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The Standard Model Part II: Charged Current weak interactions I

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Abstract:

Rough notes on ...

- Introduction
- Relation between G_F and g_W
- Leptonic CC processes, νe^- scattering

Estimated time: ~ 3 hours

Contents

1 Charged current weak interactions

1.1 Introduction

- Back in the early 1930's we physicists were puzzled by nuclear decay.
  - In particular, the nucleus was observed to decay into a nucleus with the same mass number ($A \to A$) and one atomic number higher ($Z \to Z+1$), and an emitted electron.
  - In such a two-body decay the energy of the electron in the decay rest frame is constrained by energy-momentum conservation alone to have a unique value.
  - However, it was observed to have a continuous range of values.
- In 1930 Pauli first introduced the neutrino as a way to explain the observed continuous energy spectrum of the electron emitted in nuclear beta decay
  - Pauli was proposing that the decay was not two-body but three-body and that one of the three decay products was simply able to evade detection.
- To satisfy the history police
  - We point out that when Pauli first proposed this mechanism the neutron had not yet been discovered and so Pauli had in fact named the third mystery particle a 'neutron'.
  - The neutron was discovered two years later by Chadwick (for which he was awarded the Nobel Prize shortly afterwards in 1935).
  - So with 'neutron' taken and predicting that the mystery particle had to be $\lesssim 0.01$ proton masses, it was named 'neutrino'.
  - 'Neutrino' roughly translates as 'small neutral one' in Italian and the name was in fact due to Enrico Fermi.
- Fermi was the first to develop a QFT of a weak interaction and he built it on Pauli's neutrino hypothesis.
  - The basic (correct) idea was the electron and neutrino are not nuclear constiuents but are emitted (created) particles in the decay process.
  - Not unlike the process of photon creation and emission in nuclear γ-decay.

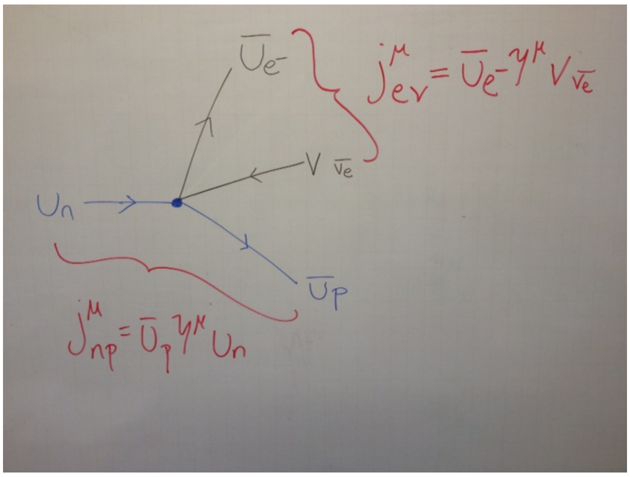

Figure 1. Original pure vector current hypothesised β-decay Feynman diagram.

- In keeping further with the photon emission analogy Fermi was guided by lessons learned from electromagnetism when putting together his candidate theory of β-decay.
- The basic process by which β-decay was assumed to (and does) proceed was $n \to pe^-\bar{\nu}_e$.
- The emission of the $e^-\bar{\nu}_e$ pair was assumed to happen *at a point* (like a photon emission in QED)
- By analogy to QED the nucleons were present in the interaction as *weak currents*
- Rather than having a charge conserving form e.g. $\bar{u}_p\gamma^\mu u_p$ the weak currents were charge non-conserving: $\bar{u}_p\gamma^\mu u_n$ and $\bar{u}_e\gamma^\mu u_{\nu_e}$
- For Lorentz invariance a *current-current* amplitude for the interaction was then proposed, of the form

$$\mathcal{M} = \hat{j}^{\mu}_{pn}\, A\, g_{\mu\nu}\, \hat{j}^{\nu}_{e\nu}\,, \tag{1.1}$$

with A a constant and the currents $\hat{j}^{\mu}_{pn}$ and $\hat{j}^{\nu}_{e\nu}$ are given by

$$\begin{aligned} \hat{j}^{\mu}_{pn} &= \bar{u}_p\gamma^\mu u_n\,, \\ \hat{j}^{\nu}_{e\nu} &= \bar{u}_e\gamma^\mu v_{\bar{\nu}_e}\,. \end{aligned}$$

- In terms of field theoretic Lagrangian, this interaction is naturally represented as

$$\mathcal{L}_{\text{Fermi}} = A\,\bar{\psi}_p(x)\,\gamma^\mu\psi_n(x)\;\bar{\psi}_e(x)\,\gamma_\mu\psi_{\nu_e}(x)\;. \tag{1.2}$$

- The discovery of positron β-decay followed and electron capture; these processes were included by adding to the Fermi interaction Lagrangian its complex conjugate

$$\mathcal{L}_{\text{Fermi}} \to A\,\bar{\psi}_p(x)\,\gamma^\mu\psi_n(x)\;\bar{\psi}_e(x)\,\gamma_\mu\psi_{\nu_e}(x) + A\,\bar{\psi}_n(x)\,\gamma^\mu\psi_p(x)\;\bar{\psi}_{\nu_e}(x)\,\gamma_\mu\psi_e(x)\;. \tag{1.3}$$

• Fermi's choice of vector-vector interaction is a very specific one among the various Lorentz invariant combinations that can be constructed.

- There is a priori no reason to use vectors — in this sense Fermi was really following the example of electromagnetic theory.
- A characteristic of Fermi's interaction type was that the spin of the nucleon can't flip.
- We can see this by noting that the energy emitted in the β-decay process is very small with respect to the masses of the nucleons involved i.e. it is essentially a non-relativistic process.
- From earlier in the course we have derived the free, positive energy, particle solutions to the Dirac equation as

$$u_p = N \begin{pmatrix} \phi_p \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\phi_p \end{pmatrix} ,$$

where u_p is a four component Dirac spinor describing an incident proton state, an analogous expression holding for the outbound neutron state, and ϕ_p a two component spinor. In the non-relativistic limit $\vec{p}/m \to 0$ and so

$$u_p \to N \begin{pmatrix} \phi_p \\ 0 \end{pmatrix} .$$

Evaluating the np current explicitly in this limit we find

$$\begin{aligned} \bar{u}_p \gamma^\mu u_n &= u_p^\dagger \gamma_0 \gamma^\mu u_n \\ &= u_p^\dagger \beta \left(\beta, \beta\vec{\alpha}\right) u_n \\ &= u_p^\dagger \left(1, \vec{\alpha}\right) u_n \\ &= u_p^\dagger \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right) u_n \\ &= N^2 \left(u_p^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_n, u_p^\dagger \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} u_n \right) \\ &= N^2 \left(\begin{pmatrix} \phi_p^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \phi_n \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_p^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \phi_n \\ 0 \end{pmatrix} \right) \\ &= N^2 \left(\begin{pmatrix} \phi_p^\dagger & 0 \end{pmatrix} \begin{pmatrix} \phi_n \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_p^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vec{\sigma}\phi_n \end{pmatrix} \right) \\ &= N^2 \left(\phi_p^\dagger \phi_n, \vec{0} \right) , \end{aligned} \tag{1.4}$$

where the only non-zero component of the current, $N^2 \phi_p^\dagger \phi_n$, is in fact zero if the spins of n and p are different, more precisely, if they are not the same eigenstate of $S_z = \frac{1}{2}\sigma_z$ (the spin-up the z-axis eigenstate of S_z is $\phi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, while the spin-down the z-axis eigenstate is $\phi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with eigenvalues $\pm\frac{1}{2}$ respectively).

