^\delta_\alpha \delta^\gamma_\beta)$$

We now evaluate $N_{\mu\nu}$. This gives

$$N_{\mu\nu} = 8 [(p_{3\mu} p_{2\nu} + p_{3\nu} p_{2\mu} + \frac{t}{2} g_{\mu\nu}) - i \epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_3^\beta]$$

For $E^{\mu\nu}$ we have a similar result (and a factor of $\frac{1}{2}$)

$$E^{\mu\nu} = 4 [(p_4^\mu p_1^\nu + p_4^\nu p_1^\mu + \frac{t}{2} g^{\mu\nu}) - i \epsilon^{\mu\nu\gamma\delta} p_{1\gamma} p_{4\delta}]$$

But we know that $q^\mu N_{\mu\nu} = q^\nu N_{\mu\nu} = 0$ where q is the momentum of the intermediate boson, so we may replace $p_4 = p_1 + q$ and then drop all terms involving $q^{\mu\nu}$ in the contraction with $N_{\mu\nu}$. In the anti symmetric term though, we have

$$\epsilon^{\mu\nu\gamma\delta} p_{1\gamma}(p_{1\delta} + q_\delta) = \epsilon^{\mu\nu\gamma\delta} p_{1\gamma} q_\delta \quad \text{from antisymmetry.}$$

Thus we obtain

$$E^{\mu\nu}_{\text{eff}} = 8 p_1^\mu p_1^\nu + 2t g^{\mu\nu} - 4i \epsilon^{\mu\nu\gamma\delta} p_{1\gamma} q_\delta .$$

We now calculate the contraction it is easiest to use the Mandelstam variables

$$s = 2 p_1 \cdot p_2$$

$$u = -2 p_3 \cdot p_1 \qquad \text{(note convention)}$$

$$t = -2 p_2 \cdot p_3 = q^2 .$$

satisfying $s + t + u = 0$

The result of performing the ~~subtraction~~ contraction.

$N_{(\alpha\beta)} E^{(\alpha\beta)} = N_{(\alpha\beta)} E^{(\alpha\beta)}_{ass}$ results in

$16(s^2+u^2) + 16(s^2-u^2)$,

where the first term is from the contraction of the two symmetric parts and the second from the parts containing ε terms. We have also used

$t = q^2 = -(s+u)$.

Hence, overall we get.

$N_{(\alpha\beta)} E^{(\alpha\beta)} = 32 s^2$.

so since $\dfrac{d\sigma}{dt} = \dfrac{1}{16\pi s^2}\left(\dfrac{G_F^2}{2}\right) N_{\alpha\beta} E^{\alpha\beta}$

we obtain $\dfrac{d\sigma}{dt} = \dfrac{G_F^2}{\pi}$

But $t = -2p^2(1-\cos\theta)$ where p is the centre of mass momentum and θ the scattering angle, so integrating over the final phase space

$$\sigma = \frac{G_F^2}{\pi} p^2 \times 4 = \underline{\underline{\frac{G_F^2 s}{\pi}}}$$

6.

$$\mathcal{L} = \frac{1}{4} \underline{F}^{\mu\nu} \cdot \underline{F}_{\mu\nu} + (D^\mu \phi)^\dagger (D_\mu \phi) - \frac{1}{2} \lambda (\phi^\dagger \phi - \tfrac{1}{2} v^2)^2$$

$$\underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + g \, \underline{A}_\mu \times \underline{A}_\nu$$

$$D_\mu \phi = \partial_\mu \phi - i g \, \underline{A}_\mu \cdot \tfrac{1}{2} \underline{\tau} \, \phi$$

minimum in V when $\phi_0 \phi_0 = v^2/2$.

May choose any minimum e.g.

