QCD Phenomenology – Final States

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Quantum ChromoDynamics QCD is the theory of the strong interaction.

Quarks (fermions) interact via the exchange of gluons (vector bosons) with the physics described by the $SU(3)$ gauge theory with Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = -1/4 F^{\mu\nu}{}_{a} F_{\mu\nu a} + \sum_{f=1}^{n_f} \bar{q}_f \big(i\gamma^\mu D_\mu - m_f \big) q_f,$$

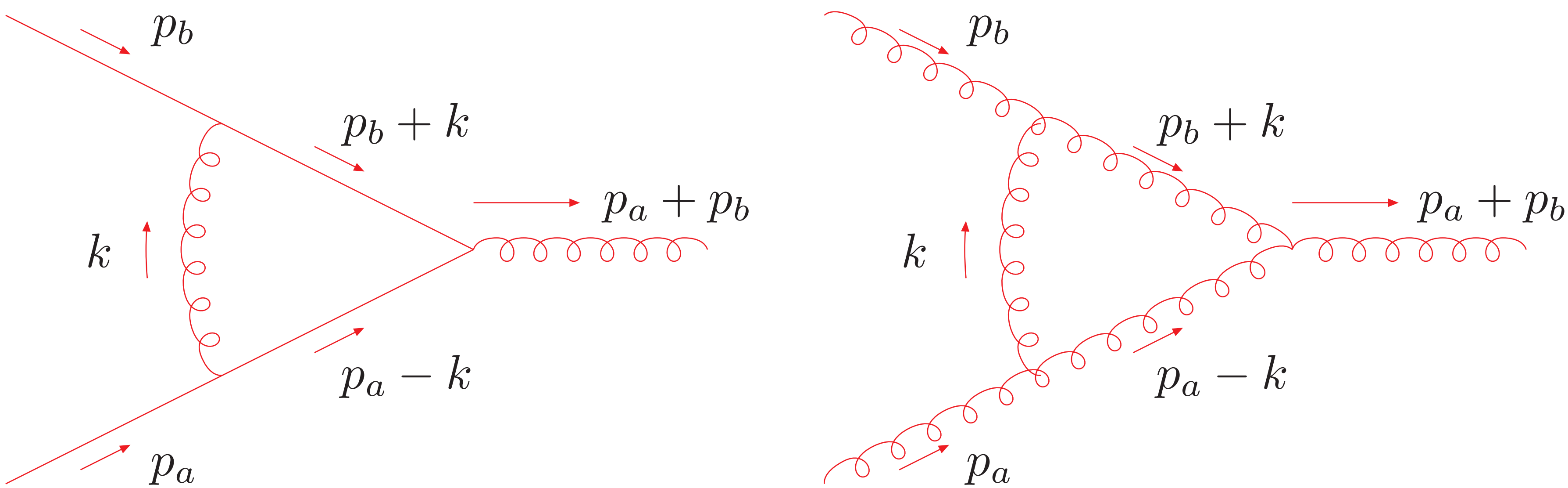
where the covariant derivative is defined by

$$D_\mu q_f = \partial_\mu q_f + i g_s A_{\mu a} 1/2 \lambda_a q_f$$

and q_f represent the fermionic quark fields and $A_{\mu,a}$ the vector boson gluon fields. (Very similar for the electroweak sector.) The sum over f is for the different quark flavours, up. down, strange, charm, bottom and top, each with different masses.

Can formulate Feynman rules to calculate particle interactions as a perturbation series in $\alpha_S = g_s^2/(4\pi)$

At first non-classical order obtain corrections to quark-gluon or gluon-gluon coupling of form



This results in integrals of the form

$$\mathcal{V} \sim \int \frac{d^4k}{(2\pi)^4} \frac{k\ k}{k^2(p_b+k)^2(p_a-k)^2} \rightarrow \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \int \frac{dk}{(2\pi)} \frac{1}{k}$$

when we consider the limit $k \rightarrow \infty$ in the loop. Leads to *ultraviolet* divergence.

In order to obtain a well-defined result must implement some *ultraviolet cutoff* Λ_0 above which QCD is no longer a reliable theory (e.g. Λ_0 is the scale of new physics).

Also introduce a physical *renormalization scale* μ_R – choose to be similar to scale of physics.

Subtract divergences like $\ln(\Lambda_0^2/\mu_R^2)$ and absorb into definition of **bare** parameters, leaving behind finite predictions in terms of physical **renormalised** parameters.

$$g_s^0 = g_s + g_s^3 C \ln(\Lambda_0^2/\mu_R^2) \qquad\qquad \sigma(\{p\}, g_s^0, \Lambda_0) \equiv \sigma(\{p\}, g_s)$$

Process known as renormalization. Long been proved that it can be applied successfully to all orders in QCD and rest of the Standard Model.

However, we have introduced artificial renormalization scale μ_R on which renormalised couplings, masses, *etc* depend, though dependence disappears (at all orders in physical quantities), e.g.

$$\frac{d}{d\ln\mu_R^2}\left(\alpha_S(\mu_R^2)\sigma_1(\{p\}, \mu_R) + \alpha_S^2(\mu_R^2)\sigma_2(\{p\}, \mu_R)\right) = \mathcal{O}(\alpha_S^3(\mu_R^2)).$$

By calculating previous diagrams representing coupling find that coupling satisfies evolution equation

$$\frac{d\alpha_S}{d\ln\mu_R^2} = -\beta_0\alpha_S^2 - \beta_1\alpha_S^3 + \cdots, \qquad \beta_0 = \frac{(11 - 2/3N_f)}{4\pi}$$

Negative β-function means strong at low scales but weaker at higher scales.

Ignoring the $\mathrm{O}(\alpha_s^3)$ corrections this may be solved

$$-\int_{\mu_0^2}^{\mu_R^2} d\,\ln\tilde{\mu}_R^2 = \frac{1}{\beta_0}\int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu_R^2)} \frac{d\,\tilde{\alpha}_s}{\tilde{\alpha}_s^2},$$

where μ_0 is some fixed scale. Hence,

$$-\ln(\mu_R^2/\mu_0^2) = \frac{1}{\beta_0}\left[\frac{1}{\alpha_s(\mu_0^2)} - \frac{1}{\alpha_s(\mu_R^2)}\right].$$

This leads to

$$\alpha_s(\mu_R^2) = \frac{1}{\beta_0} * \frac{1}{\ln(\mu_R^2/\mu_0^2) + \frac{1}{\beta_0\alpha_s(\mu_0^2)}}.$$

From this expression we can indeed see that $\alpha_s(\mu_R^2)$ decreases as μ_R^2 increases, and that $\alpha_s(\mu_R^2) \to 0$ as $\mu_R^2 \to \infty$. However, the definition relies on an arbitrary boundary condition for the coupling at some fixed scale μ_0^2.

It is simpler, and more illustrate to rewrite the solution for $\alpha_s(\mu_R^2)$ slightly. It may be expressed as

$$\alpha_s(\mu_R^2) = \frac{1}{\beta_0} * \frac{1}{\ln(\mu_R^2) - (\ln(\mu_0^2) - \frac{1}{\beta_0 \alpha_s(\mu_0^2)})}.$$

Defining a scale Λ_{QCD} by

$$\ln(\mu_0^2) - \frac{1}{\beta_0 \alpha_s(\mu_0^2)} = \ln(\Lambda_{QCD}^2),$$

Λ_{QCD} is the value of μ_0^2 for $\alpha_s(\mu_0^2) \to \infty$. Results in the solution.

$$\alpha_s(\mu_R^2) \approx \frac{4\pi}{(11 - 2/3N_f)\ln(\mu^2/\Lambda_{QCD}^2)}$$

Binds partons into hadrons at low scales, but can do perturbative calculations at higher scales.