- Soon after Fermi's theory came out it became clear that other possible Lorentz invariant interaction types were possible and indeed necessary to explain data, in particular it was the case that $\Delta J = 1$ β-decays were observed and so Fermi's theory was insufficient
  - Gamow and Teller introduced the general four-fermion interaction constructed from Lorentz invariant combinations of bilinear combinations of the np fields and lepton fields, e.g.

$$\mathcal{L}_{\mathrm{GT}}^{\Delta J=0} \propto \bar{\psi}_p\left(x\right)\psi_n\left(x\right)\,\bar{\psi}_e\left(x\right)\psi_{\nu_e}\left(x\right) ,$$

which following the exercise above for the Fermi interactions again forbids spin flips and so only allows for $\Delta J = 0$ transitions, and

$$\mathcal{L}_{\mathrm{GT}}^{\Delta J=1} \propto \bar{\psi}_p(x)\, \sigma^{\mu\nu} \psi_n(x)\; \bar{\psi}_e(x)\, \sigma_{\mu\nu} \psi_{\nu_e}(x)\ ,$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right)\ ,$$

which allows $\Delta J = 1$ nucleon transitions.

- Much more deadly though was Madame Wu's 'Wu experiment' in 1957 which demonstrated that β-decay violated parity invariance.
  - The experiment took a sample of Cobalt 60 (J=5) and cooled it to 0.01 K in a solenoid so that the spins aligned with the magnetic field giving a net polarization $\left\langle \vec{J} \right\rangle$.
  - The Cobalt 60 β-decays to Nickel 60 (J=4) a $\Delta J = 1$ transition.
  - The extent of the ^{60}Co alignment was measured from observations of the angular distribution of γ rays from the ^{60}Ni.
  - The angular distribution of electrons with respect to $\left\langle \vec{J} \right\rangle$ was measured and found to be

$$\begin{aligned} I(\theta) &= 1 - \left\langle \vec{J} \right\rangle . \vec{p}/E \\ &= 1 - Pv \cos\theta\, , \end{aligned}$$

  where v, $\vec{p}$ and E are the electron speed, momentum and energy, P is the magnitude of the polarization and θ is the angle of emission with respect to $\left\langle \vec{J} \right\rangle$.
  - A parity transformation replaces $\vec{p} \to -\vec{p}$ but leaves $\left\langle \vec{J} \right\rangle$ untouched $\left\langle \vec{J} \right\rangle \to \left\langle \vec{J} \right\rangle$, since $\vec{J}$ is an axial vector; applying this transformation to the functional form of the distribution above we obtain

$$\begin{aligned} \hat{P} I(\theta) &= 1 + \left\langle \vec{J} \right\rangle . \vec{p}/E \\ &= 1 + Pv \cos\theta\, , \end{aligned}$$

  i.e. we get a different answer, a demonstration that β-decay exhibits parity non-invariance; more frequently, less accurately, called parity violation.
  - Another way of interpreting this result is that by performing Wu's experiment we can *determine* which of the two coordinate systems we are in.
- Mathematically the distribution in the Wu experiment comprises of a scalar quantity, 1, (invariant under parity transformations) and a pseudoscalar quantity $\left\langle \vec{J} \right\rangle . \vec{p}$, that is to say a scalar quantity but one which is not invariant under a parity transformation.
- To accommodate parity violation Fermi's theory making use of only vector currents needed to be extended to include some axial-vector (parity invariant vector) component.

– In particular it was suggested to replace each vector current by a combination of vector and axial-vector currents

$$\bar{u}\gamma^{\mu}u \to \bar{u}\gamma^{\mu}\left(1 - r\gamma_5\right)u\,,$$

where $r = 1$ for the leptonic current and for the $n - p$ current r was empirically determined tp be ~ 1.2. In fact, neglecting CKM flavour mixing effects (to be discussed shortly), knowing, as we do now, that in fact the β-decay process corresponds at the quark level via a $d \to u$ transition, the interaction comprises of two such currents with $r = 1$.

– Thus the structure of the weak interaction currents, then and in the full blown standard model is of the type vector minus axial-vector, routinely shortened to just 'V-A'.

- The (parity violating version of) Fermi theory was also employed in the describing leptonic processes such as muon decay.

  – In this case the nuclear $n - p$ current is simply replaced by a $\mu - \nu_\mu$ one in the amplitude / Lagrangian:

$$\mathcal{M} = j^{\mu}_{\nu_\mu \mu, L}\, 2\sqrt{2}G_F\, g_{\mu\nu}\, j^{\nu}_{e\nu_e, L}\,, \tag{1.5}$$

where (for μ^-)

$$\begin{aligned} j^{\mu}_{\nu_\mu \mu, L} &= \bar{u}_{\nu_\mu}\,\frac{1}{2}\gamma^{\mu}\left(1 - \gamma_5\right)\,u_{\mu}\,, \\ j^{\nu}_{e\nu_e, L} &= \bar{u}_{e}\,\frac{1}{2}\gamma^{\nu}\left(1 - \gamma_5\right)\,v_{\bar{\nu}_e}\,, \end{aligned}$$

G_F being the so-called Fermi-constant (indeed the value of the Fermi constant is determined from measurements of the muon decay process).

  – Nowadays we know that the Fermi interaction is not in fact point like but that only appear point-like due to the fact that the weak interation force is mediated by weak bosons with very large masses $\mathcal{O}\left(100\,\mathrm{GeV}\right)$. As we heard at length in the introduction to this course the very large masses of the exchanged particles manifest as a very short range of the interaction, in this case making it appear point-like in relatively low-energy phenomena such as muon and β-decay. In higher energy processes where the momentum transfer in the scattering processes approaches $\mathcal{O}\left(100\,\mathrm{GeV}\right)$, the particle-like nature of the exchange effectively becomes 'resolved' by the incoming and outgoing particles.

  – In particular we know that associated with the W boson exchange in these processes there are coupling constant factors associated with the vertices where the currents couple to the exchanged particles. In fact, in the Standard Model the muon decay amplitude above is rather given by

$$\mathcal{M} = i \times j^{\mu}_{\nu_\mu \mu, L}\left(-i\frac{g_W}{\sqrt{2}}\right)\left(\frac{-i\left(g_{\mu\nu} - p_\mu p_\nu/m_W^2\right)}{p^2 - m_W^2}\right)\left(-i\frac{g_W}{\sqrt{2}}\right)j^{\nu}_{e\nu_e, L}\,, \tag{1.6}$$

where the central bracketed term is the propagator factor for a W boson of (on-shell) mass m_W carrying momentum p, while the remaining two factors in round brackets are the relevant coupling constant factors attaching the currents to the propagator. The overall factor of i comes from the conventions for the Feynman rules (Halzen and Martin).

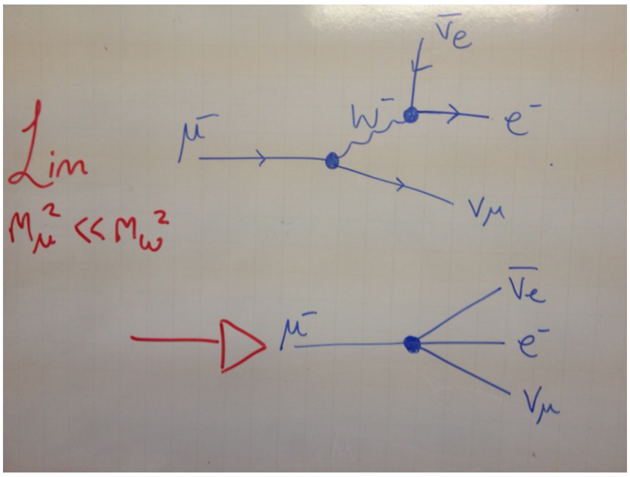

Figure 2. Muon decay neglecting terms suppressed by $q^2/m_W^2 \ll 1$. This limit relates the Fermi constant G_F to the more fundamental weak coupling constant g_W.

