Standard Model Part II

1. $[C^\dagger C^{-1}, \gamma^{\mu}] = C^\dagger C^{-1}\gamma^{\mu} - \gamma^{\mu} C^\dagger C^{-1}$
$= -C^\dagger \gamma^{\mu t} C^{-1} + C^\dagger \gamma^{\mu t} C^{-1} = 0$

since $C\gamma^{\mu t} = -\gamma^{\mu} C$ $\therefore$ $\gamma^{\mu} C^t = -C^t \gamma^{\mu t}$
and $\gamma^{\mu t} C^{-1} = -\cancel{C}^{-1}\gamma^{\mu}$

$\therefore$ $\underline{C^t C^{-1} \propto I = cI}$

$\therefore$ $C^t = cC$

$(C^t)^t = \underline{cC^t = C}$

But since $C^t = cC$ $C = C^t/c$

$\therefore$ $\underline{c^2 = 1 \quad \therefore c = \pm 1}$

$(\gamma^{\mu} C)^t = C^t \gamma^{\mu t} = cC\gamma^{\mu t} = \underline{c(-\gamma^{\mu} C)}$

$(\gamma_5 C)^t = C^t \gamma_5^t$
$= cC\, i\gamma^{3t}\gamma^{2t}\gamma^{1t}\gamma^{0t}$
$= ci\,(C\gamma^{3t})\gamma^{2t}\gamma^{1t}\gamma^{0t}$
$= ci\,(-\gamma^3 C)\gamma^{2t}\gamma^{1t}\gamma^{0t}$ repeat
$= ci\,\gamma^3\gamma^2\gamma^1\gamma^0(-1)^4 C$
$= ci\,\gamma^0\gamma^1\gamma^2\gamma^3 C$
$= \underline{c\gamma_5 C}$

$([\gamma^{\mu}, \gamma^{\nu}]C)^t = (\gamma^{\mu}\gamma^{\nu} C)^t - (\gamma^{\nu}\gamma^{\mu} C)^t$
$= C^t\gamma^{\nu t}\gamma^{\mu t} - C^t\gamma^{\nu t}\gamma^{\mu t}$
$= c(-\gamma^{\nu} C\gamma^{\mu t}) - c(-\gamma^{\mu} C\gamma^{\nu t})$
$= -c\,\gamma^{\nu}(-\gamma^{\mu} C) + c\gamma^{\mu}(-\gamma^{\nu} C)$
$= -c\,(-\gamma^{\nu}\gamma^{\mu} C + \gamma^{\mu}\gamma^{\nu} C]$
$= \underline{-c\,[\gamma^{\mu}, \gamma^{\nu}]C}$

$1,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ \gamma_5,\ [\gamma^\mu,\gamma^\nu]$ are independent matrices
$1 \quad 4 \quad 4 \quad 1 \quad 6$

if $c = +1$ $\gamma^\mu C$ and $[\gamma^\mu,\gamma^\nu]C$ are antisymmetric
10 in all $\Rightarrow$ too many.

If $c = -1$ they are symmetric and C, $\gamma_5 C$ and $\gamma^\mu\gamma_5 C$ are 6 antisymmetric matrices.

$[\gamma^\mu, CC^\dagger] = \gamma^\mu CC^\dagger - CC^\dagger\gamma^\mu$

Use $C\gamma^{\mu\dagger} = -\gamma^\mu C \Rightarrow \gamma^{\mu *} C^\dagger = -C^\dagger \gamma^{\mu\dagger}$

So $\gamma^{0\dagger} C^\dagger = -C^\dagger\gamma^0$ and $-\gamma^{i\dagger}C^\dagger = +C^\dagger\gamma^i$

$\therefore \gamma^{\mu\dagger} C^\dagger = -C^\dagger\gamma^\mu$

$$\therefore \quad \gamma^\mu CC^\dagger - CC^\dagger\gamma^\mu = -C\gamma^{\mu\dagger}C^\dagger - CC^\dagger\gamma^\mu$$
$$= -C(-C^\dagger\gamma^\mu) - CC^\dagger\gamma^\mu$$
$$= \underline{0}$$

2. $\psi_P(x_P) = \hat{P}\,\psi(x)\,\hat{P}^{-1} = \gamma^0 \psi(x_P)$

$\mathcal{L}_I(x) = g\,\bar{\psi}\psi\phi + g'\,\bar{\psi}\,i\gamma_5\psi\phi$ not $\int d^4x = \int d^4x_P$.

$\hat{P}\,\bar{\psi}(x)\,\hat{P}^{-1} = \psi^\dagger(x_P)\gamma^{0\dagger}\gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$
$= \bar{\psi}(x_P)\gamma^0$

$\therefore\ \hat{P}\,\bar{\psi}(x)\,A\,\psi(x_P)\,\hat{P}^{-1} = \bar{\psi}(x_P)\gamma^0 A \gamma^0 \psi(x)$

If $g' = 0$ $\quad \bar{\psi}(x)\psi(x) \Rightarrow \bar{\psi}(x_P)\gamma^{0\,2}\psi(x_P) \quad A = 1$
$= \bar{\psi}(x_P)\psi(x_P)$.

