2018 Standard Model Part II R.S. Thorne

1. The matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^2$. Prove that $(\gamma^5)^2 = 1$ independent of representation.
In the Weyl representation for the γ-matrices $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$, where each entry is a 2×2 sub-matrix and the σ_i are the Pauli spin matrices. Find the plane wave solutions of the Dirac equation in this representation. Construct the helicity projection operators $P_R = (1+\gamma^5)/2$ and $P_L = (1-\gamma^5)/2$, and consider the solutions for case $m = 0$ and explain why this representation is particularly useful for the massless limit.

2. Starting from the matrix element for the process

$$\nu_\mu + e^- \to \mu^- + \nu_e$$

given by the current-current (four-fermion) theory of weak interactions,

$$M = (G_F/\sqrt{2})\bar{u}(\mu)\gamma^\mu(1-\gamma^5)u(\nu_\mu)g_{\mu\nu}\bar{u}(\nu_e)\gamma^\nu(1-\gamma^5)u(e)$$

and using the result

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}|M|^2$$

where $s = (p_1+p_2)^2$ and $t = (p_1-p_3)^2$, i.e. the usual Mandelstam variables, show that

$$\sigma = G_F^2 s/\pi,$$

i.e. is inconsistent with perturbative unitarity. You will find the results $\mathrm{tr}(\gamma^5\gamma\cdot a\gamma\cdot b\gamma\cdot c\gamma\cdot d) = 4i\epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta$ and $\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\gamma\delta} = -2(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta)$ useful in intermediate stages. Average over initial electron spins and sum over final muon spins.

3. Consider an $SU(2)$ gauge theory coupled to a complex two-component "doublet" scalar field ϕ acting on which the $SU(2)$ generators are represented by $\frac{1}{2}\tau$, for $\tau = \{\tau_1, \tau_2, \tau_3\}$ the usual Pauli matrices, and

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \tfrac{1}{2}\lambda\big(\phi^\dagger\phi - \tfrac{1}{2}v^2\big)^2\,,$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\,\mathbf{A}_\mu\times\mathbf{A}_\nu\,,\quad D_\mu\phi = \partial_\mu\phi - ig\,\mathbf{A}_\mu\cdot\tfrac{1}{2}\tau\phi\,.$$

$\times$ denotes the 3-vector cross product, which is equivalent to using the $SU(2)$ structure constants. Explain why we may choose $\phi = \frac{1}{\sqrt{2}}(v+f)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and that the $SU(2)$ gauge symmetry is completely broken. What are the masses of the elementary particle states neglecting any quantum corrections?

4. The gauge part of the Lagrangian density for the Electro-Weak theory with gauge fields $\mathbf{A}_\mu, B_\mu$, may be written as

$$\mathcal{L} = -\tfrac{1}{4}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu}$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\mathbf{A}_\mu\times\mathbf{A}_\nu\,,\quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

Show that the coupling of the W to the electromagnetic field A is described by

$$\mathcal{L}_{W,A} = -\tfrac{1}{2}F^{W\mu\nu\dagger}F^{W}{}_{\mu\nu} + ie\,W^{\mu}W^{\nu\dagger}F_{\mu\nu}\ ,$$
$$F^{W}{}_{\mu\nu} = d_\mu W_\nu - d_\nu W_\mu\ , \quad d_\mu = \partial_\mu - ieA_\mu\ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ .$$

Hence determine the contribution of the W field to the electromagnetic current j^μ (this is identified from the term of the form $j^\mu A_\mu$ in $\mathcal{L}_{W,A}$).

5. In order for symmetries present at the classical level to be preserved at the quantum level, and hence to avoid for so-called anomalies destroying the renormalizability of a theory, the quantity

$$\mathrm{tr}(\{T^R_a(R), T^R_b(R)\}T^R_c(R)) - \mathrm{tr}(\{T^L_a(R), T^L_b(R)\}T^L_c(R))$$

must vanish. $T^R_a(R)$ and $T^L_a(R)$ are the generators for the weak charge and hypercharge in the appropriate representation for each right-handed and left-handed particle respectively, the trace includes the sum over all fermions, and $\{T^R_a(R), T^R_b(R)\}$ denotes the anticommutator of the two generators. Show that for a single family the standard model is indeed anomaly free, provided one uses a piece of knowledge about quarks from outside the electroweak sector.

6. For any complex matrix M, show that $MM^\dagger$ is positive and hermitian. Let $V^\dagger MM^\dagger V = \Lambda$, for unitary V, and Λ diagonal with real positive eigenvalues. Writing $\Lambda = M_d M_d^\dagger$, where M_d is diagonal, show that remaining freedom to redefine V up to a diagonal matrix of phases can be used to set the eigenvalues of M_d to be real and positive. Define hermitian matrix H by

$$H \equiv V M_d V^\dagger$$

and $U \equiv H^{-1}M$. Show that U is unitary and that

$$V^\dagger M W = M_d$$

where $W = U^\dagger V$ is a unitary matrix. (This is the bi-unitary transformation that is needed to diagonalise quark mass matrices in general).

For two generations of quarks with masses m_d, m_s, m_u, m_c, suppose that for the Lagrangian density $\mathcal{L}_m = -\big(\overline{q}_+ m_+ \tfrac{1}{2}(1+\gamma_5)q_+ + \overline{q}_- m_- \tfrac{1}{2}(1+\gamma_5)q_- + \text{hermitian conjugate}\big)$, the mass matrices take the form

$$q_+ = \begin{pmatrix} u' \\ c' \end{pmatrix}, \quad q_- = \begin{pmatrix} d' \\ s' \end{pmatrix}, \quad m_+ = \begin{pmatrix} 0 & A \\ A^* & B \end{pmatrix}, \quad m_- = \begin{pmatrix} 0 & C \\ C^* & D \end{pmatrix}$$

where u', d', c', s' are quark fields (the prime indicates they diagonalize the weak charged current interaction). If $R_\mp(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \mp\sin\theta & \pm\cos\theta \end{pmatrix}$, define θ_+ by

$$R_+(\theta_+)\begin{pmatrix} 0 & |A| \\ |A| & B \end{pmatrix} R_-(\theta_+)^{-1} = \begin{pmatrix} m_u & 0 \\ 0 & m_c \end{pmatrix}$$

with θ_- similarly defined in terms of m_s, m_d. Show that after suitable rephasing of quark fields, one can use $R_\pm$ to diagonalize $m_\pm$. Hence, from the mixing matrix so generated in the quark weak charged current interaction, show that the Cabbibo angle in this case is given by

$$\theta_C = \theta_- - \theta_+ = \tan^{-1}\sqrt{\frac{m_d}{m_s}} - \tan^{-1}\sqrt{\frac{m_u}{m_c}}\ .$$

Check this relation against experimental data.