- I point out that in a text book approach to writing down the amplitude for muon decay you would rather write it down with the $-i\frac{g_W}{\sqrt{2}}$ coupling factors contained within the currents and more generically refer to the combination

$$-i\frac{g_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu\left(1-\gamma_5\right) ,$$

as the Feynman rule for the lepton-lepton-W-boson vertex. Hopefully it is very obvious that this is simply a trivial reorganisation of $\mathcal{M}$ as written in Eq. 1.6.

- Let us now take the limit $p^2/m_W^2 \to 0$ in Eq. 1.6
  - In this limit the propagator factor becomes simply $ig_{\mu\nu}/m_W^2$ and we obtain (see figure 2)

$$\lim_{p^2/m_W^2\to 0}\mathcal{M} \to j^\mu_{\nu_\mu\mu,L}\left(\frac{g_W^2}{2m_W^2}\right) g_{\mu\nu}\, j^\nu_{e\nu_e,L} ,$$

Comparing the full Standard Model form for the amplitude with the one expected assuming a Fermi four-point interaction we identify

$$\frac{g_W^2}{2m_W^2} = 2\sqrt{2}G_F$$

i.e.

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2} .$$

- Last but not least we point out that the V-A weak interaction vertices strictly couple only left-handed particles and anti-particles
  - To this end recall that

$$P_L = \frac{1}{2}\left(1-\gamma_5\right)$$

is a left-handed projection operator. Suppose we take some arbitrary spinor u we can write

$$\begin{aligned} u &= 1 \times u \\ &= \frac{1}{2}(1-\gamma_5)\,u + \frac{1}{2}(1+\gamma_5)\,u \\ &= u_L + u_R\,. \end{aligned}$$

If we now act on this state with $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ we get

$$\begin{aligned} P_{L/R}u &= \frac{1}{2}(1 \mp \gamma_5)\,u_L + \frac{1}{2}(1 \mp \gamma_5)\,u_R \\ &= \frac{1}{2}(1 \mp \gamma_5)\frac{1}{2}(1-\gamma_5)\,u + \frac{1}{2}(1 \mp \gamma_5)\frac{1}{2}(1+\gamma_5)\,u \\ &= \frac{1}{4}(1-\gamma_5 \mp \gamma_5(1-\gamma_5))\,u + \frac{1}{4}(1+\gamma_5 \mp \gamma_5(1+\gamma_5))\,u \\ &= \frac{1}{4}\left(1-\gamma_5 \mp \left(\gamma_5-\gamma_5^2\right)\right)u + \frac{1}{4}\left(1+\gamma_5 \mp \left(\gamma_5+\gamma_5^2\right)\right)u \\ &= \frac{1}{4}(1-\gamma_5 \pm (1-\gamma_5))\,u + \frac{1}{4}(1+\gamma_5 \mp (1+\gamma_5))\,u \\ &= \frac{1}{4}(1-\gamma_5)(1\pm 1)\,u + \frac{1}{4}(1+\gamma_5)(1\mp 1)\,u \\ &= \frac{1}{2}(1\pm 1)\,u_L + \frac{1}{2}(1\mp 1)\,u_R\,, \end{aligned}$$

so $P_L u$ returns u_L and $P_R u$ returns u_R i.e. the left-hand projections operator deletes everything not left-handed in a state and the right-handed one deletes everything in it that's not right-handed.

– Equally this means (obviously I hope) $P_L(P_L u) = P_L u$ and $P_R(P_R u) = P_R u$.

– So in the currents we can rewrite

$$\begin{aligned} j^{\mu}_{\nu_\mu\mu,L} &= \bar{u}_{\nu_\mu}\frac{1}{2}\gamma^\mu(1-\gamma_5)\;u_\mu\,, \\ &= \bar{u}_{\nu_\mu}\,\gamma^\mu P_L\,u_\mu\,, \\ &= \bar{u}_{\nu_\mu}\,\gamma^\mu P_L P_L\,u_\mu\,, \\ &= \bar{u}_{\nu_\mu}\,\gamma^\mu P_L\,u_{\mu,L}\,, \\ &= u^\dagger_{\nu_\mu}\gamma_0\,\gamma^\mu P_L\,u_{\mu,L}\,, \end{aligned}$$

then make use of the γ_5 anticommutor relation $\{\gamma_\mu,\gamma_5\}=0$ to write

$$\begin{aligned} \gamma_0\gamma^\mu P_L &= \gamma_0\gamma^\mu\frac{1}{2}(1-\gamma_5) \\ &= \gamma_0\frac{1}{2}(1+\gamma_5)\,\gamma^\mu \\ &= \frac{1}{2}(1-\gamma_5)\,\gamma_0\gamma^\mu \\ &= P_L\gamma_0\gamma^\mu\,, \end{aligned}$$

and the hermiticity of γ_5 ($\gamma_5 = \gamma_5^\dagger$) to write

$$\gamma_0\gamma^\mu P_L = P_L^\dagger\gamma_0\gamma^\mu\,,$$

giving

$$
\begin{aligned}
j^{\mu}_{\nu_\mu \mu, L} &= u^{\dagger}_{\nu_\mu} P^{\dagger}_{L} \gamma_0 \gamma^{\mu}\, u_{\mu,L} \\
&= \left(P_L u_{\nu_\mu}\right)^{\dagger} \gamma_0 \gamma^{\mu}\, u_{\mu,L} \\
&= \overline{P_L u_{\nu_\mu}} \gamma^{\mu}\, u_{\mu,L} \\
&= \bar{u}_{\nu_\mu, L}\, \gamma^{\mu}\, u_{\mu,L}\,.
\end{aligned}
$$

- Thus we have the remarkably simple (and fundamental) result that only the left-chiral / left-handed parts of spinors enter the weak interactions. The Standard Model is a chiral-theory (distinguishing left from right).

- Right-handed matter simply doesn't feel the weak interaction mediated by the W-boson, more accurately and more generally speaking, right-handed fermions are not charged under the $SU\left(2\right)_L$ gauge group of the Standard Model.

- Left-handed fermions carry weak isospin-$\frac{1}{2}$, a quantum number / charge with mathematical characteristics much the same as conventional spin angular momentum.

- W bosons carry weak isospin-1, and the leptonic interactions of W bosons with leptons, transforming leptons to neutrinos and vice-versa are transitions where the W transforms the weak isospin $T^3 = -\frac{1}{2}$ leptons into $T^3 = +\frac{1}{2}$ neutrinos.

- Accordingly, the left-handed and leptons and neutrinos are organised in the Standard Model Lagrangian as $SU\left(2\right)_L$ doublets, just as neutron and proton are part of the same strong $SU\left(2\right)$ isospin doublet:

$$
\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \,.
$$

- Right-handed fermions carry no weak isospin, they are weak isospin singlet states in the Standard Model Lagrangian

$$
e^-_R\,.
$$

1.2 Leptonic charge current process

- Muon decay has been exhaustively studied theoretically and experimentally since the late 1940s.

  - By studying the angular distribution of the produced leptons and / or their (scaled) energy spectrum one is able to probe and the V-A nature of the decay.

  - As far as I am aware the agreement these analyses show with the V-A predictions is striking, showing agreement at the level of 1-2%, either consistent with or well within the experimental uncertainties.

  - Measurements of muon decay also offers a precise determination of the Fermi constant, in particular the muon decay lifetime is determined with very high precision and from it G_F.

- In the years since Madame Wu's experiment parity violation and the V-A structure of weak interactions is observable much more directly.

  - In particular these days we are able to prepare neutrinos effectively as intense beams which we typically like to blast at hadronic or sometimes leptonic targets.

  - This makes it possible to probe the V-A structure by looking at the angular distributions of final-state particles in e.g. $\nu_e e^-$ or $\bar{\nu}_e e^-$ scattering.

  - It turns out, somewhat surprisingly I would say, that $\nu_e e^- \to \nu_e e^-$ scattering produces an isotropic distribution in the rest frame of the interaction.