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Because we are expanding about a non-zero vacuum expectation value can write

$$\phi = e^{i \underline{\tau} \cdot \underline{\theta}(x)/v} \begin{pmatrix} 0 \\ \frac{v+\xi}{\sqrt{2}} \end{pmatrix}$$

since this corresponds to the 4 independent degrees of freedom in the scalar doublet, but each field is a perturbation about the vacuum, i.e. for small $\underline{\theta}$

$$\phi \sim \begin{pmatrix} 0 \\ \frac{v+\xi}{\sqrt{2}} \end{pmatrix} + i \frac{\underline{\tau} \cdot \underline{\theta}(x)}{v} \begin{pmatrix} 0 \\ \frac{v+\xi}{\sqrt{2}} \end{pmatrix} + \ldots$$

$$= \begin{pmatrix} 0 \\ \frac{v+\xi}{\sqrt{2}} \end{pmatrix} + i \frac{\underline{\tau} \cdot \underline{\theta}(x)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

+ non linear (interaction) terms.

However since $e^{i\underline{\tau}\cdot\theta(x)/v}$ is of the form of an $SU(2)$ gauge transformation we can choose the gauge by letting

$$\phi \to \phi' = u\,\phi$$

where $u = e^{-i\underline{\tau}\cdot\underline{\theta}(x)/2}$.

In this "unitary gauge"

$$\phi = \begin{pmatrix} 0 \\ \frac{v+\xi}{\sqrt{2}} \end{pmatrix}$$

and we fix the gauge to be this in which case the particle content is explicit.

In this gauge

$$\tfrac{1}{2}\,\underline{\tau}\cdot\underline{A_\mu} \to u\,\tfrac{1}{2}\underline{\tau}\cdot\underline{A_\mu}\,u^{-1} - \frac{i}{g}(\partial_\mu u)\,u^{-1}$$

which we define by $\frac{1}{2}\,\underline{\tau}\cdot\underline{B_\mu}$

and so $D_\mu\phi = \partial_\mu\phi - ig\,\underline{B_\mu}\cdot\frac{1}{2}\underline{\tau}\,\phi$

The vacuum $\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$ does not satisfy $\underline{\tau}\,\phi_0 = 0$ for any linear combination of the $\underline{\tau}$'s so the ~~vacuum~~ symmetry is completely broken.

Now $\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \frac{1}{4} \underline{F} \cdot \underline{F}$

$$\underbrace{- \frac{1}{2} \lambda \left(\phi^\dagger \phi - \frac{1}{2} v^2 \right)^2}$$

$$- \frac{1}{2} \lambda \left(\frac{1}{2} (v + \xi)^2 - \frac{1}{2} v^2 \right)^2 = - \frac{\lambda v^2}{2} \xi^2 + O(\xi^3)$$

$\therefore$ mass of scalar $= \underline{\underline{\sqrt{\lambda}\, v}}$.

$(D_\mu \phi)^\dagger (D^\mu \phi)$ contains terms quadratic in B_μ.

$$\Rightarrow \frac{ig}{2} (0, v+\xi) \left(\frac{1}{2} \underline{\tau} \cdot \underline{B}_\mu \right) \left(-ig \frac{1}{2} \underline{\tau} \cdot \underline{B}^\mu \right) \begin{pmatrix} 0 \\ v+\xi \end{pmatrix}$$

$$\therefore \text{mass term} = \frac{g^2 v^2}{8} (\underline{\tau} \cdot \underline{B}_\mu)(\underline{\tau} \cdot \underline{B}^\mu).$$

$$= \frac{g^2 v^2}{8} \left[(\tau_i \cdot B_i^\mu)(\tau_j B_{j\mu}) \right]$$

$$= \frac{g^2 v^2}{8} \left[\frac{1}{2} B_i^\mu B_{\mu j} \underbrace{(\tau_i \tau_j + \tau_j \tau_i)}_{2\delta_{ij}} \right]$$

$$= \frac{g^2 v^2}{8} \underline{B}^\mu \cdot \underline{B}_\mu \equiv \frac{1}{2} M_V^2 \underline{B}^\mu \cdot \underline{B}_\mu$$

$$\therefore M_V = \underline{\underline{\frac{g v}{2}}}$$

4. The j^μ couples to A_μ via a term $j^\mu A_\mu$ if j^μ is the electromagnetic current.

Hence we wish to identify the A_μ term in $\mathcal{L}_{W,A}$.