Even in processes involving no incoming hadrons the final state is often dominated by hadrons, and thus a full understanding of the physics requires an understanding of this hadronic final state. This means we must use QCD and understand it as well as possible. The most obvious features in the hadronic final states are jets.

General form of Perturbative Expansion

Suppose we calculate a total cross-section with one variable, e.g. centre of mass energy $\sqrt{s}$. Since the coupling depends on the renormalization scale μ the cross-section is scale-dependent. At LO in α_S

$$\sigma(s) = A\alpha_S(\mu^2).$$

This automatically leads to

$$\frac{d\sigma(s)}{d\ln\mu^2} = -A\beta_0\alpha_S^2(\mu^2).$$

At NLO in α_S renormalisation leads to explicit scale dependence

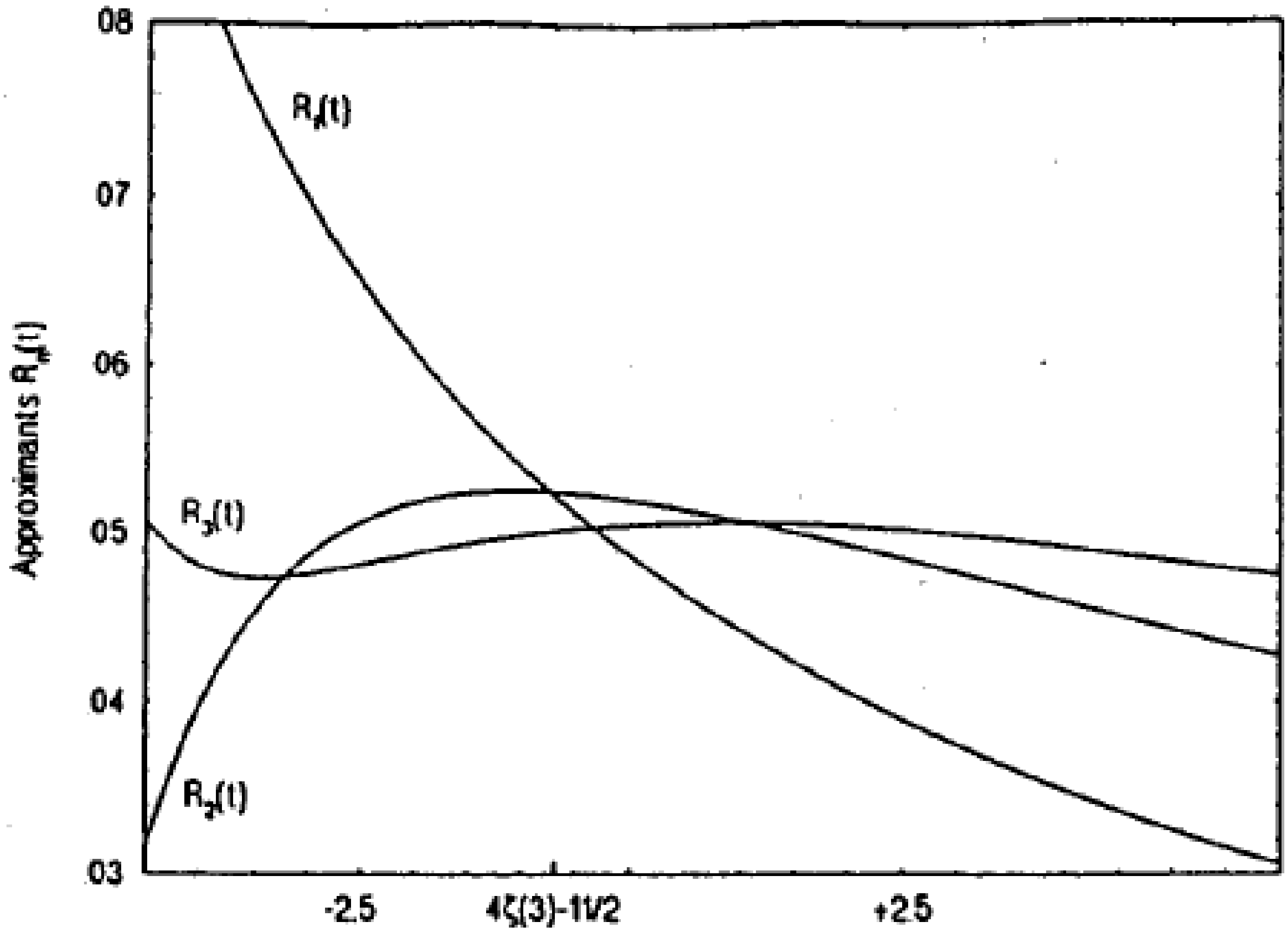
$$\sigma(s) = A\alpha_S(\mu^2) + \alpha_S^2(\mu^2)(B + b\ln(\mu^2/s)).$$

In general the scale dependence is

$$\frac{d\sigma(s)}{d\ln\mu^2} = -A\beta_0\alpha_S^2(\mu^2) + b\alpha_S^2(\mu^2) + \mathcal{O}(\alpha_S^3).$$

The scale dependence must decrease as we go to higher orders.

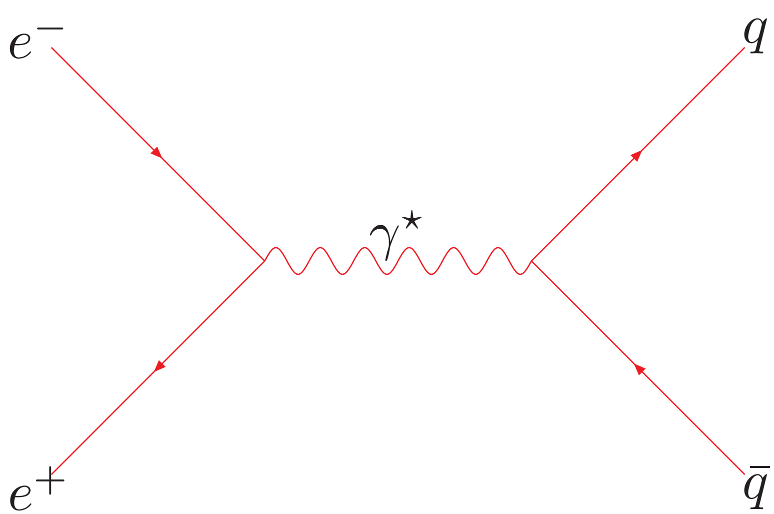
Achieved if $b = A\beta_0$, i.e. scale dependent part of NLO correction determined by lower orders and running of the coupling. Constant B has to be calculated explicitly.

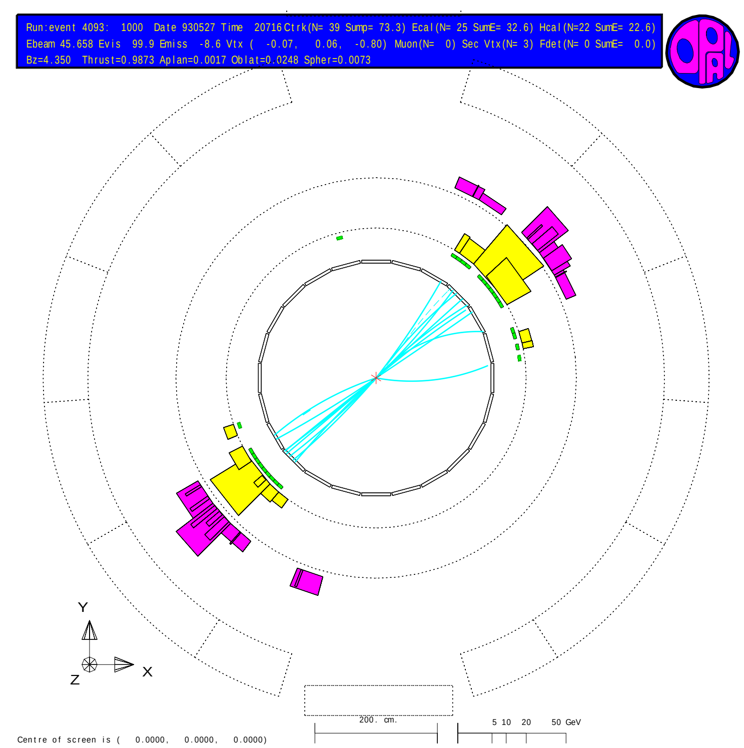


Renormalisation scheme dependence at LO, NLO and NNLO, for ratio of $e^+ + e^- \rightarrow \text{hadrons/leptons}$ (Samuel and Surguladze).