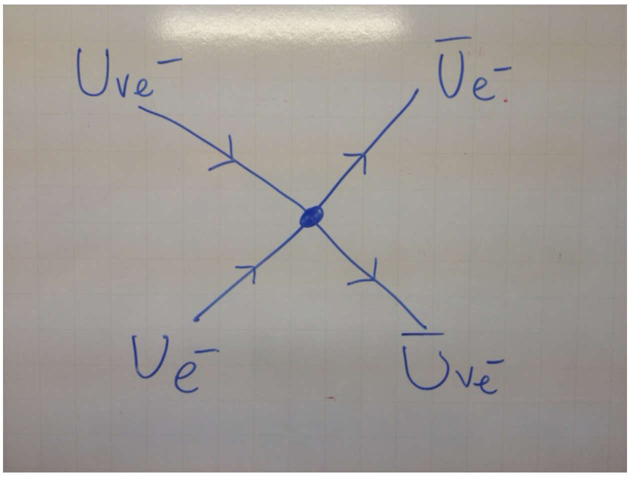

Figure 3. Neutrino electron scattering via $V-A$ current-current four point interaction ($(p_e - q_{\nu_e})^2 = (p_{\nu_e} - q_e)^2 \ll m_W^2$).

- We shall now calculate the differential cross section for $\nu_e e^- \to \nu_e e^-$ and show this is indeed the case (at least for low energies $\ll m_W$).
  - With small modifications the same calculation can be recycled and applied to neutrino-quark scattering, which will be useful later on in the lectures.
  - As intimated above, the calculation is carried out here for $q^2 \ll m_W^2$ i.e. we work in the limit of the Fermi four-point interaction where W propagator effects are neglected.
  - In fact typical dedicated neutrino experiments do not directly probe high enough energy scales to see W propagator effects.
  - This is in contrast to collider experiments, e.g. HERA, where the W boson exchange in the reaction $e^- q \to \nu q'$ has observable consequences.
  - Nowadays, the inclusion of W propagator effects is vital in describing the decays of the Higgs boson into W pairs as well as in the precision predictions needed for the pure weak diboson background to that Higgs signal process (something I worked a lot on once upon a time).
  - We will label in the incident neutrino and electron momenta p_{ν_e} and p_e respectively, similarly we label the final-state neutrino and electron momenta q_{ν_e} and q_e.
  - In contrast to antineutrino-electron scattering which we will mention later on, neutrino electron scattering takes place via the exchange of a *t-channel* W boson.
  - In the point-like limit:

$$\mathcal{M} = j^{\mu}_{e\nu_e,L}\, 2\sqrt{2}G_F\, g_{\mu\nu}\, j^{\nu}_{\nu_e e,L}\,, \tag{1.7}$$

$$j^{\mu}_{e\nu_e,L} = \bar{u}_e\,(q_e)\,\frac{1}{2}\gamma^{\mu}\,(1-\gamma_5)\;u_{\nu_e}\,(p_{\nu_e})\;, \tag{1.8}$$

$$j^{\nu}_{\nu_e e,L} = \bar{u}_{\nu_e}\,(q_{\nu_e})\,\frac{1}{2}\gamma^{\nu}\,(1-\gamma_5)\;u_e\,(p_e)\;, \tag{1.9}$$

where the p's and q's refer to incoming and outgoing momenta respectively (Fig. 3).

– We have been careful and labelled the particle subscript labels in the currents in the same order that the corresponding particle / anti-particle spinors would appear in the relevant Dirac chains of those currents.

– The first thing we want to do is get rid of the spinors by squaring the amplitude, more specifically by multiplying it by its complex conjugate:

$$\mathcal{M}^* = j^{\alpha *}_{e\nu_e,L}\, 2\sqrt{2} G_F\, g_{\alpha\beta}\, j^{\beta *}_{\nu_e e,L}\,, \tag{1.10}$$

where

$$\begin{aligned}
j^{\alpha *}_{e\nu_e,L} &= \left(\bar{u}_e\left(q_e\right)\, \frac{1}{2}\gamma^\alpha\left(1-\gamma_5\right)\, u_{\nu_e}\left(p_{\nu_e}\right)\right)^* && (1.11)\\
&= \left(\bar{u}_e\left(q_e\right)\, \frac{1}{2}\gamma^\alpha\left(1-\gamma_5\right)\, u_{\nu_e}\left(p_{\nu_e}\right)\right)^\dagger && (1.12)\\
&= u^\dagger_{\nu_e}\left(p_{\nu_e}\right)\left(\frac{1}{2}\left(1-\gamma_5\right)\right)^\dagger \gamma^{\alpha\dagger}\bar{u}^\dagger_e\left(q_e\right) && (1.13)\\
&= u^\dagger_{\nu_e}\left(p_{\nu_e}\right)\frac{1}{2}\left(1-\gamma_5\right)\gamma_0\gamma_0\gamma^{\alpha\dagger}\gamma_0 u_e\left(q_e\right) && (1.14)\\
&= u^\dagger_{\nu_e}\left(p_{\nu_e}\right)\frac{1}{2}\left(1-\gamma_5\right)\gamma_0\gamma^\alpha u_e\left(q_e\right) && (1.15)\\
&= \bar{u}_{\nu_e}\left(p_{\nu_e}\right)\frac{1}{2}\left(1+\gamma_5\right)\gamma^\alpha u_e\left(q_e\right) && (1.16)\\
&= \bar{u}_{\nu_e}\left(p_{\nu_e}\right)\frac{1}{2}\gamma^\alpha\left(1-\gamma_5\right)u_e\left(q_e\right)\,, && (1.17)
\end{aligned}$$

where in the 4th line we made use of the very handy identity $\gamma_0\gamma^{\mu\dagger}\gamma_0 = \gamma^\mu$ (coming from $\gamma_0^2 = 1$, $\gamma_0^\dagger = \gamma_0$, $\gamma_k^\dagger = -\gamma_k$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$) and in other places $\gamma_5 = \gamma_5^\dagger$.

– The structure of the second current is identical to the first one and so it follows without thinking (pattern substitution) that

$$j^{\beta *}_{\nu_e e,L} = \bar{u}_e\left(p_e\right)\, \frac{1}{2}\gamma^\beta\left(1-\gamma_5\right)\, u_{\nu_e}\left(q_{\nu_e}\right)\,.$$

– Squaring the amplitude we then have

$$\begin{aligned}
\left|\mathcal{M}\right|^2 &= j^\mu_{e\nu_e,L}\, 2\sqrt{2}G_F\, g_{\mu\nu}\, j^\nu_{\nu_e e,L}\, j^{\alpha *}_{e\nu_e,L}\, 2\sqrt{2}G_F\, g_{\alpha\beta}\, j^{\beta *}_{\nu_e e,L},\\
&= 8G_F^2\; g_{\alpha\beta}g_{\mu\nu}\; \left(j^\mu_{e\nu_e,L}\, j^{\alpha *}_{e\nu_e,L}\right)\; \left(j^\nu_{\nu_e e,L}\, j^{\beta *}_{\nu_e e,L}\right).
\end{aligned}$$