$$F^W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ie A_\mu W_\nu + ie A_\nu W_\mu. \quad \text{as in notes}$$

$$\therefore -\tfrac{1}{2} F_{\mu\nu}^{W+} F^{W\mu\nu} = -\tfrac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + ie A_\mu W_\nu^+ - ie A_\nu W_\mu^+)$$
$$\times (\partial^\mu W^\nu - \partial^\nu W^\mu - ie A^\mu W^\nu + ie A^\nu W^\mu)$$

$$= -\tfrac{1}{2}\Big([\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+][-ie A^\mu W^\nu + ie A^\nu W^\mu]$$

$$+ [ie A_\mu W_\nu^+ - ie A_\nu W_\mu^+][\partial^\mu W^\nu - \partial^\nu W^\mu] + \text{other}$$

terms which do not contribute to term linear in A^μ $\Big)$.

$$= -\tfrac{1}{2} A^\mu \Big[-ie\,(W^\nu \partial_\mu W_\nu^+ - W^\nu \partial_\nu W_\mu^+)\, W^\mu$$
$$+ ie(\partial_\nu W_\mu^+ - \partial_\mu W_\nu^+) W^\nu + ie W^{\nu +}(\partial_\mu W^\nu - \partial^\nu W_\mu)$$
$$- ie\, W_\nu^+ (\partial^\nu W_\mu - \partial_\mu W^\nu) \Big].$$

$$= -\frac{ie}{2} A^\mu \Big[2\,(\partial_\nu W_\mu^+ W^\nu - \partial_\mu W_\nu^+ W^\nu)$$
$$+ 2\,(W_\nu^+ \partial_\mu W^\nu - W_\nu^+ \partial^\nu W_\mu) \Big] \quad \left(A \overleftrightarrow{\partial_\mu} B = A\partial_\mu B - (\partial_\mu A) B \right).$$

$\therefore$ contribution of W to j_μ from $(F^W)^2$ is

$$-ie\,(W_\nu^+ \overleftrightarrow{\partial_\mu} W^\nu - W_\nu^+ \partial^\nu W_\mu + \partial_\nu W_\mu^+ W^\nu)$$

But we also have parts from the term $i W^\mu W^{\nu +} e F^A_{\mu\nu}$ in eq. 29 in part C of the notes.

$$ie W^\mu W^{\nu +} F^A_{\mu\nu} = ie W^\mu W^{\nu +}(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

integrate by parts under the $\int d^4x$ integral.

$$\rightarrow -ie\, \partial_\mu (W^\mu W^{\nu +}) A_\nu + ie\, \partial_\nu (W^\mu W^{\nu +}) A_\mu.$$

$$= (-ie\, \partial_\nu (W^\nu W^{\mu +}) + ie\, \partial_\nu (W^\mu W^{\nu +})) A_\mu$$

$$= \big(-ie\,(\partial_\nu W^\nu W^{\mu +} + W^\nu \partial_\nu W^{\mu +})$$
$$+ ie\,(\partial_\nu W^\mu\, W^{\nu +} + W^\mu \partial_\nu W^{\nu +})\big) A_\mu.$$

Hence $j^\mu = -ie\, W_\nu^\dagger \overleftrightarrow{\partial_\mu} W^\nu + 2ie\,(W_\nu^\dagger \partial^\nu W_\mu - \partial_\nu W_\mu^\dagger W^\nu)$

$+ ie\,(W_\mu \partial_\nu W^{\nu\dagger} - \partial_\nu W^\nu W_\mu^\dagger)$.

$= -ie\, W_\nu^\dagger \overleftrightarrow{\partial_\mu} W^\nu + 2ie\,\partial_\nu (W^{\nu\dagger} W_\mu - W_\mu^\dagger W^\nu)$

$- ie\,(\partial_\nu W^{\nu\dagger} W_\mu - W_\mu^\dagger \partial_\nu W^\nu)$.

(Note, if the W is "on mass shell", i.e. describes a physical particle, then $\partial_\mu W^\mu = 0$ and the last term disappears).