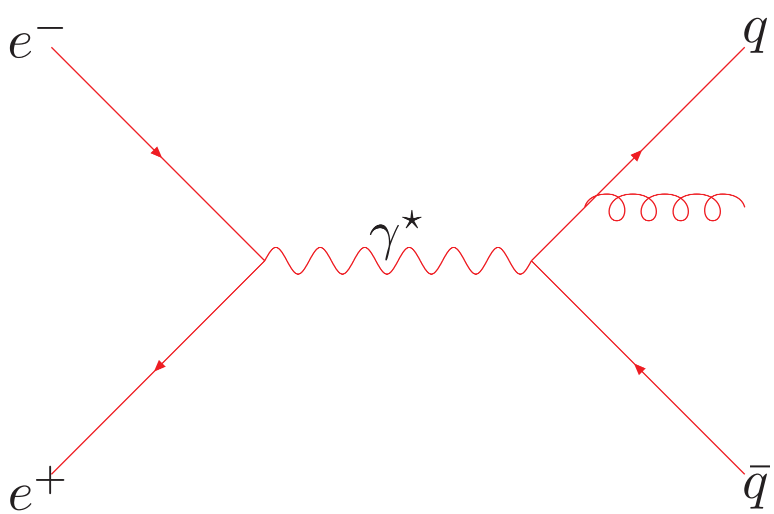
Jet Events

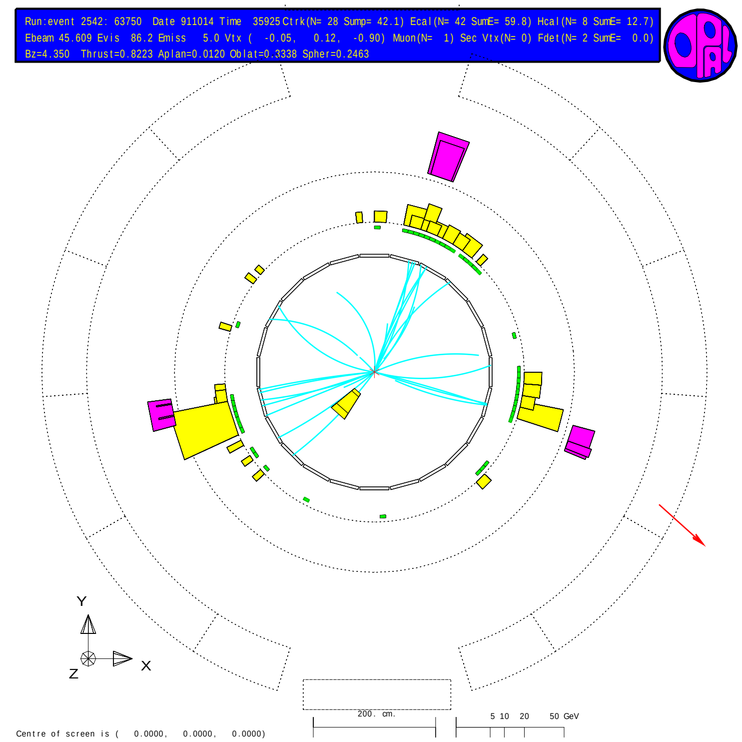
Consider the simplest case of e^+e^- annihilation into a photon producing a hadronic final state. At zeroth order in QCD this will simply be a quark-antiquark pair at parton level, and each quark will hadronize into at jet. Hence we have a 2-jet final state.





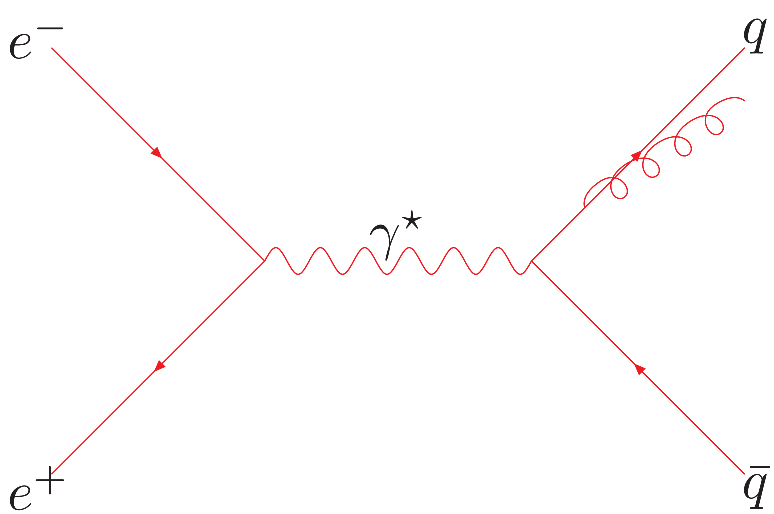
At first order in α_S we can also have the emission of a gluon from the quark or anti-quark, e.g.

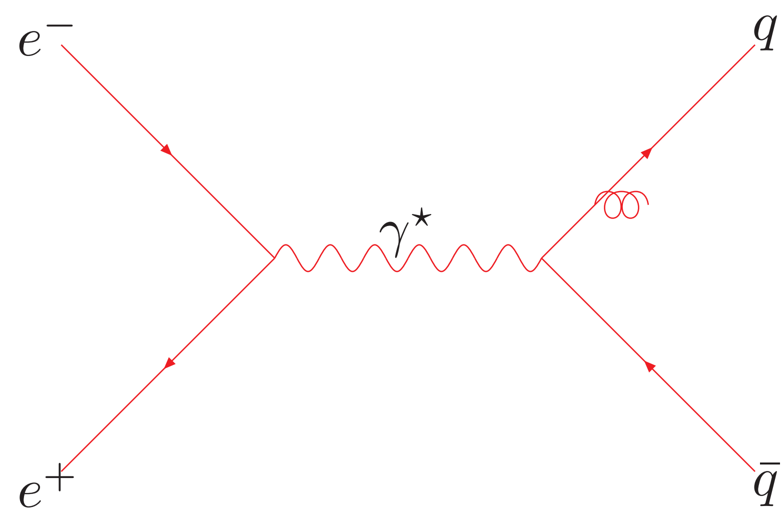




In general this will lead to a 3-jet final state.

At order α_S^2 we could have two gluons, or the one gluon could fragment into a quark-antiquark pair, and we could obtain a 4-jet final state. This continues to higher orders, i.e. at n_{th}-order α_S we can have a $(n+2)$-jet state. However, if a final state parton is sufficiently collinear with another parton or is sufficiently unenergetic (soft) it will simply go into the jet of the initial parton. i.e.



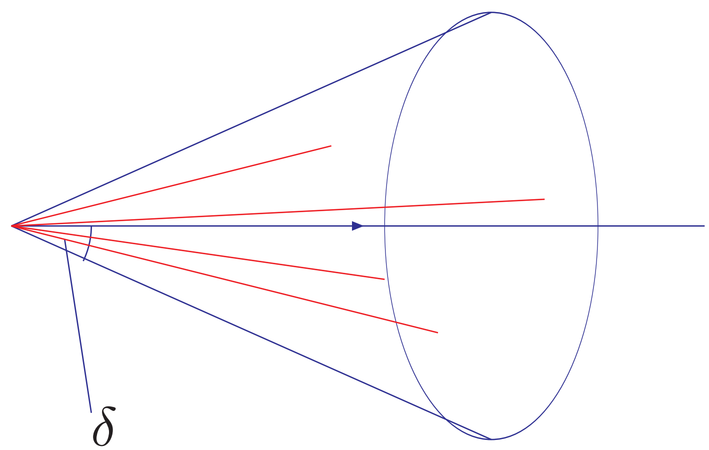


will contribute to the 2-jet rate rather than the 3-jet rate.