– The currents and their complex conjugates have no loose Dirac indices. Using the cyclicity of the traces we are able to rewrite them in a way which groups the barred and un-barred fermion spinors together in the form where it is obvious how to perform spin sums with them:

$$\begin{aligned}
j^\mu_{e\nu_e,L}\, j^{\alpha *}_{e\nu_e,L} &= \bar{u}_e\left(q_e\right)\, \gamma^\mu\, P_L\, u_{\nu_e}\left(p_{\nu_e}\right)\bar{u}_{\nu_e}\left(p_{\nu_e}\right)\, \gamma^\alpha\, P_L\, u_e\left(q_e\right)\\
&= \mathrm{Tr}\left[\bar{u}_e\left(q_e\right)\, \gamma^\mu\, P_L\, u_{\nu_e}\left(p_{\nu_e}\right)\bar{u}_{\nu_e}\left(p_{\nu_e}\right)\, \gamma^\alpha\, P_L\, u_e\left(q_e\right)\right]\\
&= \mathrm{Tr}\left[\left(u_e\left(q_e\right)\bar{u}_e\left(q_e\right)\right)\, \gamma^\mu\, P_L\, \left(u_{\nu_e}\left(p_{\nu_e}\right)\bar{u}_{\nu_e}\left(p_{\nu_e}\right)\right)\gamma^\alpha\, P_L\,\right]
\end{aligned}$$

$$
\begin{aligned}
j^{\nu}_{\nu_e e,L}\, j^{\beta *}_{\nu_e e,L} &= \bar{u}_{\nu_e}\left(q_{\nu_e}\right)\, \gamma^{\nu}\, P_L\, u_e\left(p_e\right) \bar{u}_e\left(p_e\right)\, \gamma^{\beta}\, P_L\, u_{\nu_e}\left(q_{\nu_e}\right) \\
&= \mathrm{Tr}\left[\bar{u}_{\nu_e}\left(q_{\nu_e}\right)\, \gamma^{\nu}\, P_L\, u_e\left(p_e\right) \bar{u}_e\left(p_e\right)\, \gamma^{\beta}\, P_L\, u_{\nu_e}\left(q_{\nu_e}\right)\right] \\
&= \mathrm{Tr}\left[\left(u_{\nu_e}\left(q_{\nu_e}\right) \bar{u}_{\nu_e}\left(q_{\nu_e}\right)\right)\, \gamma^{\nu}\, P_L\, \left(u_e\left(p_e\right) \bar{u}_e\left(p_e\right)\right)\, \gamma^{\beta}\, P_L\right]
\end{aligned}
$$

– We now sum the squared amplitude over the fermion spins which allows us to apply the identity (for particle spinors)

$$
\sum_s u\left(p,s\right) \bar{u}\left(p,s\right) = \not{p} + m\,.
$$

We find

$$
\begin{aligned}
\sum_{\text{spins}} j^{\mu}_{e\nu_e,L}\, j^{\alpha *}_{e\nu_e,L} &= \mathrm{Tr}\left[\left(u_e\left(q_e\right) \bar{u}_e\left(q_e\right)\right) \gamma^{\mu}\, P_L\, \left(u_{\nu_e}\left(p_{\nu_e}\right) \bar{u}_{\nu_e}\left(p_{\nu_e}\right)\right) \gamma^{\alpha}\, P_L\, \right] \\
&= \mathrm{Tr}\left[\, \not{q}_e\, \gamma^{\mu}\, P_L\, \; \not{p}_{\nu_e} \gamma^{\alpha}\, P_L\, \right] \\
&= \mathrm{Tr}\left[\, \not{q}_e\, \gamma^{\mu}\, \; \not{p}_{\nu_e} \gamma^{\alpha}\, P_L\, \right]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\text{spins}} j^{\nu}_{\nu_e e,L}\, j^{\beta *}_{\nu_e e,L} &= \mathrm{Tr}\left[\left(u_{\nu_e}\left(q_{\nu_e}\right) \bar{u}_{\nu_e}\left(q_{\nu_e}\right)\right)\, \gamma^{\nu}\, P_L\, \left(u_e\left(p_e\right) \bar{u}_e\left(p_e\right)\right)\, \gamma^{\beta}\, P_L\right] \\
&= \mathrm{Tr}\left[\, \not{q}_{\nu_e}\, \gamma^{\nu}\, P_L\, \; \not{p}_e\, \gamma^{\beta}\, P_L\right] \\
&= \mathrm{Tr}\left[\, \not{q}_{\nu_e}\, \gamma^{\nu}\, \; \not{p}_e\, \gamma^{\beta}\, P_L\right]
\end{aligned}
$$

– Thus

$$
\begin{aligned}
\sum_{\text{spins}} \left|\mathcal{M}\right|^2 &= \sum_{\text{spins}} 8G_F^2\; g_{\alpha\beta} g_{\mu\nu}\; \left(j^{\mu}_{e\nu_e,L}\, j^{\alpha *}_{e\nu_e,L}\right)\; \left(j^{\nu}_{\nu_e e,L}\, j^{\beta *}_{\nu_e e,L}\right) \\
&= 8G_F^2\, \mathrm{Tr}\left[\, \not{q}_e\, \gamma^{\mu}\, \; \not{p}_{\nu_e} \gamma^{\alpha}\, P_L\right]\, \mathrm{Tr}\left[\, \not{q}_{\nu_e}\, \gamma_{\mu}\, \; \not{p}_e\, \gamma_{\alpha}\, P_L\right]\,.
\end{aligned}
$$

– We now apply the trace relation

$$
\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\left(1-\gamma_5\right)\right] = 4g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4i\epsilon^{\mu\nu\rho\sigma}\,,
$$