So $\underline{\hat{P}\,\phi(x)\,\hat{P}^{-1} = \phi(x_P)}$ for invariance.

If $g = 0$ $\quad \bar{\psi}\,i\gamma_5\psi \Rightarrow \hat{P}\,\psi(x)\,i\gamma_5\psi(x)\,\hat{P}^{-1}$

$= \bar{\psi}(x_P)\gamma^0 i\gamma_5\gamma^0\psi(x_P)$, $\{\gamma^5, \gamma^0\} = 0$

$\rightarrow -\bar{\psi}(x_P)\gamma^0\gamma^0 i\gamma_5\psi(x_P)$

$= -\bar{\psi}(x_P)\,i\gamma_5\psi(x_P)$

$\therefore\ \underline{\hat{P}\,\phi(x)\,\hat{P}^{-1} = -\phi(x_P)}$ for invariance of $\mathcal{L}_I(x)$.

$\therefore$ Parity cannot be conserved if both $g, g' \neq 0$.

$\sigma^\mu_5 = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \rightarrow \hat{P}\,\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\,\hat{P}^{-1}$

$= \bar{\psi}(x_P)\gamma^0\gamma^\mu\gamma_5\gamma^0\psi(x_P)$

$= -\bar{\psi}(x_P)\gamma^0\gamma^0\gamma^0\gamma_5\psi(x_P)$

$$= -\bar{\psi}(x_p)\left[-\gamma^\mu\gamma^0 + 2g^{\mu 0}\right]\gamma^0\gamma^5\psi(x_p)$$

$$= \bar{\psi}(x_p)\gamma^\mu\gamma_5\psi(x_p) - 2\bar{\psi}(x_p)g^{\mu 0}\gamma^0\gamma_5\psi(x_p)$$

$$= -\bar{\psi}(x_p)\gamma^0\gamma_5\psi(x_p) \quad \mu = 0$$
$$= +\bar{\psi}(x_p)\gamma^\mu\gamma_5\psi(x_p) \quad \mu \neq 0.$$

$$= -\bar{\psi}(x_p)\gamma_\mu\gamma_5\psi(x_p) = -j_{5\mu}.$$

$\therefore$ need $\hat{P}^{-1}V_\mu(x)\hat{P} = -V^\mu(x_p)$ if $g' \neq 0$.

J^μ the same up to $-$ sign as $\gamma^\mu\gamma_5$; $\gamma_5\gamma_0 \to -\gamma_0\gamma_5$

$\therefore$ need $\hat{P}^{-1}V_\mu(x_p)\hat{P} \to V^\mu(x_p)$ if $g \neq 0$.

3.
7. First part gone through precisely in notes.

i.e. at first order

$$\sim -\frac{g^2}{8} J^{\mu +} D^{w}_{\mu\nu} J^{\nu}_{w} \sim -\frac{g^2}{8} J^{\mu +} \frac{g_{\mu\nu}}{m_w^2} J^{\nu}$$

$$= -\frac{g^2}{8 m_w^2} J^{\mu +} J_\mu = -\frac{G_F}{\sqrt{2}} J^{\mu +} J_\mu$$

Matrix element $\mathcal{M}$ for $W^+ \to e^+ \nu_e$

$$\mathcal{M} = \langle e^+(p_1\lambda_1)\, \nu_e(p_2\lambda_2) | -\frac{g}{2\sqrt{2}} \bar{\nu}_e \gamma^\mu (1-\gamma^5) e W_\mu | W(q) \rangle$$

$$= -\frac{g}{2\sqrt{2}} \bar{u}(p_2\lambda_2) \gamma^\mu (1-\gamma^5) v(p_1\lambda_1)\, \epsilon_\mu(q,\lambda)$$

$\therefore$ Averaging over 3 polarizations of W

$$\frac{1}{3}\sum_\lambda |\mathcal{M}|^2 = \frac{g^2}{24} \sum_\lambda \bar{u}(p_2,\lambda_2) \gamma^\mu (1-\gamma^5) v(p_1,\lambda_1) \times \bar{v}(p_1,\lambda_1) \gamma^\nu (1-\gamma^5) u(p_2,\lambda_2)\, \epsilon_\mu(q,\lambda) \epsilon^*_\nu(q,\lambda)$$

$$= \frac{g^2}{24} \operatorname{tr}\left(\gamma\cdot p_2 \gamma^\mu (1-\gamma^5) \gamma\cdot p_1 \gamma^\nu (1-\gamma^5)\right) \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_w^2}\right)$$

$$\operatorname{tr}(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) = 4(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\beta\mu})$$

and $\operatorname{tr}(\gamma^5 \gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) = 4i\,\epsilon^{\alpha\mu\beta\nu}$

which contributes zero since it contracts with the symmetric $\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_w^2}\right)$.