5. Right-handed fermions are $SU(2)$ singlets so

$Tr(\{T_a^R, T_b^R\} T_c^R) = 2\,Tr(Y^{R3})$

For left-handed fermions,

$Tr(\{\sigma_a, \sigma_b\}, \sigma_c) = 0$

from tracelessness of $SU(2)$ generators

Same result if two of the T^L's are hypercharge generators.

Leaves

$2\,Tr(Y^{L3})$ and $Tr(\{\sigma^a, \sigma^b\} Y)$ and $Tr(\{\sigma^a, Y\}\sigma^b)$

The last two are identical due to cyclic property of trace and Y commuting with σ.

$\therefore$ need

$2\,Tr(Y^{R3}) - 2\,Tr(Y^{L3}) = 0$

and $Tr(Y^L\{\sigma^a, \sigma^b\}) = 0$

{Only need linear sum $= 0$ in fact, but each are true separately.}

$Tr(Y^L\{\sigma^a, \sigma^b\}) = Tr(Q^L\{\sigma^a, \sigma^b\})$

since $Q = Y + T^3$

$Tr\{\sigma^a, \sigma^b\} = 0 \quad a \neq b$

$\therefore$ reduces to $Tr(Q^L) = 0$

Remembering 3 colours of quark

$Tr(Q^L) = 3(Q_u + Q_d) + Q_e$

$= 3\left(\frac{2}{3} - \frac{1}{3}\right) - 1 = 0$

Similar for $Tr(Y^{R3}) - Tr(Y^{L3})$.

though can rewrite as

$$Tr\,(Y_L\,(\sigma^3)^2) = 0$$

since

$$Tr\,(Q_L^3 - Q_R^3) = 0 \quad \text{since } Q_L = Q_R$$

$$\therefore \quad Tr\,((\sigma^3 + Y^L)^3 - Y^{R3}) = 0$$

$$\therefore \quad \underbrace{Tr\,((\sigma^3)^3)}_{=0} + 3\,Tr\,(Y^L(\sigma^3)^2) + \underbrace{3\,Tr\,(Y^{L2}\sigma^3)}_{=0}$$

$$+\ Tr\,(Y^{L3} - Y^{R3}) = 0 \quad \text{automatically}$$

$$\therefore \quad Tr\,(Y^{L3} - Y^{R3}) = 0 \ \text{ implies } \ Tr\,(Y^L(\sigma^3)^2) = 0$$

but $Y^L = Q^L \mp \sigma^3$

$$\therefore \quad Tr\,(Y^L(\sigma^3)^2) = Tr\,(Y^L(\sigma^3)^2) \mp \underbrace{Tr\,((\sigma^3)^3)}_{=0}$$

$$\rightarrow \quad Tr\,Q_L = 0$$

$\therefore$ two conditions equivalent.

6. Consider MM^+ for complex matrix M.

$$(MM^+)^+ = (M^+)^+ M^+ = MM^+$$

$\therefore$ MM^+ is hermitian

Let $\underline{V}$ be an eigenvector of MM^+

i.e $MM^+\underline{V} = \lambda \underline{V}$

$\therefore$ $\underline{V}^+ MM^+ \underline{V} = \lambda \underline{V}^+\underline{V}$

For a hermitian matrix eigenvalues are real and eigenvectors orthogonal.

$\therefore$ $\underline{V}^+ MM^+ \underline{V} = \lambda$

But $\underline{M^+ V} =$ vector $\underline{u}$

$\therefore$ $\underline{u}^+ = (M^+\underline{V})^+ = \underline{V}^+ M$

$\therefore$ $\lambda = \underline{u}^+ . \underline{u}$ for some vector $\underline{u}$

$\therefore$ $\underline{\underline{\lambda \geqslant 0.}}$

We let $V^+ MM^+ V = \Delta$ for unitary V and Δ is the matrix of eigenvalues, i.e. V is the matrix of eigenvectors $\underline{V}$

Define $M^+ V = M_d^+$ so $M_d = V^+ M$.