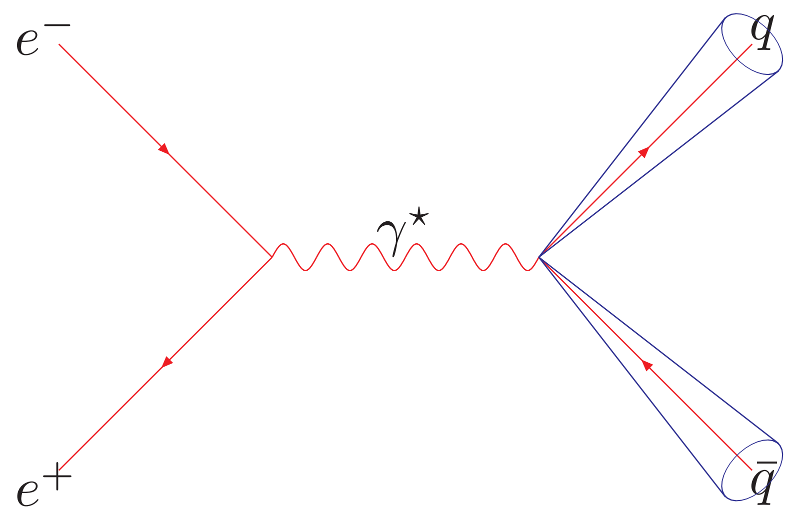
Jet Definitions

In order to be quantitative we need a precise definition of what constitutes an n-jet event, i.e. we need a jet definition.

The simplest definition (in principle) is the cone algorithm. Define a cone with an opening angle δ. The jet is made up of all partons within the cone, with the axis chosen such that the energy within the cone is maximised, and the momentum of the jet is the sum of the hadron momenta.

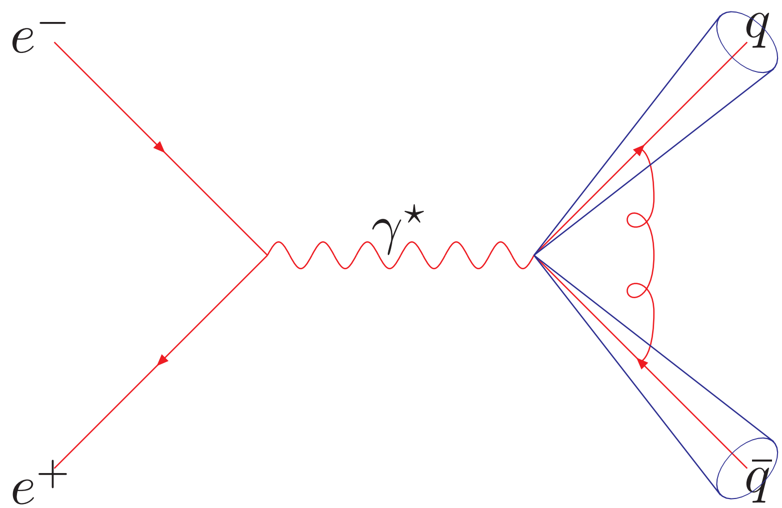


The same definition must be used at parton level in order to predict the theoretical jet rate. At leading order



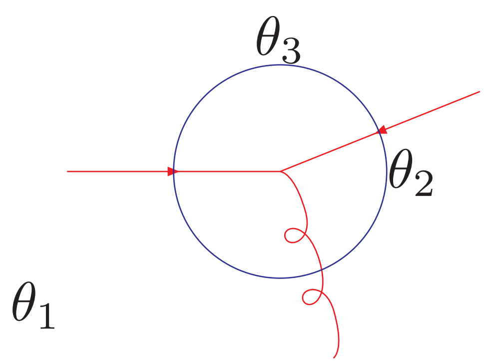
Hence at this order $\sigma_{2jet} = \sigma_{tot}$, and every event has two jets each of energy $1/2\sqrt{s}$.

When working to order α_S we must consider the virtual correction



which always leads to a 2-jet event. However, the amplitude for this diverges. We must also consider the emission of a gluon off the quark (antiquark) line.

Real emission of a gluon will lead to a 3-jet event if $\theta_1, \theta_2, \theta_3 > \delta$ (in the limit $\theta_i \to 0$ the 3-parton amplitude also diverges).



Defining $x_{1,2,3} = \frac{2E_{q,\bar{q},g}}{\sqrt{s}}$ the three parton cross section is calculated as

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

where using $\underline{p}_1 = \underline{p}_2 + \underline{p}_3$ and $E_1 = \sqrt{s} - E_2 - E_3$

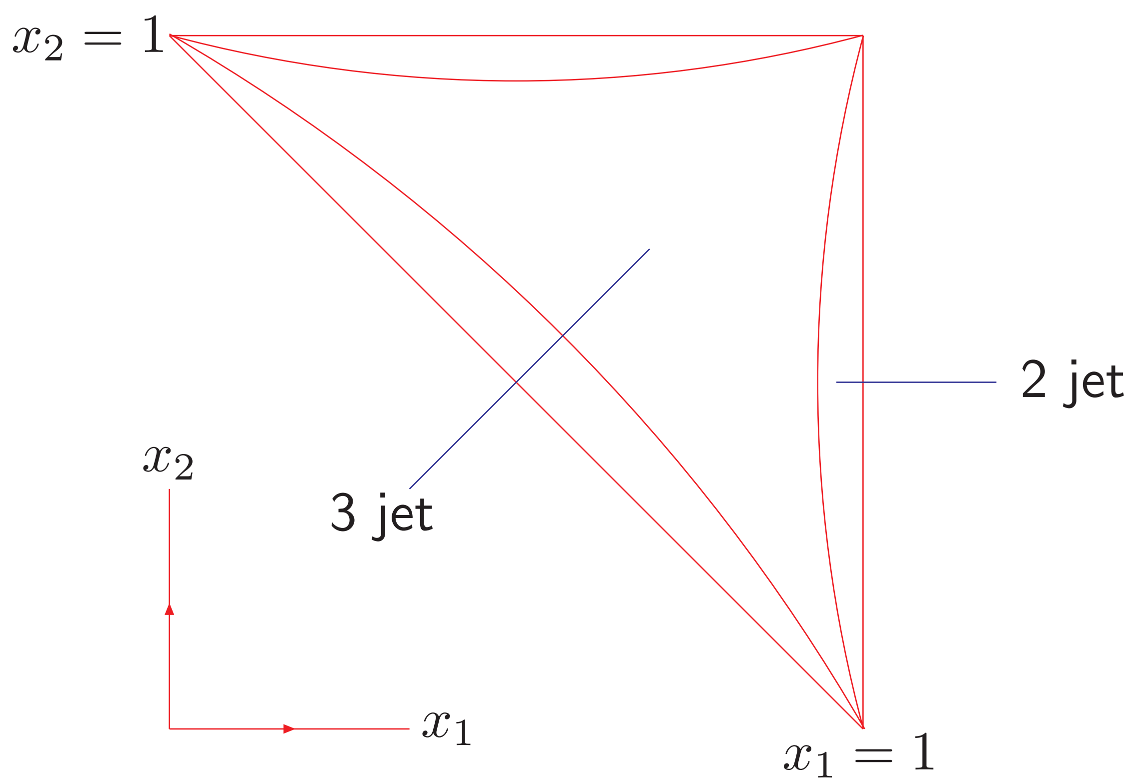
$$(1-x_1) = \frac{1}{2} x_2 x_3 (1 - \cos\theta_{qg}) + \text{permutations.}$$

Hence divergences when $\theta_i \to 0$.

$$\theta_1 > \delta \to \frac{2(1-x_1)}{x_2(2-x_1-x_2)} > 1 - \cos(\delta)$$

So if $\delta << 1$ we have $x_1 > 1 - 0.25\delta^2 x_2(1-x_2)$.