to write

$$
\begin{aligned}
& \mathrm{Tr}\left[\not{q}_e\,\gamma^\mu\ \ \not{p}_{\nu_e}\,\gamma^\alpha\,P_L\right]\ \mathrm{Tr}\left[\not{q}_{\nu_e}\,\gamma_\mu\ \ \not{p}_e\,\gamma_\alpha\,P_L\right] \\
=\ & \frac{1}{4}\left(4q_e^\mu p_{\nu_e}^\alpha+4q_e^\alpha p_{\nu_e}^\mu-4g^{\mu\alpha}\left(q_e p_{\nu_e}\right)-4i\epsilon^{\mu\alpha\rho\sigma}q_{e,\rho}p_{\nu_e,\sigma}\right) \\
\times\ & \left(4q_{\nu_e,\mu}p_{e,\alpha}+4q_{\nu_e,\alpha}p_{e,\mu}-4g_{\mu\alpha}\left(q_{\nu_e}p_e\right)-4i\epsilon_{\mu\alpha\kappa\lambda}q_{\nu_e}^\kappa p_e^\lambda\right) \\
=\ & \frac{1}{4}\left(4q_e^\mu p_{\nu_e}^\alpha+4q_e^\alpha p_{\nu_e}^\mu-4g^{\mu\alpha}\left(q_e p_{\nu_e}\right)\right)\left(4q_{\nu_e,\mu}p_{e,\alpha}+4q_{\nu_e,\alpha}p_{e,\mu}-4g_{\mu\alpha}\left(q_{\nu_e}p_e\right)\right) \\
-\ & \frac{1}{4}\left(16\epsilon^{\mu\alpha\rho\sigma}\epsilon_{\mu\alpha\kappa\lambda}q_{\nu_e}^\kappa p_e^\lambda q_{e,\rho}p_{\nu_e,\sigma}\right)+\text{imaginary stuff} \\
=\ & \frac{1}{4}\left(4q_e^\mu p_{\nu_e}^\alpha+4q_e^\alpha p_{\nu_e}^\mu-4g^{\mu\alpha}\left(q_e p_{\nu_e}\right)\right)\left(4q_{\nu_e,\mu}p_{e,\alpha}+4q_{\nu_e,\alpha}p_{e,\mu}-4g_{\mu\alpha}\left(q_{\nu_e}p_e\right)\right) \\
+\ & \frac{1}{4}\left(32\left(\delta_\kappa^\rho\delta_\lambda^\sigma-\delta_\lambda^\rho\delta_\kappa^\sigma\right)q_{\nu_e}^\kappa p_e^\lambda q_{e,\rho}p_{\nu_e,\sigma}\right) \\
=\ & \frac{1}{4}\left(4q_e^\mu p_{\nu_e}^\alpha+4q_e^\alpha p_{\nu_e}^\mu-4g^{\mu\alpha}\left(q_e p_{\nu_e}\right)\right)\left(4q_{\nu_e,\mu}p_{e,\alpha}+4q_{\nu_e,\alpha}p_{e,\mu}-4g_{\mu\alpha}\left(q_{\nu_e}p_e\right)\right) \\
+\ & 8\left(\left(p_e p_{\nu_e}\right)\left(q_{\nu_e}q_e\right)-\left(p_e q_{e,\rho}\right)\left(p_{\nu_e}q_{\nu_e}\right)\right) \\
=\ & 4\left(q_e^\mu p_{\nu_e}^\alpha q_{\nu_e,\mu}p_{e,\alpha}+q_e^\alpha p_{\nu_e}^\mu q_{\nu_e,\mu}p_{e,\alpha}-g^{\mu\alpha}\left(q_e p_{\nu_e}\right)q_{\nu_e,\mu}p_{e,\alpha}\right) \\
+\ & 4\left(q_{\nu_e,\alpha}p_{e,\mu}q_e^\mu p_{\nu_e}^\alpha+q_{\nu_e,\alpha}p_{e,\mu}q_e^\alpha p_{\nu_e}^\mu-g^{\mu\alpha}q_{\nu_e,\alpha}p_{e,\mu}\left(q_e p_{\nu_e}\right)\right) \\
+\ & 4\left(-q_e^\mu p_{\nu_e}^\alpha g_{\mu\alpha}\left(q_{\nu_e}p_e\right)-q_e^\alpha p_{\nu_e}^\mu g_{\mu\alpha}\left(q_{\nu_e}p_e\right)+g_{\mu\alpha}\left(q_{\nu_e}p_e\right)g^{\mu\alpha}\left(q_e p_{\nu_e}\right)\right) \\
+\ & 8\left(\left(p_e p_{\nu_e}\right)\left(q_{\nu_e}q_e\right)-\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)\right) \\
=\ & 4\left(\left(p_e p_{\nu_e}\right)\left(q_e q_{\nu_e}\right)+\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)-\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)\right) \\
+\ & 4\left(\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)+\left(p_e p_{\nu_e}\right)\left(q_e q_{\nu_e}\right)-\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)\right) \\
+\ & 4\left(-\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)-\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)+g_{\mu\alpha}g^{\mu\alpha}\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)\right) \\
+\ & 8\left(\left(p_e p_{\nu_e}\right)\left(q_{\nu_e}q_e\right)-\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)\right) \\
=\ & 4\left(2\left(p_e p_{\nu_e}\right)\left(q_e q_{\nu_e}\right)+2\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)-4\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)+g_{\mu\alpha}g^{\mu\alpha}\left(p_e q_{\nu_e}\right)\left(p_{\nu_e}q_e\right)\right) \\
+\ & 8\left(\left(p_e p_{\nu_e}\right)\left(q_{\nu_e}q_e\right)-\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)\right) \\
=\ & 8\left(\left(p_e p_{\nu_e}\right)\left(q_e q_{\nu_e}\right)+\left(p_e, q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)\right) \\
+\ & 8\left(\left(p_e p_{\nu_e}\right)\left(q_{\nu_e}q_e\right)-\left(p_e q_e\right)\left(p_{\nu_e}q_{\nu_e}\right)\right) \\
=\ & 16\left(p_e p_{\nu_e}\right)\left(q_e q_{\nu_e}\right)
\end{aligned}
$$

thus finally we have

$$
\begin{aligned}
\overline{\sum_{\text{spins}}} |\mathcal{M}|^2 \;&=\; \frac{1}{2}\, 8\, G_F^2\, \mathrm{Tr}\left[\, \not{q}_e\, \gamma^\mu \;\; \not{p}_{\nu_e} \gamma^\alpha\, P_L \right]\, \mathrm{Tr}\left[\, \not{q}_{\nu_e}\, \gamma_\mu \;\; \not{p}_e\, \gamma_\alpha\, P_L \right] \\
&=\; 64\, G_F^2\, \left(p_e p_{\nu_e}\right) \left(q_e q_{\nu_e}\right) \,,
\end{aligned}
$$

where the factor of $\frac{1}{2}$ in the first line is the spin averaging factor (the incident electron has two spin states, the neutrino has one).

– Note that the "imaginary stuff" in the trace expression comes from the $i\epsilon^{\mu\alpha\rho\sigma}$ multiplying a bunch of four momenta and metric tensors, and the $i\epsilon_{\mu\alpha\kappa\lambda}$ in the other bracket doing similarly. We know however, that we are going to multiply these traces by a real number proportional to $8G_F^2$ to obtain the spin summed matrix element, and if we are sure of anything, we are sure that a number ($\mathcal{M}$) times its complex conjugate gives a real number, so we know that any imaginary stuff in this expression therefore has to cancel against itself! We can see already though that the only imaginary terms in the squared amplitude come from what we just explained "imaginary stuff" was, everything else that's not in "imaginary stuff" you can clearly see is all real, it is just products of four momenta (real) with metric tensors (all real) and / or epsilon tensors (all reall), hence we can simply proceed in the calculation throwing away the "imaginary stuff" immediately.

– Note also I have been a bit blasé when it comes to writing "."'s in dot products, tending to leave them out altogether. Essentially the dot product should be understood when you see a pair of adjacent four vectors without Lorentz indices (this is actually sufficient to completely distinguish them notationally here), however, to be really easy on the eye I hope I have demarcated all such four vector products with round brackets.

– Defining the kinematic invariant $s \equiv \left(p_e + p_{\nu_e}\right)^2$ and expanding the right-hand side we have

$$
\begin{aligned}
s \;&=\; \left(p_e + p_{\nu_e}\right)^2 \\
&=\; p_e^2 + p_{\nu_e}^2 + 2p_e.p_{\nu_e} \\
&=\; 2p_e.p_{\nu_e} \,.
\end{aligned}
$$

Furthermore, by momentum conservation we also have $s \equiv \left(p_e + p_{\nu_e}\right)^2 = \left(q_e + q_{\nu_e}\right)^2$, which also yields on expansion (again using the fact that we treat the electron and neutrino as massless)

$$
s \;=\; 2q_e.q_{\nu_e} \,.
$$

Substituting these relations into our expression for the spin averaged squared amplitude gives,

$$
\begin{aligned}
\overline{\sum_{\text{spins}}} |\mathcal{M}|^2 \;&=\; 64\, G_F^2\, \left(p_e p_{\nu_e}\right) \left(q_e q_{\nu_e}\right) \,, \\
&=\; 16\, G_F^2\, s^2 \,.
\end{aligned}
$$

– To go from the spin summed and average matrix element to the differential cross section we include the Lorentz invariant phase space factor, which in the case of a $2 \to 2$ process is given by

$$
d\mathrm{LIPS} = \frac{1}{4\pi^2}\, \frac{p_f}{4\sqrt{s}}\, d\Omega \,,
$$

where p_f is the magnitude of the three-momentum of either final-state particle in their combined rest frame, and then we divide by the flux factor

$$F = 4p_i\sqrt{s}\,,$$

where p_i is the magnitude of the three-momentum of either initial-state particle in the rest frame. Given we ignore the lepton and neutrino masses

$$\begin{aligned}
s &= 2p_e.p_{\nu_e} \\
&= 2\left(E_e E_{\nu_e} - |\boldsymbol{p}_e|\left|\boldsymbol{p}_{\nu_e}\right|\cos\theta_{e\nu_e}\right) \\
&= 2\left(|\boldsymbol{p}_e|\left|\boldsymbol{p}_{\nu_e}\right| - |\boldsymbol{p}_e|\left|\boldsymbol{p}_{\nu_e}\right|\cos\pi\right) \\
&= 4\,|\boldsymbol{p}_e|\left|\boldsymbol{p}_{\nu_e}\right| \\
&= 4\,|\boldsymbol{p}|^2 \\
\Rightarrow |p_i| &= |\boldsymbol{p}| = \frac{1}{2}\sqrt{s}\,.
\end{aligned}$$

A completely analogous computation holds for the $|p_f|$, $|p_f| = \frac{1}{2}\sqrt{s}$.