$\therefore$ $\Delta = M_d M_d^+ = \begin{pmatrix} m_1^2 & & 0 \\ & m_2^2 & \\ 0 & & \ddots \; m_n^2 \end{pmatrix} = M_d^2$

We can then introduce a matrix $S = \begin{pmatrix} e^{i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_n} \end{pmatrix}$

and $(VS)^+ (MM^+)(SV) = M_d^2 = \Delta$

Defining $H = V M_d V^+$ and $U = H^{-1} M$.

$U^+ = M^+ H^{-1}$ since H is hermitian.

$\therefore$ $UU^+ = H^{-1} MM^+ H^{-1}$

$= H^{-1} V M_d^2 V^+ H^{-1}$

$$= H^{-1} (V\, m_d\, V^{\dagger} V\ V\, m_d\, V^{\dagger}) H^{-1}$$

$$= H^{-1} (V m_d V^{\dagger}) (V m_d V^{\dagger}) H^{-1}$$

$$= \underline{\underline{H^{-1}\ H H\ H^{-1} = 1}}$$

$$V^{\dagger} M W = V^{\dagger} M u^{\dagger} V$$

But $H u = M$ (since $u = H^{-1} M$)

$\therefore$ $H = M u^{\dagger}$

$$\therefore\ V^{\dagger} M W = V^{\dagger} H V$$

$$= V^{\dagger} V\ m_d V^{\dagger} V$$

$$= m_d \qquad \text{where } W = u^{\dagger} V$$

So $W^{\dagger} = V^{\dagger} u$

$$\therefore\ W^{\dagger} W = V^{\dagger} u u^{\dagger} V = V^{\dagger} V = 1$$

Hence any complex M can be diagonalised to have real positive eigenvalues using two unitary matrices V and W.

$$\mathcal{L}_m = -\left(\left(\bar{q}_+ m_+ \tfrac{1}{2}(1+\gamma_5) q_+\right) + \bar{q}_- m_- \tfrac{1}{2}(1+\gamma_5) q_-\right) + \text{hermitian conjugate.}$$

But $\frac{1}{2}(1+\gamma^5) = \frac{1}{2}(1+\gamma^5) \cdot \frac{1}{2}(1+\gamma^5) = P_R^2$

and $\gamma^0 P_R = P_L \gamma^0$

$\therefore \mathcal{L}_m = -(\bar{q}_{L+} m_+ q_{R+} + \bar{q}_{L-} m_- q_{R-}) + h.c.$

where $q_{R+} = \begin{pmatrix} u_R' \\ c_R' \end{pmatrix}$ and $q_{R-} = \begin{pmatrix} d_R' \\ s_R' \end{pmatrix}$

where u' etc are weak eigenstates.

We can make a redefinition

$A = |A| e^{i\lambda_A}$, $C = |C| e^{i\lambda_C}$

and redefine fields $c' \to c' e^{-i\lambda_A}$ $(\bar{c}' \to \bar{c}' e^{+i\lambda_A})$

$s' \to s' e^{-i\lambda_C}$ $(\bar{s}' \to \bar{s}' e^{+i\lambda_A})$

which means $m_+ \to \begin{pmatrix} 0 & |A| \\ |A| & B \end{pmatrix}$ $m_- \to \begin{pmatrix} 0 & |C| \\ |C| & D \end{pmatrix}$.

Let us ~~equate~~. Look at

$$R_+(\theta_+) \begin{pmatrix} 0 & |A| \\ |A| & B \end{pmatrix} R_-(\theta_+)^{-1}$$

$$= \begin{pmatrix} \cos\theta_+ & \sin\theta_+ \\ \sin\theta_+ & -\cos\theta_+ \end{pmatrix} \begin{pmatrix} 0 & |A| \\ |A| & B \end{pmatrix} \begin{pmatrix} \cos\theta_+ & -\sin\theta_+ \\ \sin\theta_+ & \cos\theta_+ \end{pmatrix}$$

$$= \begin{pmatrix} |A|\sin\theta_+ & |A|\cos\theta_+ + B\sin\theta_+ \\ -|A|\cos\theta_+ & |A|\sin\theta_+ - B\cos\theta_+ \end{pmatrix} \begin{pmatrix} \cos\theta_+ & -\sin\theta_+ \\ \sin\theta_+ & \cos\theta_+ \end{pmatrix}$$

we want this to be equal to $\begin{pmatrix} m_u & 0 \\ 0 & m_c \end{pmatrix}$

$\therefore 2|A|\cos\theta_+ \sin\theta_+ + B\sin^2\theta_+ = m_u$ (1)