Using the kinematic constraint that $x_1, x_2 \leq 1$, $x_1 + x_2 \geq 1$ and $x_1 + x_2 \leq 2$, the phase space can be pictured as below



However, this single constraint does not render results for the 2-jet and 3-jet cross-sections finite, i.e. there is not a complete cancellation of divergence between the real and virtual contributions. This is because we still have divergences coming from the situation when one of the 3 jets has vanishingly small momenta, i.e. when the corresponding parton is sufficiently soft.

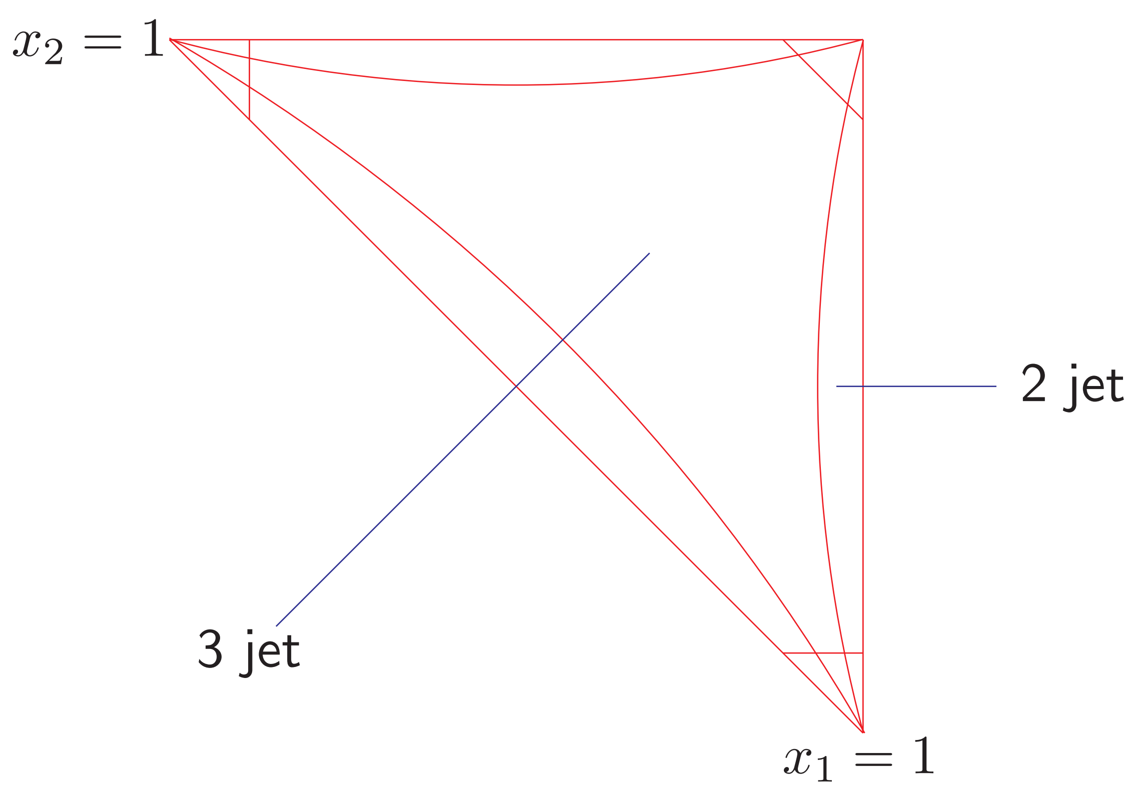
Infrared Safety

To obtain a finite answer we must ensure that both soft and collinear real emission is treated the same way as virtual contributions.

$\rightarrow$ as well as finite angle δ we need an energy cut-off in the jet definition.

$$E_{jet} > \epsilon \times 1/2\sqrt{s}$$

This modifies our previous diagram.



The cancellation of divergences between real and virtual contributions is now complete and the fraction of 3-jet events R_3 at first order in α_S is now given by.

$$R_3 = C_F \frac{\alpha_S}{2\pi} \left[\log(1/\delta) \log(1/\epsilon) + \log(1/\epsilon) + \log(1/\delta) + C \right]$$

$$R_2 = 1 - R_3$$

where the latter definition avoids complications in regularising the divergent contributions.

Note: as $\epsilon \longrightarrow 0$ or $\delta \longrightarrow 0$, R_3 can become very large, i.e. cancellation of divergences is ceasing to be effective. Possible for $R_3 > 1$ i.e. $R_2 < 0$ - clearly not sensible.

Intuitively simple jet definition. But suffers from some problems in details, e.g. need two completely separate cuts-off; possible for two cones to overlap. What happens in this case?

$\longrightarrow$ alternative definitions.

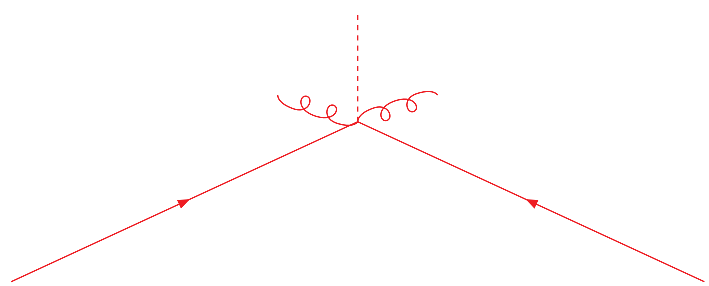
Cluster Algorithms.

Consider a system of two hadrons in the experimental case, or two partons in the theoretical case. JADE algorithm will combine them into a single jet if the invariant mass m^2 satisfies.

$$m^2 = (p_1 + p_2)^2 < y_{cut} s.$$

$m^2 = 2E_1 E_2 (1 - \cos\theta_{12})$ and so $\rightarrow 0$ if $E_i \rightarrow 0$ (soft) or $\theta_{12} \rightarrow 0$ (collinear).

Works pretty well, but has some problems for small y_{cut}. Exhibited by below event.



The two gluons may be combined into a spurious jet indicated by the dashed line. This may be avoided by a simple modification.

K_T (Durham) Algorithm.

In this case the algorithm will combine into a single jet if

$$K_T^2 < y_{cut} s.$$

where $K_T^2 = 2min(E_1^2, E_2^2)(1 - \cos\theta_{12})$ and so $\rightarrow 0$ if $E_i \rightarrow 0$ (soft) or $\theta_{12} \rightarrow 0$ (collinear).