– Combining all pieces we find

$$\begin{aligned}
d\sigma &= \frac{1}{F}\overline{\sum_{\text{spins}}}\,|\mathcal{M}|^2\, d\text{LIPS} \\
&= \frac{1}{4p_i\sqrt{s}}\,16\,G_F^2\,s^2\,\frac{1}{4\pi^2}\,\frac{p_f}{4\sqrt{s}}\,d\Omega \\
&= \frac{G_F^2\,s}{4\pi^2}\,d\Omega\,.
\end{aligned}$$

N.B. the expression for the Lorentz invariant phase space can be looked up in e.g. Halzen & Martin (back cover), where it is also derived in detail on pg. 91.

– Since the scattering is isotropic ($s = 4\,|\boldsymbol{p}|^2$ and therefore has no angular dependence) the integral over the solid angle just gives an overall factor of 4π:

$$\sigma = \frac{G_F^2\,s}{\pi}\,.$$

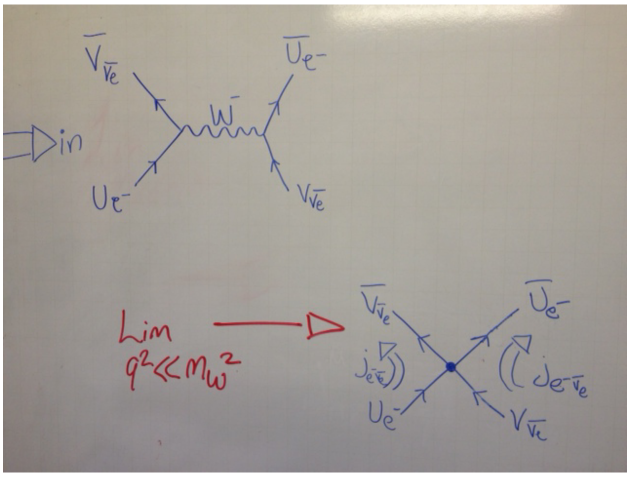

Figure 4. Anti-neutrino electron scattering via $V - A$ current-current four point interaction.

- An analogous calculation can be performed for anti-neutrino-electron scattering,

$$e^- \bar{\nu}_e \to e^- \bar{\nu}_e$$

(see Fig. 4)

  - For the scattering amplitude we have the same structure as before, all we do is take into account the fact that in the spinorial part of the currents we now have some antiparticle rather than particle spinors:

$$\mathcal{M} = j^{\mu}_{\bar{\nu}_e e, L}\, 2\sqrt{2} G_F\, g_{\mu\nu}\, j^{\nu}_{e\bar{\nu}_e, L}\,, \tag{1.18}$$

$$j^{\mu}_{\bar{\nu}_e e, L} = \bar{v}_{\bar{\nu}_e}\left(p_{\bar{\nu}_e}\right) \frac{1}{2} \gamma^{\mu} \left(1 - \gamma_5\right) u_e\left(p_e\right)\,, \tag{1.19}$$

$$j^{\nu}_{e\bar{\nu}_e, L} = \bar{u}_e\left(q_e\right) \frac{1}{2} \gamma^{\nu} \left(1 - \gamma_5\right) v_{\bar{\nu}_e}\left(q_{\bar{\nu}_e}\right)\,, \tag{1.20}$$

  where the p's and q's refer to incoming and outgoing momenta respectively (Fig. 3).

  - To square the amplitude we compute the complex conjugate, as before, in electron-neutrino scattering, noting that the only difference is that a couple of u particle spinrs are now replaced by v antiparticle spinors, but that's all. This being the case the manipulations used to derive the complex conjugate currents goes exactly as it did before — nothing in those manipulations referenced whether the spinors involved were particle of antiparticle spinors!
  - Essentially then, to get the conjugate currents, we just take the the spinors at the ends of the expressions for the currents and swap them around inside the current, from the front to the back and vice-versa, and make the 'barred' spinor the one now appearing at the front of the current.

– The first thing we want to do is get rid of the spinors by squaring the amplitude, more specifically by multiplying it by its complex conjugate:

$$\mathcal{M}^* = j^{\alpha *}_{\bar{\nu}_e e,L}\, 2\sqrt{2} G_F\, g_{\alpha\beta}\, j^{\beta *}_{e\bar{\nu}_e,L}\,, \tag{1.21}$$

where

$$j^{\alpha *}_{\bar{\nu}_e e,L} = \bar{u}_e\,(p_e)\,\frac{1}{2}\gamma^\alpha\,(1-\gamma_5)\;v_{\bar{\nu}_e}\,(p_{\bar{\nu}_e})\;, \tag{1.22}$$

$$j^{\beta *}_{e\bar{\nu}_e,L} = \bar{v}_{\bar{\nu}_e}\,(q_{\bar{\nu}_e})\,\frac{1}{2}\gamma^\beta\,(1-\gamma_5)\;u_e\,(q_e)\;. \tag{1.23}$$

– Squaring the amplitude and summing it over the fermion spins we then have

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}|^2 &= \sum_{\text{spins}} j^{\mu}_{\bar{\nu}_e e,L}\, 2\sqrt{2} G_F\, g_{\mu\nu}\, j^{\nu}_{e\bar{\nu}_e,L}\, j^{\alpha *}_{\bar{\nu}_e e,L}\, 2\sqrt{2} G_F\, g_{\alpha\beta}\, j^{\beta *}_{e\bar{\nu}_e,L}, \\ &= \sum_{\text{spins}} 8G_F^2\; g_{\alpha\beta} g_{\mu\nu}\, \left(j^{\mu}_{\bar{\nu}_e e,L}\, j^{\alpha *}_{\bar{\nu}_e e,L}\right)\, \left(j^{\nu}_{e\bar{\nu}_e,L}\, j^{\beta *}_{e\bar{\nu}_e,L}\right).\end{aligned}$$

– The currents and their complex conjugates have no loose Dirac indices. Using the cyclicity of the traces we are able to rewrite them in a way which groups the barred and un-barred fermion spinors together in the form where it is obvious how to perform spin sums with them (note we also use the Fermion spin sum relations with the fermion masses set to zero i.e. $\sum_s u\,(p,s)\,\bar{u}\,(p,s) = \sum_s v\,(p,s)\,\bar{v}\,(p,s) = \not{p}$)

$$\begin{aligned}\sum_{\text{spins}} j^{\mu}_{\bar{\nu}_e e,L}\, j^{\alpha *}_{\bar{\nu}_e e,L} &= \sum_{\text{spins}} \bar{v}_{\bar{\nu}_e}\,(p_{\bar{\nu}_e})\;\gamma^\mu P_L\, u_e\,(p_e)\,\bar{u}_e\,(p_e)\;\gamma^\alpha P_L\, v_{\bar{\nu}_e}\,(p_{\bar{\nu}_e}) \\ &= \sum_{\text{spins}} \mathrm{Tr}\left[\bar{v}_{\bar{\nu}_e}\,(p_{\bar{\nu}_e})\;\gamma^\mu P_L\, u_e\,(p_e)\,\bar{u}_e\,(p_e)\;\gamma^\alpha P_L\, v_{\bar{\nu}_e}\,(p_{\bar{\nu}_e})\right] \\ &= \mathrm{Tr}\left[\not{p}_{\bar{\nu}_e}\,\gamma^\mu\;\;\not{p}_e\,\gamma^\alpha P_L\right]\end{aligned}$$

$$\begin{aligned}\sum_{\text{spins}} j^{\nu}_{e\bar{\nu}_e,L}\, j^{\beta *}_{e\bar{\nu}_e,L} &= \sum_{\text{spins}} \bar{u}_e\,(q_e)\;\gamma^\nu P_L\, v_{\bar{\nu}_e}\,(q_{\bar{\nu}_e})\,\bar{v}_{\bar{\nu}_e}\,(q_{\bar{\nu}_e})\;\gamma^\beta P_L\, u_e\,(q_e) \\ &= \sum_{\text{spins}} \mathrm{Tr}\left[\bar{u}_e\,(q_e)\;\gamma^\nu P_L\, v_{\bar{\nu}_e}\,(q_{\bar{\nu}_e})\,\bar{v}_{\bar{\nu}_e}\,(q_{\bar{\nu}_e})\;\gamma^\beta P_L\, u_e\,(q_e)\right] \\ &= \mathrm{Tr}\left[\not{q}_e\,\gamma^\nu\;\;\not{q}_{\bar{\nu}_e}\,\gamma^\beta P_L\right]\end{aligned}$$

