1) Exponentials of Operators [6 marks]

F(α) = \exp(\imath \alpha A) \ B \ \exp(\imath \alpha A), \ where \ A \ and \ B \ are \ operators, \ satisfying \ [A,B] = A

- Find a simplified expression for F(α) by
  a) Expanding one of the exponentials as a power series.
  b) Considering derivatives.

2) SU(2) Singlet [4 marks]

- Write down an SU(2)_{spin} singlet state for a quark-quark pair.
- Consider the rotation of the state about the y-axis through an angle θ.
- Find the new state and comment.

Formulae:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

3) A Group of Transformations [7 marks]

Consider a set of transformations in the x-y plane \{I, X, Y, R\}:

- I Identity
- X Reflection in x=0
- Y Reflection in y=0
- R Rotation by 180°

- Identify the 2x2 matrices which correspond to these transformations.
- Show that this set is a group under the operation “follows”.
- When addressing the issues of Associativity, prove that this is valid for all square matrices of the same order.
- Prove that all diagonal matrices of the same order commute, and hence that the group is Abelian.
- Identify all subgroups.
4) Product of Groups [3 marks]

This question looks scary, but actually it is pretty straightforward. It is effectively a simplified version of the studies we made of general SU(2) transformations on spin states for two spin-$\frac{1}{2}$ particles (see also Question 2) – but expressed more formally. When we looked at SU(2) we didn’t explicitly refer to a product of groups, but that is effectively what we had.

Consider the product of two groups: $Z_2 \otimes Z_2$, where $Z_2$ is the cyclic group of order 2 (this subscript “2” relates to the order of the group).

$Z_2$ could be represented by two transformations $\{g = i, f\}$ operating on spin states $\chi$: $i$ leaves the spin unchanged; $f$ flips the spin.

If we have two spin states (labelled with subscripts 1 and 2), we can imagine a product operator $g_1 \times g_2$ belonging to $Z_2 \otimes Z_2$, operating on the combined spin state $\chi_1 \chi_2$, where $g_1$ only operates on $\chi_1$ and $g_2$ only operates on $\chi_2$.

Now consider the set of these products of two transformations. We could indicate this set by $G = \{i \times i, f \times i, i \times f, f \times f\}$, where $\times$ denotes the product of the members of the two groups.

If $\cdot$ denotes the combination of two members of a group, with the operation “follows”, then transformations associated with the first particle are combined separately, and likewise for the transformations associated with the second particle:

$$(g'_1 \times g'_2) \cdot (g_1 \times g_2) \text{ corresponds to } (g'_1 \cdot g_1) \times (g'_2 \cdot g_2),$$

which in turn can be written $g'_1 \cdot g'_2$.

So for example $(i \times i) \cdot (i \times f) = (i \cdot i) \times (i \cdot f) = i \times f$.

- Find the 4x4 combination table for the members of the set $G$.
- There is an obvious Isomorphism between this group and the group in Question 3.
- Identify the isomorphism and give some simple rationale as to how this can be understood.

(In this question, distinguish carefully between the product of two groups, and the combination with an operation of members of a given group.)