Use an iterative procedure

1. Find the pair with the smallest y_{ij}.

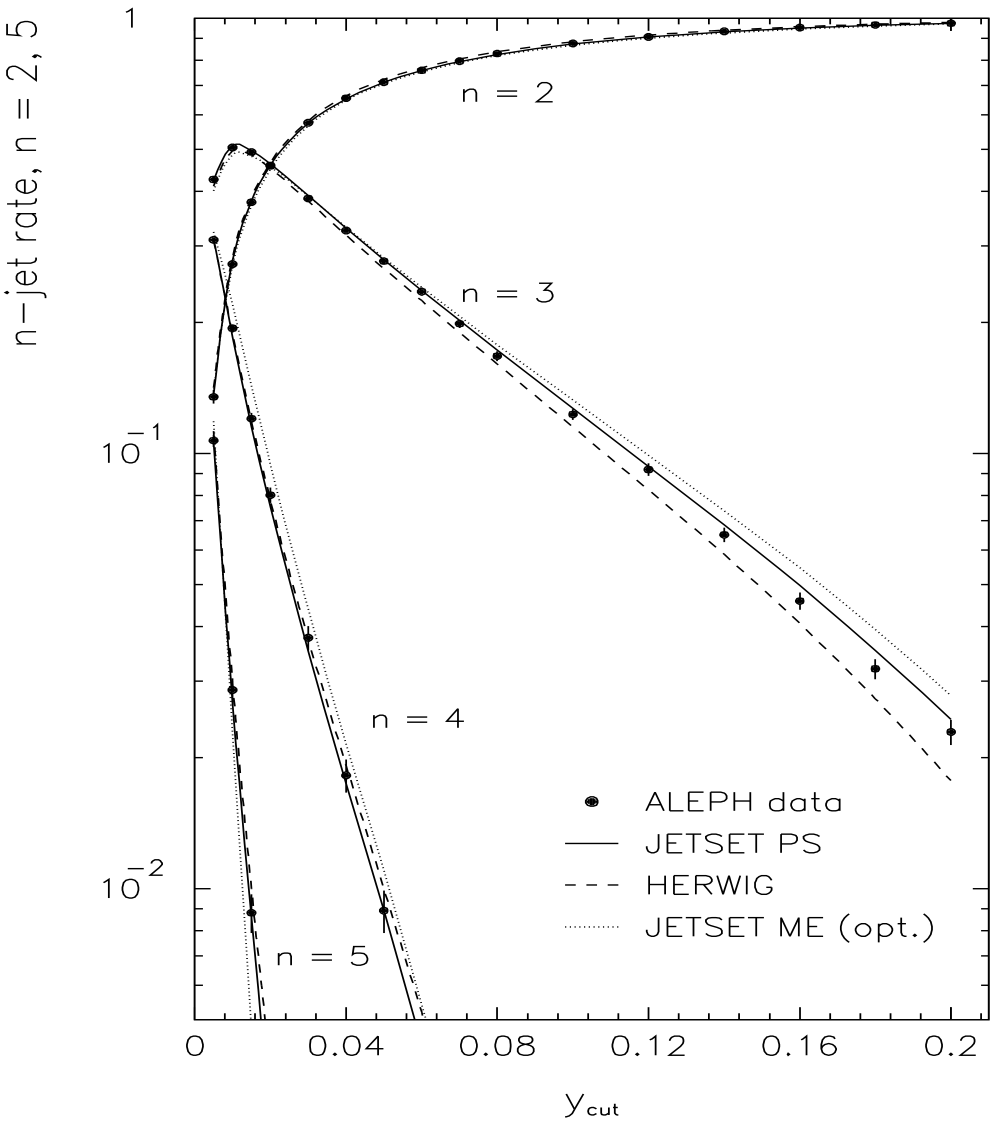
2. If $y_{ij,min} < y_{cut}$ combine i and j, e.g. $p_{ij} = p_i + p_j$ and go back to 1.

3. If $y_{ij,min} > y_{cut}$ stop. All remaining momenta are called jets.

(Avoids previous problem because gluons combined with quarks before each other.)

In practice simpler than cone algorithm. Can be applied at any order. $\rightarrow$ powerful test of QCD.

Comparison with n-jet rates at ALEPH.



Anti-K_T Algorithm

"anti-K_T algorithm" combines all soft partons within "cone" with hard parton to produce cone-like jet definition.

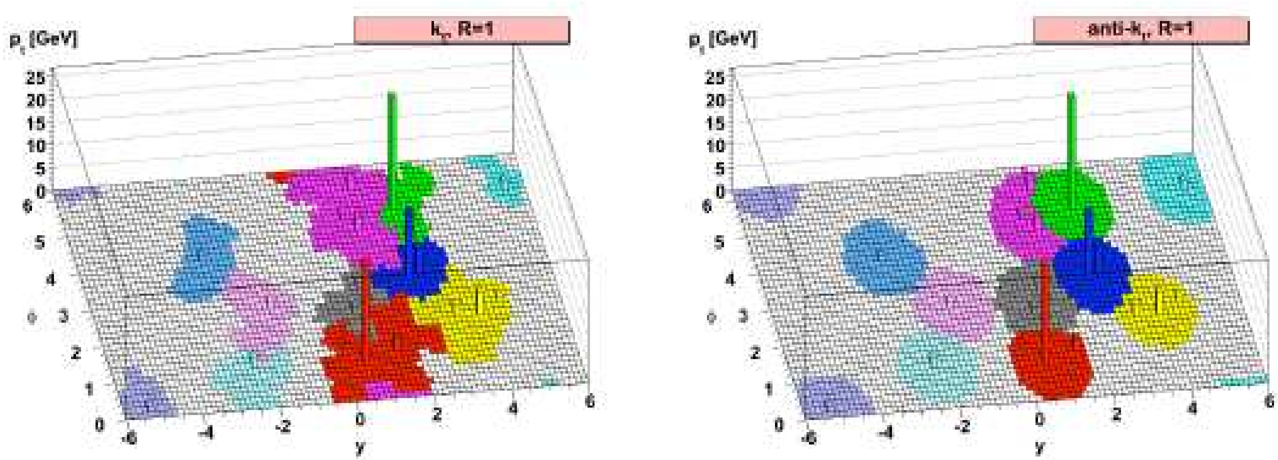
Come back to recombination-type algorithms:

Soyez

$$d_{ij} = \min(k_{t,i}^{2p}, k_{t,j}^{2p})\ (\Delta\phi_{ij}^2 + \Delta\eta_{ij}^2)$$

- $p = 1$: k_t algorithm
- $p = 0$: Aachen/Cambridge algorithm
- $p = -1$: anti-k_t algorithm [M.Cacciari, G.Salam, G.S.,JHEP 04 (08) 063]

Hard event + homogeneous soft background



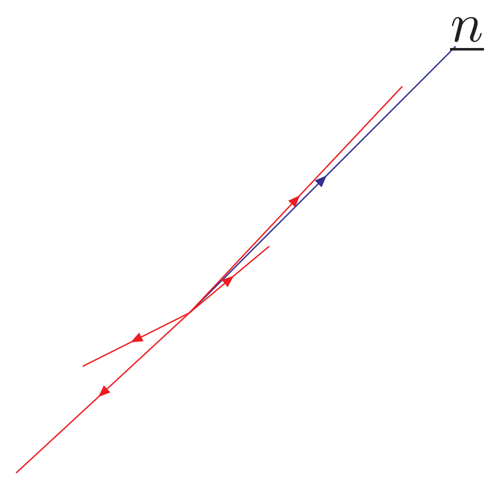
Most commonly used at the LHC.

Event Shapes.