$\therefore \quad tr(\gamma\cdot p_2 \gamma^\mu (1-\gamma^5) \gamma\cdot p_1 \gamma^\nu (1-\gamma^5))$

$= 4\times 2 \, (p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - p_1\cdot p_2 g^{\mu\nu})$. effectively.

$\rightarrow \frac{1}{3} \sum_\lambda |M|^2 = \frac{g^2}{3} (p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - p_1\cdot p_2 g^{\mu\nu}) \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_w^2}\right)$

$= \frac{g^2}{3} \left(-2p_1\cdot p_2 + 4p_1\cdot p_2 + \frac{2\, p_2\cdot q\, p_1\cdot q}{m_w^2} - p_1\cdot p_2\right)$

since $q^2 = m_w^2$

$= \frac{g^2}{3} \left(p_1\cdot p_2 + \frac{2(p_2\cdot q)(p_1\cdot q)}{m_w^2}\right)$

In rest frame of w. $q = (m_w, 0)$

$p_1 = (\frac{m_w}{2}, \vec{p}) = (|\vec{p}|, \vec{p})$

$p_2 = (\frac{m_w}{2}, -\vec{p}) = (|\vec{p}|, -\vec{p})$

$\rightarrow \frac{g^2}{3} (2|p|^2 + 2|p|^2) = \frac{4}{3} g^2 |p|^2$.

Decay rate $= \frac{1}{2m_w} \iint |M|^2 \frac{d^3 p_1}{(2\pi)^3 2E_{p_1}} \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} (2\pi)^4 \delta^4(q-p_1-p_2)$

$= \frac{2g^2}{3m_w} \int |p|^2 \frac{d|p| \, 4\pi |p|^2}{4|p|^2 (2\pi)^2} \delta(m_w - 2|p|)$

$= \frac{2g^2}{3m_w} \int \frac{|p|^2}{4\pi} d|p| \, \delta(m_w - 2|p|)$

$= \frac{1 \, g^2}{12 m_w \pi} \cdot \left(\frac{m_w}{2}\right)^2$

$$= \frac{1}{48\pi} g^2 m_W = \frac{M_W^2}{6\pi} \cdot \frac{g^2}{8 m_W^2}$$

$$= \frac{G_F}{\sqrt{2}} \frac{m_W^3}{6\pi} = \Gamma_{W^+ \to \nu_e e^+}$$

$\Gamma_{W^+ \to e^+ \nu_e} \simeq \Gamma_{W^+ \to \mu^+ \nu_\mu}$ because there is no helicity suppression and in the limit of vanishing masses the rates are identical.

The π^+ is spinless, so in any decay, since the lepton and neutrino have opposite directions they must have the same helicity. In the absence of lepton mass the particles are both purely left-handed so the decay rate would be zero, and with mass present is $\propto m^2$ for the lepton.

$\therefore$ although because $m_\mu \gg m_e$ the phase space for $\pi^+ \to \mu^+ \nu_\mu$ is supressed a little compared to $\pi^+ \to e^+ \nu_e$, the overall factor of m_μ^2 instead of m_e^2 far out weighs this and

$$\Gamma_{\pi^+ \to \mu^+ \nu_\mu} \gg \Gamma_{\pi^+ \to e^+ \nu_e}.$$

4. Parity $\hat{P}\, e\bar{\psi}\gamma^{\mu}\psi\hat{P}^{-1} = e\,\bar{\psi}(x_p)\gamma_{\mu}\psi(x_p)$

So $\hat{P}^{-1}\,\partial_{\nu}F^{\mu\nu}(x)\,\hat{P}^{-1} = \partial_{\nu}F_{\mu}{}^{\nu}(x_p)$. to keep equation the same.

$$= \frac{\partial}{\partial x_p^{\nu}}\left(\frac{\partial}{\partial x_p^{\mu}}A^{\nu}(x_p) - \frac{\partial}{\partial x_{p\nu}}A_{\mu}(x_p)\right)$$

$$= \frac{\partial}{\partial x_{p\nu}}\left(\frac{\partial}{\partial x_p^{\mu}}A_{\nu}(x_p) - \frac{\partial}{\partial x_{p\nu}}A_{\mu}(x_p)\right)$$

$$= \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial}{\partial x_{\mu}}A_{\nu}(x_p) - \frac{\partial}{\partial x_{\nu}}A_{\mu}(x_p)\right).$$

$$= \partial_{\nu}\left(\partial^{\mu}A_{\nu}(x_p) - \partial^{\nu}A_{\mu}(x_p)\right)$$