– Thus

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}|^2 &= \sum_{\text{spins}} 8G_F^2\; g_{\alpha\beta} g_{\mu\nu}\, \left(j^{\mu}_{\bar{\nu}_e e,L}\, j^{\alpha *}_{\bar{\nu}_e e,L}\right)\, \left(j^{\nu}_{e\bar{\nu}_e,L}\, j^{\beta *}_{e\bar{\nu}_e,L}\right) \\ &= 8G_F^2\, \mathrm{Tr}\left[\not{p}_{\bar{\nu}_e}\,\gamma^\mu\;\;\not{p}_e\,\gamma^\alpha P_L\right]\, \mathrm{Tr}\left[\not{q}_e\,\gamma_\mu\;\;\not{q}_{\bar{\nu}_e}\,\gamma_\alpha P_L\right]\;.\end{aligned}$$

– Now, from the neutrino-electron scattering calculation we evaluated the following very lengthy product of traces

$$\mathrm{Tr}\left[\not{q}_e\,\gamma^\mu\;\;\not{p}_{\nu_e}\gamma^\alpha\,P_L\right]\,\mathrm{Tr}\left[\not{q}_{\nu_e}\,\gamma_\mu\;\;\not{p}_e\,\gamma_\alpha\,P_L\right] = 16\,(p_e p_{\nu_e})\,(q_e q_{\nu_e})\;.$$

Crucially, in doing so, we made no assumptions at all about the nature of the four-momenta involved in the trace, i.e. this trace relation is true for arbitrary four-vectors a, b, c and d,

$$\mathrm{Tr}\,[\not{a}\,\gamma^\mu \;\; \not{b}\gamma^\alpha\,P_L]\;\mathrm{Tr}\,[\not{c}\,\gamma_\mu \;\; \not{d}\,\gamma_\alpha\,P_L] \;=\; 16\,(ac)\,(bd)\;,$$

and so we can use it to write

$$\begin{aligned}\overline{\sum}_{\text{spins}}|\mathcal{M}|^2 \;&=\; \frac{1}{2}\,8G_F^2\,\mathrm{Tr}\,[\not{p}_{\bar{\nu}_e}\,\gamma^\mu \;\; \not{p}_e\,\gamma^\alpha P_L]\;\mathrm{Tr}\,[\not{q}_e\,\gamma_\mu \;\; \not{q}_{\bar{\nu}_e}\,\gamma_\alpha P_L]\;. \\ &=\; 64G_F^2\;(p_{\bar{\nu}_e}q_e)\,(p_e q_{\bar{\nu}_e})\;.\end{aligned}$$

where the factor of $\frac{1}{2}$ appearing at the front goes with the bar over the spin sum to indicate that we are summing over all spins and averaging over the initial-state spin states (the incident neutrino has one possible spin state and the incident electron has two).

- Defining the kinematic invariant $s \equiv (p_e + p_{\bar{\nu}_e})^2$, and expanding the right-hand side we have

$$\begin{aligned}s \;&=\; (p_e + p_{\bar{\nu}_e})^2 \\ &=\; p_e^2 + p_{\bar{\nu}_e}^2 + 2p_e.p_{\bar{\nu}_e} \\ &=\; 2p_e.p_{\bar{\nu}_e} \\ &=\; 2\,(|\vec{p}|\,,0,0,-|\vec{p}|)\,(|\vec{p}|\,,0,0,|\vec{p}|) \\ &=\; 4\,|\vec{p}|^2\end{aligned}$$

  where $\vec{p}$ is the 3-momenta of the incoming anti-neutrino in the collision centre-of-mass frame. Thus we have the relation

$$|\vec{p}| = \frac{1}{2}\sqrt{s}\,.$$

- Repeating the kinematics exercise for the scalar product in our squared matrix element, defining θ_e and $\theta_{\bar{\nu}_e}$ as the polar angle of the outgoing eletron and neutrino, respectively, with the incident $\bar{\nu}_e$ defining the +z-axis, we get

$$\begin{aligned}p_{\bar{\nu}_e}.q_e \;&=\; (|\vec{p}|\,,0,0,|\vec{p}|)\,(|\vec{p}|\,,\vec{q}_e) \\ &=\; |\vec{p}|^2\,(1-\cos\theta_e) \\ &=\; \frac{1}{4}s\;(1-\cos\theta_e)\end{aligned}$$

  and

$$\begin{aligned}p_e q_{\bar{\nu}_e} \;&=\; (|\vec{p}|\,,0,0,-|\vec{p}|)\,(|\vec{p}|\,,\vec{q}_{\bar{\nu}_e}) \\ &=\; |\vec{p}|^2\,(1-\cos{(180-\theta_{\bar{\nu}_e})}) \\ &=\; |\vec{p}|^2\,(1-\cos\theta_e) \\ &=\; \frac{1}{4}s\;(1-\cos\theta_e)\;.\end{aligned}$$

In terms of the polar angle of the scattered electron we are then left with

$$\overline{\sum}_{\text{spins}} |\mathcal{M}|^2 = 64G_F^2\, (p_{\bar{\nu}_e} q_e)\, (p_e q_{\bar{\nu}_e}) \\ = 4G_F^2\, s^2\, (1-\cos\theta_e)^2$$

– As in neutrino-electron scattering, here, to go from the spin summed and average matrix element to the differential cross section we include the Lorentz invariant phase space factor, which in the case of a $2 \to 2$ process is given by

$$d\text{LIPS} = \frac{1}{4\pi^2}\, \frac{p_f}{4\sqrt{s}}\, d\Omega\,,$$

where p_f is the magnitude of the three-momentum of either final-state particle in their combined rest frame, and then we divide by the flux factor

$$F = 4p_i\sqrt{s}\,.$$

– Combining all pieces we find

$$\begin{aligned} d\sigma &= \frac{1}{F} \overline{\sum_{\text{spins}}} |\mathcal{M}|^2\, d\text{LIPS} \\ &= \frac{1}{4p_i\sqrt{s}}\, 16G_F^2\, s^2\, \frac{1}{4}\, (1-\cos\theta_e)^2\, \frac{1}{4\pi^2}\, \frac{p_f}{4\sqrt{s}}\, d\Omega \\ &= \frac{G_F^2\, s}{4\pi^2}\, \frac{1}{4}\, (1-\cos\theta_e)^2\, d\Omega\,. \end{aligned}$$

N.B. the expression for the Lorentz invariant phase space can be looked up in e.g. Halzen & Martin (back cover), where it is also derived in detail on pg. 91.

– Up to the factor $\frac{1}{4}(1-\cos\theta_e)^2$ this differential cross section is exactly the same as the one obtained for $e^-\nu_e \to e^-\nu_e$ scattering. Strikingly though here we have an angular dependence of the differential cross section in the rest frame whereas in the latter case the angular distribution of the scattered electron was isotropic there!