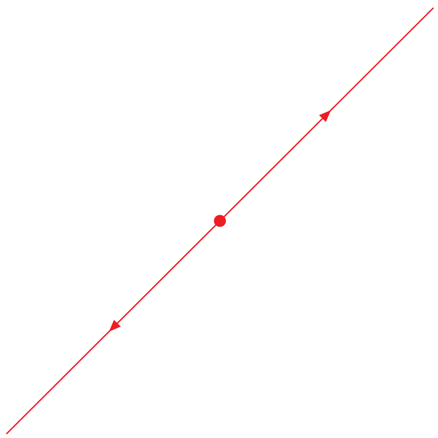
These are quantities which contain more detailed information about the final hadronic state.

e.g. Thrust $T = max_{\underline{\mathsf{n}}} \frac{\sum_i |\underline{\mathsf{p}}_i \cdot \underline{\mathsf{n}}|}{\sum_i |\underline{\mathsf{p}}_i|}$

this measures how unidirectional a set of jets are,

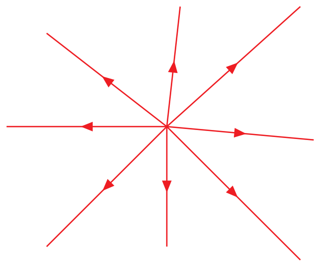


Has extreme limits



$T = 1$

$T = 1/2$



isotropic

Other variable such as spherocity S

$$S = \left(\frac{4}{\pi}\right)^2 min_{\underline{\mathsf{n}}} \left(\frac{\sum_i |\underline{\mathsf{p}}_i \times \underline{\mathsf{n}}|}{\sum_i |\underline{\mathsf{p}}_i|}\right)^2$$

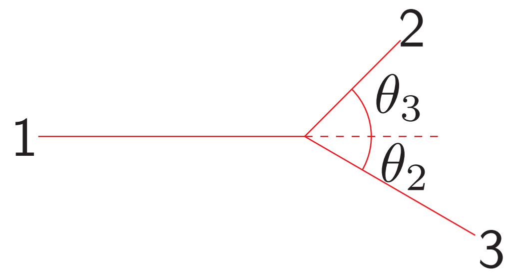
which has value $S = 1$ for an isotropic event and $S = 0$ for a linear event. Also $C - \mathrm{parameter}$

$$C - \mathrm{parameter} = \frac{3}{2}\frac{\sum_{i,j} |\underline{\mathsf{p}}_i||\underline{\mathsf{p}}_j| \sin^2\theta_{ij}}{(\sum_i |\underline{\mathsf{p}}_i|)^2}$$

which is similar to spherocity.

Concentrate on Thrust as example.

Order α_S - three partons



It is not too difficult to see that n must lie along the axis of a particle.

$$T = max\left(\frac{x_1 + x_2\cos\theta_2 + x_3\cos\theta_3}{x_1 + x_2 + x_3}, \cdots\right)$$

$$T = max\{x_1, x_2, x_3\}$$

Leads to a constraint $\rightarrow T > 2/3$ at this order

From the previous calculation of the three parton rate

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi} \int dx_1\, dx_2 \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T - max\{x_1, x_2, x_3\})$$

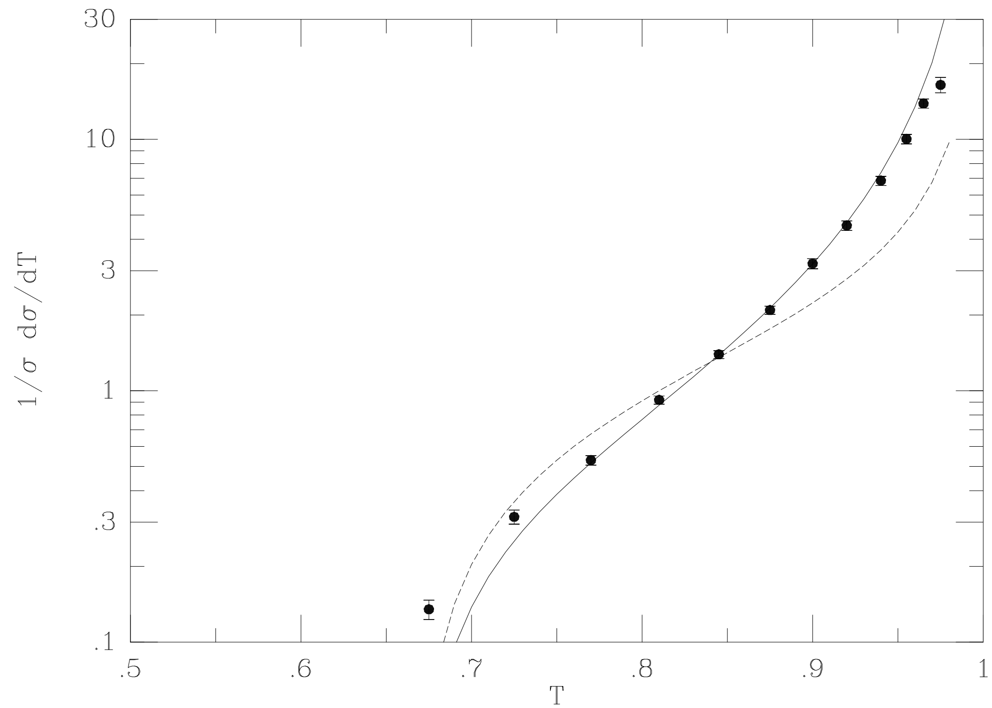
We obtain a cross-section which diverges in the limit $T \to 1$ tending to

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi} \left[\frac{4}{(1-T)} \log\left(\frac{1}{1-T}\right) - \frac{3}{1-T} \right]$$

This is due to soft and collinear singularities. The $\mathcal{O}(\alpha_S)$ virtual correction $\propto \delta(1-T)$ such that a finite total cross-section is obtained. Of course in the limit $T \to 1$ we should use a proper jet definition.

The figure shows the thrust distribution measured at DELPHI compared with the theory (solid line) and for scalar gluon (dashed line).

Deficiency at small T due to kinematic bound for 3-parton process. Just starting to fail as $T \to 1$.



Useful to define an effective **two jet rate** using the thrust distribution.

$$R(\tau) \equiv \int_{1-\tau}^{1} dT \, \frac{1}{\sigma}\frac{d\sigma}{dT}$$

where σ is now summed over all n-parton final states.

This is like the previous two-jet fraction, but with $\tau = 1 - T$ being the jet resolution y. As $\tau \longrightarrow 0$ find up to NNLO

$$R(\tau) \to 1 - C_F\frac{\alpha_S}{2\pi} 2\log^2\tau + \left(C_F\frac{\alpha_S}{2\pi}\right)^2 2\log^4\tau$$

Extra $\log^2\tau$ at every order of α_S. Negative at NLO as $\tau \to 0$. Oscillates at NNLO.

Fortunately terms in $\log^2\tau$ can be very neatly summed to all orders (resummed) $\longrightarrow$ Sudakov form factor

$$R(\tau) \to \exp\left[-C_F\frac{\alpha_S}{2\pi} 2\log^2\tau\right]$$

Hence the probability to produce a $q\bar{q}$ pair with no accompanying gluon $= 0$. (Similar to QED.)

$$P(\text{no emission}) = \exp\big(-P_{simple}(\text{emission})\big).$$

Higher order resummations.

Exponentiation removes unphysical behaviour, but still problems with perturbative convergence due to large $\log\tau$ terms and corresponding μ^2-dependence.

