Projectile problem

A particle of mass m is fired with an initial speed v_0 at an initial angle θ to the horizontal Earth's surface. Neglecting any drag forces the equations of motion are well known:

$$m\frac{d^2x}{dt^2}=0; \qquad m\frac{d^2y}{dt^2}=-mg,$$

where x, y are the horizontal and vertical positions relative to the origin at time t. These differential equations have solutions for x and y as functions of t as

$$x(t)=v_0 t\cos\theta; \qquad y(t)=v_0 t\sin\theta-\frac{1}{2}gt^2$$

giving a trajectory

$$y(x)=x\tan\theta-\frac{gx^2}{2v_0{}^2\cos^2\theta}.$$

The range R is given by $R=\dfrac{2v_0{}^2\sin\theta\cos\theta}{g}=\dfrac{v_0{}^2\sin2\theta}{g}$ and the time, T, to the highest point is $T=\dfrac{v_0\sin\theta}{g}$.

The present task is to simulate this motion in a spreadsheet. To make it a little more like the "real world" you will include in the simulation a drag force ($-\gamma mv$) that is proportional to the instantaneous speed v. The equations of motion then become

$$m\frac{d^2x}{dt^2}=-m\gamma v\cos\theta=-m\gamma\frac{dx}{dt},$$

$$m\frac{d^2y}{dt^2}=-mg-m\gamma v\sin\theta=-mg-m\gamma\frac{dy}{dt},$$

where γ is a drag coefficient per unit mass.

The earlier exercise on derivatives showed how to get a numerical approximation to the first derivative. For y tabulated at intervals of δt we have

$$\left(\frac{dy}{dt}\right)\approx\frac{y_{n+1}-y_n}{\delta t} \qquad \text{or} \qquad \left(\frac{dy}{dt}\right)\approx\frac{y_{n+1}-y_{n-1}}{2\delta t}.$$

The second form was found to converge more quickly so we will use it here.
The second derivative can be approximated by

$$\frac{d^2y}{dt^2}\approx\frac{\left(\frac{dy}{dt}\right)_n-\left(\frac{dy}{dt}\right)_{n-1}}{\delta t}$$

$$\frac{d^2y}{dt^2}\approx\frac{\left(\frac{y_{n+1}-y_n}{\delta t}\right)-\left(\frac{y_n-y_{n-1}}{\delta t}\right)}{\delta t}=\frac{y_{n+1}-2y_n+y_{n-1}}{(\delta t)^2}.$$

Thus the equations of motion may be approximated by

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{(dt)^2} = -g\left(\frac{x_{n+1} - x_{n-1}}{2dt}\right),$$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{(dt)^2} = -g - g\left(\frac{y_{n+1} - y_{n-1}}{2dt}\right).$$

Thus we obtain

$$x_{n+1} = \frac{2x_n - x_{n-1}(1 - gdt/2)}{(1 + gdt/2)},$$

$$y_{n+1} = \frac{2y_n - y_{n-1}(1 - gdt/2) - g(dt)^2}{(1 + gdt/2)}.$$

These equations can be used to generate the positions (x_{n+1}, y_{n+1}) from (x_n, y_n) and (x_{n-1}, y_{n-1}). Thus we can obtain positions (x_2, y_2) and onwards. However (x_1, y_1) cannot be obtained this way as it requires a position (x_{-1}, y_{-1}) before the start! The effective initial accelerations in the x and y directions are $-g v_x(0)$ and $-g - g v_y(0)$. Thus using the equation "$s = ut + \frac{1}{2}ft^2$" we can approximate

$$x_1 = x_0 + v_x(0)dt - \left(\frac{1}{2}g\, v_x(0)\right)(dt)^2,$$

$$y_1 = y_0 + v_y(0)dt - \frac{1}{2}\left(g + g\, v_y(0)\right)(dt)^2.$$

where $v_x(0)$ and $v_y(0)$ are the initial components of velocity along the $x-$axis and $y-$axis respectively.

The task is to model the motion of the particle and to investigate some properties of it for some choice of the drag coefficient.

- Construct a spreadsheet which contains labelled cells for the quantities v_0, q, g, g and δt. You may find it convenient to calculate the subsidiary quantities $v_x(0), v_y(0)$, $(dt)^2$, $(1 + gdt/2)$, $(1 - gdt/2)$ to avoid repeated calculation of these quantities for each position. If these are positioned near the top-left of the spreadsheet, then below them under headings *n*, *t*, *x* and *y* compute the position of the particle for integer values of *n* from 0 to 1500. Take an initial speed $v_0 = 100$ ms^{-1}, δt = 0.01 s and initially set γ = 0 s^{-1}, i.e. no drag. For simplicity take g = 10 ms^{-2}.
- Plot *y* against *x*.
- The range (when y = 0) and the time of flight (= 2*T*, for this range) should agree with the values calculable from the formulae above. Do they? If not try and find the source of your error. Does the plot look like a parabola?
- If all is correct then set γ = 0.4 s^{-1} and see how the plot of the trajectory changes.
- For this value of γ construct a table of range *R* against the angle of firing θ.
- Plot these ranges against θ and determine the angle to achieve maximum range. (It is unlikely that you will have a data point at the maximum, so try

fitting a polynomial trend line to the data points and deduce the maximum from the equation of this line).

Save your spreadsheet as *username-projectile.xls*.