$2|A| \cos\theta_+ \sin\theta_+ - B\cos^2\theta_+ = m_c$ (2)

$\pm(-|A|\sin^2\theta_+ + |A|\cos^2\theta_+ + B\cos\theta_+\sin\theta_+) = 0$ (3)

From (3) $|A| = \dfrac{B\sin\theta_+\cos\theta_+}{\sin^2\theta_+ - \cos^2\theta_+}$

Substituting in (1) + (2).

$$2B \frac{\sin^2\theta_+ \times \cos^2\theta_+}{\sin^2\theta_+ - \cos^2\theta_+} + B\sin^2\theta_+ = m_u$$

$$2B \frac{\sin^2\theta_+ \times \cos^2\theta_+}{\sin^2\theta_+ - \cos^2\theta_+} - B\cos^2\theta_+ = m_c$$

$$\therefore \frac{m_c}{m_u} = \frac{2\frac{\sin^2\theta_+ \times \cos^2\theta_+}{\sin^2\theta_+ - \cos^2\theta_+} - \cos^2\theta_+}{2\frac{\sin^2\theta_+ \times \cos^2\theta_+}{\sin^2\theta_+ - \cos^2\theta_+} + \sin^2\theta_+}$$

$$= \frac{\sin^2\theta_+ \cos^2\theta_+ + \cos^4\theta_+}{\sin^2\theta_+ \cos^2\theta_+ + \sin^4\theta_+}$$

$$= \frac{1 + \cos^2\theta_+/\sin^2\theta_+}{1 + \sin^2\theta_+/\cos^2\theta_+} = \frac{1+\cot^2\theta_+}{1+\tan^2\theta_+}$$

$$= \frac{\sec^2\theta_+}{\operatorname{cosec}^2\theta_+} = \frac{1}{\tan^2\theta_+}$$

$$\therefore \frac{m_u}{m_c} = \tan^2\theta_+.$$

$$\theta^+ = \tan^{-1}\left(\frac{m_u}{m_c}\right)^{\frac{1}{2}}.$$

But the Cabibbo matrix is

$$V = R_+(\theta_+) R_+^{+}(\theta_-).$$

and similar arguments to above show that

$$\theta^- = \tan^{-1}\left(\frac{m_d}{m_s}\right)^{\frac{1}{2}}$$

$$V = R_+(\theta_+) R_+^{+}(\theta_-)$$

$$= \begin{pmatrix} \cos\theta_+ & \sin\theta_+ \\ \sin\theta_+ & -\cos\theta_+ \end{pmatrix}\begin{pmatrix} \cos\theta_- & \sin\theta_- \\ \sin\theta_- & -\cos\theta_- \end{pmatrix} = \Big(\cancel{\cos\theta_+\cos\theta_- + \sin\theta_+\sin\theta_-}$$

$$= \begin{pmatrix} \cos\theta_+\cos\theta_- + \sin\theta_+\sin\theta_- & \cos\theta_+\sin\theta_- - \sin\theta_+\cos\theta_- \\ \sin\theta_+\cos\theta_- - \cos\theta_+\sin\theta_- & \sin\theta_+\sin\theta_- + \cos\theta_+\cos\theta_- \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_+ - \theta_-) & \sin(\theta_- - \theta_+) \\ -\sin(\theta_- - \theta_+) & \cos(\theta_+ - \theta_-) \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

where θ_c is the Cabibbo Angle.

$$\therefore \quad \theta_c = \theta_- - \theta_+$$

$$= \tan^{-1}\left(\frac{m_d}{m_s}\right)^{\frac{1}{2}} - \tan^{-1}\left(\frac{m_u}{m_c}\right)^{\frac{1}{2}}$$

$$\approx \tan^{-1}(0.26) - \tan^{-1}(0.058)$$

$$\approx 11.5^\circ$$ so not that close to 22°.