2017 Standard Model Part II R.S. Thorne

1. The four dimensional 4×4 Dirac matrices are defined uniquely up to an equivalence by $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1$, with 1 the unit matrix. We also usually require that if $\gamma = (\gamma^0, \boldsymbol{\gamma})$ then $\gamma^\dagger = (\gamma^0, -\boldsymbol{\gamma})$. If $[X, \gamma^\mu] = 0$ for all μ then $X \propto 1$. Define the charge conjugation matrix C by $C\gamma^{\mu\,t}C^{-1} = -\gamma^\mu$, where t denotes the matrix transpose. Show that $[C^tC^{-1}, \gamma^\mu] = 0$ and hence that $C^t = cC,\ c = \pm 1$. Derive the results

$$\left(\gamma^\mu C\right)^t = -c\gamma^\mu C\,, \qquad \left(\gamma_5 C\right)^t = c\gamma_5 C\,,$$
$$\left(\gamma^\mu\gamma_5 C\right)^t = c\gamma^\mu\gamma_5 C\,, \qquad \left([\gamma^\mu,\gamma^\nu]C\right)^t = -c[\gamma^\mu,\gamma^\nu]C\,.$$

Hence, since there are 6 independent antisymmetric and 10 symmetric 4×4 matrices, show that we must take $c = -1$ (note: the 16 matrices $1, \gamma^\mu, \gamma^\mu\gamma_5, \gamma_5, [\gamma^\mu, \gamma^\nu]$ form a basis of all 4 x 4 matrices, as does multiplying them by the non-singular matrix C). Show also, using the assumed hermiticity properties of the Dirac matrices, that $[\gamma^\mu, CC^\dagger] = 0$, so that we may take $CC^\dagger = 1$.

2. A Dirac quantum field transforms under parity so that

$$\hat{P}\psi(x)\hat{P}^{-1} = \gamma^0\psi(x_P)\,, \quad x_P^\mu = (x^0, -\mathbf{x})\,.$$

Suppose it has an interaction Lagrangian density with a scalar field $\phi(x)$

$$\mathcal{L}_I(x) = g\,\overline{\psi}(x)\psi(x)\phi(x) + g'\overline{\psi}(x)i\gamma_5\psi(x)\phi(x)\,.$$

How must $\phi(x)$ transform under parity, i.e. obtain the necessary form for $\hat{P}\phi(x)\hat{P}^{-1}$, to ensure that the Lagrangian is invariant under parity if $g' = 0$? What are the transformation properties of $\phi(x)$ for parity invariance when $g = 0$. Can parity be conserved in a theory if both g, g' are non zero? Repeat this kind of analysis for the interaction with a vector field V_μ

$$\mathcal{L}_I(x) = g\,\overline{\psi}(x)\gamma^\mu\psi(x)V_\mu(x) + g'\overline{\psi}(x)i\gamma_5\gamma^\mu\psi(x)V_\mu(x)\,.$$

3. In the Standard Model the coupling of leptons to the charged W field is

$$\mathcal{L}_I = -\frac{g}{2\sqrt{2}}(J^\mu W_\mu + \text{hermitian conjugate})$$

$$J^\mu = \bar{\nu}_e\gamma^\mu(1-\gamma^5)e + \bar{\nu}_\mu\gamma^\mu(1-\gamma^5)\mu + \bar{\nu}_\tau\gamma^\mu(1-\gamma^5)\tau.$$

Describe how at low energies this leads to the standard form for lepton weak interactions with an overall coupling $G_F/\sqrt{2} = g^2/8M_W^2$.
Calculate the total decay rate for $W^+ \to e^+\nu_e$, neglecting m_e, and using the result for the sum over polarizations $\sum_\lambda \epsilon^*_\mu(p,\lambda)\epsilon_\nu(p,\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}$, obtaining

$$\Gamma_{W^+\to e^+\nu_e} = \frac{G_F}{\sqrt{2}}\frac{M_W^3}{6\pi}$$

Explain why $\Gamma_{W^+\to e^+\nu_e} \sim \Gamma_{W^+\to\mu^+\nu_\mu}$ although $\Gamma_{K^+\to e^+\nu_e} \ll \Gamma_{K^+\to\mu^+\nu_\mu}$.

4. From Maxwell's equations $\partial_\nu F^{\mu\nu} = e\overline{\psi}\gamma^\mu\psi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, derive the required transformation properties of $A_\mu(x)$ to ensure invariance of the form of these equations under parity, charge conjugation and time reversal.

5. Beyond leading order the proton structure functions can be written as

$$F(x,Q^2) = \sum_{i=q_f,\bar{q}_f,G} \int_x^1 \frac{dy}{y}\, C_i\Big(\frac{x}{y};\alpha_s(Q^2)\Big)\, f_i(y,Q^2)\,,$$

and the parton distributions evolve according to

$$Q\frac{d}{dQ} f_i(y,Q^2) = \sum_{j=q_f,\bar{q}_f,G} \int_y^1 \frac{d}{z} z\, P_{ij}\Big(\frac{y}{z};\alpha_s(Q^2)\Big)\, f_j(z,Q^2)\,.$$

If $f(x)$ is defined on $0 \le x \le 1$ then its moments f^N are defined by

$$\int_0^1 dx\, x^{N-1} f(x) = f^N\,, \quad N = 0,1,2,\ldots\,.$$

If $g(x)$ is similarly defined show that

$$\int_0^1 dx\, x^{N-1} \int_x^1 \frac{dy}{y}\, f(x/y)\, g(y) = f^N g^N\,,$$

and if we therefore let

$$F(x,Q^2) \to F^N(Q^2)\,, \quad f_i(x,Q^2) \to f_i^N(Q^2)\,, \quad P_{ij}(x;\alpha_s(Q^2)) \to P_{ij}^N(\alpha_s(Q^2))\,,$$

then

$$F^N(Q^2) = \sum_{i=q_f,\bar{q}_f,G} C_i^N(\alpha_s(Q^2)) O_i^N(Q^2)$$

$$Q\frac{d}{dQ} f_i^N(Q^2) = \sum_{j=q_f,\bar{q}_f,G} P_{ij}^N(\alpha_s(Q^2)) f_j^N(Q^2)\,.$$

For the flavour-nonsinglet valence distributions $q_V(x,Q^2)$ there is no mixing in the parton evolution so at leading order in $\alpha_S(Q^2)$ the evolution equation can be written

$$Q\frac{d}{dQ}\, q_V^N(Q^2) = -\frac{\alpha_s(Q^2)}{4\pi}\,\gamma_q^N\, q_V^N(Q^2)\,, \qquad \alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln\frac{Q^2}{\Lambda^2}}\,.$$

Show that the solution is then of the form

$$q_V^N(Q^2) = \left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{\frac{\gamma_q^N}{2\beta_0}} q_V^N(Q_0^2)\,.$$

What particular condition must γ_q^N satisfy in order that the number of valence quarks is conserved by evolution?