So $\underline{\hat{P}A^{\mu}(x)\hat{P}^{-1} = A_{\mu}(x_p)}$

Charge Conjugation $\hat{C}\, e\bar{\psi}\gamma^{\mu}\psi\,\hat{C}^{-1}$
$= e\bar{\psi}\,C\gamma^{\mu t}C^{-1}\,\psi$
$= -\,e\bar{\psi}\gamma^{\mu}\psi.$

$\therefore$ $\underline{\hat{C}A_{\mu}\hat{C}^{-1} = -A_{\mu}}$

Time reversal $\hat{T}\,e\bar{\psi}\gamma^{\mu}\psi\hat{T}^{-1} = e\,\bar{\psi}(x_T)B\gamma^{\mu*}B^{-1}\psi(x_T)$

$= e\,\bar{\psi}(x_T)\gamma_{\mu}\psi(x_T)$

$\therefore$ $\underline{\hat{T}A_{\mu}\hat{T}^{-1} = A^{\mu}(x_T)}$ from the same argument as for parity.

5. $$F(x, Q^2) = \sum_i \int_x^1 \frac{dy}{y} C_i\left(\frac{x}{y}\right) f_i(y). \qquad (1)$$

$$Q \frac{d f_i}{dQ} = \sum_j \int_y^1 \frac{dz}{z} P_i\left(\frac{y}{z}\right) f_j(z). \qquad (2)$$

$$\int_0^1 dx\, x^{N-1} f(x) = f^N$$

$$\int_0^1 dx\, x^{N-1} \int_x^1 \frac{dy}{y} f(x/y)\, g(y)$$

$$= \int_0^1 \int_x^1 dx\, dy \left(\frac{x}{y}\right)^{N-1} y^{N-2} f(x/y)\, g(y).$$

let $z = \frac{x}{y}$ $\quad \therefore \quad dz\, y = dx$

$$\therefore \quad \left(\frac{x}{y}\right)^{N-1} y^{N-2}\, dx\, dy = z^{N-1} y^{N-1}\, dz\, dy.$$

region $x = 0..1$, $y = x..1$ maps to
$z = 0..1$ $y = 0...1$.

$$\therefore \int_0^1 dx\, x^{N-1} \int_x^1 \frac{dy}{y} f(x/y)\, g(y)$$

$$= \int_0^1 dz\, z^{N-1} f(z) \int_0^1 dy\, y^{N-1} g(y)$$

$$= \underline{\underline{f^N g^N}}.$$

Both (1) and (2) are convolutions of this form, so taking moments of both sides.

$$F^N(Q^2) = \sum_i C_i^N(\alpha_s(Q^2))\, f_i^N(Q^2)$$

$$Q\frac{d f_i^N}{dQ} = \sum_j P_{ij}^N(\alpha_s(Q^2))\, f_j^N(Q^2).$$

$$Q\frac{d}{dQ} q_v^N = -\frac{\alpha_s(Q^2)}{4\pi}\gamma_q^N q_v^N$$

$$= -\frac{1}{\beta_0 \ln Q^2/\Lambda^2}\gamma_q^N q_v^N$$

$$\therefore \frac{d q_v^N}{d\ln Q^2} = -\frac{1}{2\beta_0 \ln Q^2/\Lambda^2}\gamma_q^N q_v^N$$

$$\therefore \int_{q_v^N(Q_0^2)}^{q_v^N(Q^2)} \frac{d q_v^N}{q_v^N} = -\frac{\gamma_q^N}{2\beta_0}\int_{\ln Q_0^2/\Lambda^2}^{\ln Q^2/\Lambda^2} \frac{d\ln(\tilde{Q}^2/\Lambda^2)}{\ln(\tilde{Q}^2/\Lambda^2)}$$

$$\therefore \ln\left(\frac{q^N(Q^2)}{q^N(Q_0^2)}\right) = -\frac{\gamma^N}{2\beta_0}\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)$$

$$= -\frac{\gamma^N}{2\beta_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)$$

$$= \ln\left(\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)^{\frac{\gamma^N}{2\beta_0}}\right)$$

$$\therefore q^N(Q^2) = q^N(Q_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{\gamma_q^N/2\beta_0}$$

Consider $N = 1$

$$\rightarrow \int_0^1 dx\, x^{N-1} f(x) = \int_0^1 dx\, f(x)$$

= total number of quarks. $= f^1$

$\therefore$ conservation of valence quark number requires

$$Q \frac{dq^1_v}{dQ} = 0. = -\frac{\alpha_S(Q^2)}{4\pi} \gamma^1_q q^1_v$$

Only satisfied if $\gamma^1_q = 0$ which is indeed found to be true (up to NNLO i.e. $O(\alpha_S^3)$) in practice. ($O(\alpha_S^4)$ not yet known).