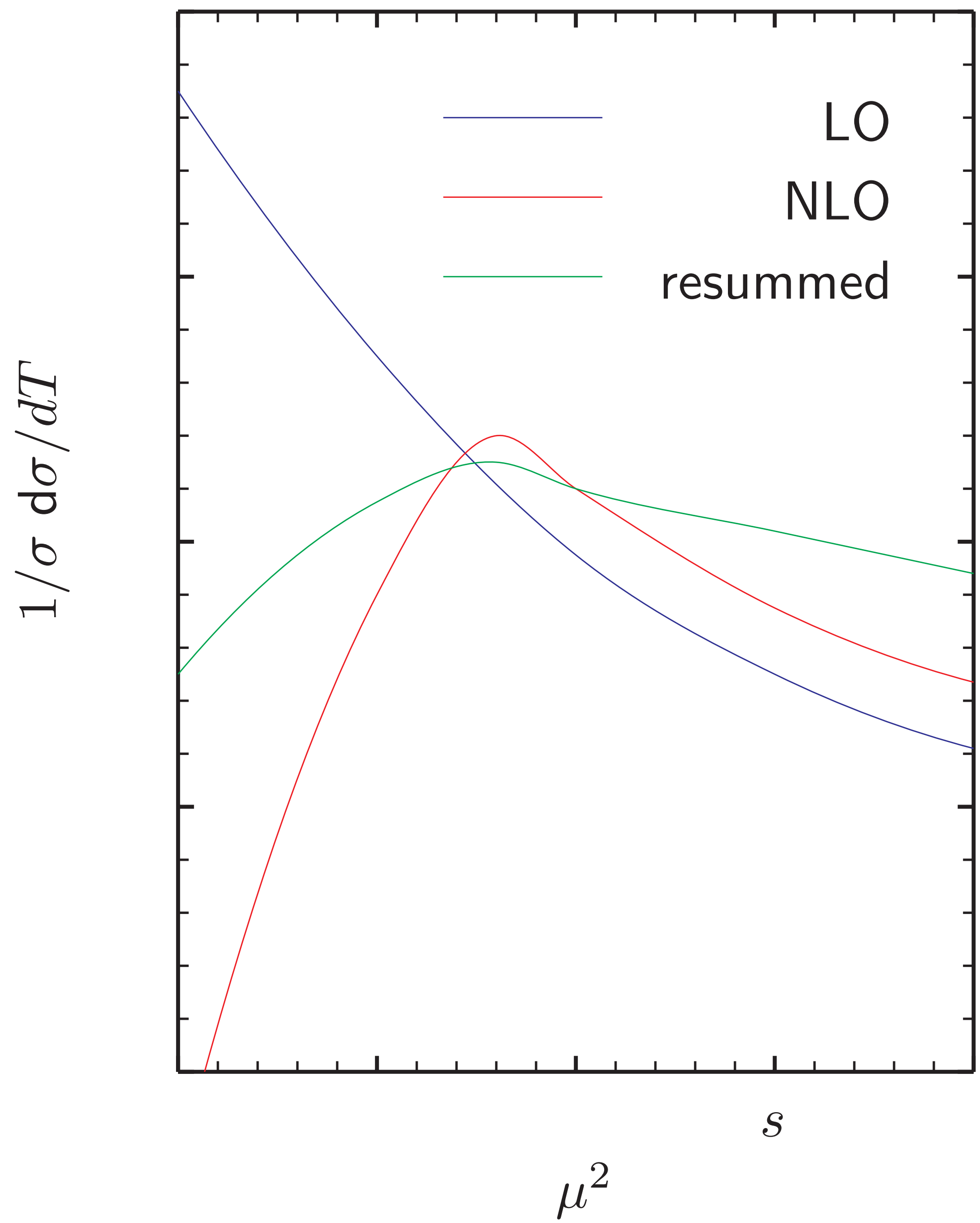
By iterative means able to even resum exponent.

$$R(\tau) \to \exp\left[-C_F\frac{\alpha_S}{2\pi}2\log^2\tau + a_2\alpha_S^2\log^3\tau + a_3\alpha_S^3\log^4\tau + \cdots\right]$$

where we have the notation

$\alpha_S^n\log^{n+1}\tau$ are leading logs.

Resummation $\longrightarrow$ much improved scale dependence (and convergence).



Warning, order by order expansion in α_S not always enough in perturbative QCD. Exponentiation sometimes necessary for correct probabilistic interpretation. Possible for various processes.

Also - if scale dependence large usually a good reason.

In this case not clear if scale is s or τs or some other combination.

Resummation of large $\log \tau$ terms removes ambiguity, e.g.

$$\alpha_S(s\tau) = \alpha_S(s) - \beta_0 \alpha_S^2(s) \log \tau + \cdots .$$

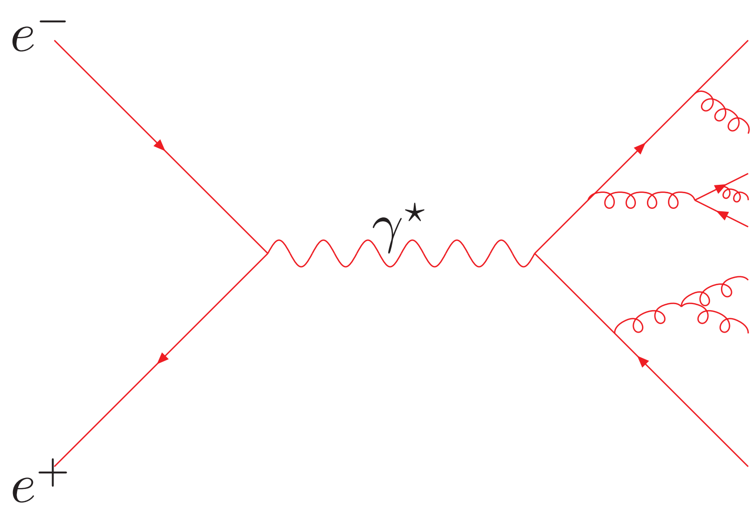
Similar procedure for other processes - C-parameter, jet rates ....

Renormalization scale variation ($s/4 \leq \mu^2 \leq 4s$) is definitely **NOT** a good estimate of theory error unless details of higher orders are understood and if necessary included.

Parton Showers.

Very often it is extremely difficult to account for the enhancements in the perturbative expansion in such a neat analytic manner. However, we have seen that in a partonic final state Sudakov form factor $\rightarrow$ probability of each parton being unaccompanied $= 0$.

Phase space fills with partons



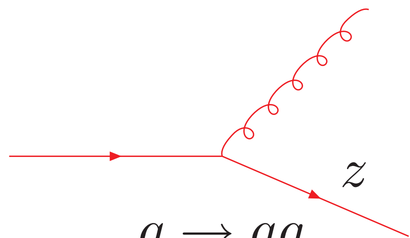
Dominated by soft and collinear partons.

Must calculate parton shower.

Collinear limit. In the limit that branching angle $\rightarrow 0$

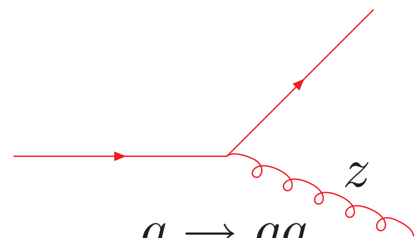
$$d\sigma = \sigma_0 \frac{\alpha_S}{2\pi} \frac{d\theta^2}{\theta^2} dz\, P(z)$$

$P(z)$ = splitting function.



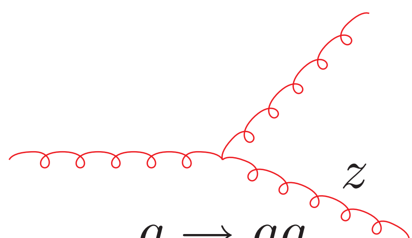
$q \longrightarrow qg$

$C_F \frac{1+z^2}{1-z}$



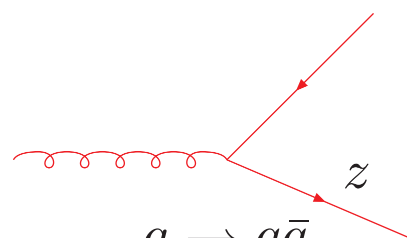
$q \longrightarrow gq$

$C_F \frac{1+(1-z)^2}{z}$



$g \longrightarrow gg$

$C_A \frac{z^4+1+(1-z)^4}{z(1-z)}$



$g \longrightarrow q\bar{q}$

$T_R(z^2 + (1-z)^2)$

Account for running coupling constant. Higher orders suggest that

$$\alpha_S \to \alpha_S(k_T^2).$$

k_T^2 is parton transverse momentum $k_T^2 = z(1-z)Q^2$ where Q^2 is initial parton virtuality. $\to$ enhancement for low k_T partons.

However, need emitted parton to be resolvable, i.e. collinear parton pair indistinguishable from single parton.

Introduce resolution criterion $k_T > Q_0$ where $Q_0 \leq 1\mathrm{GeV}$.

$$\to z, \, (1-z) > Q_0^2/Q^2$$

Virtual corrections combined with unresolvable emissions $\to$ cancellation of divergences.

Unitarity: resolved $+$ unresolved $= 1$.

Sudakov Form Factor.

Probability of resolvable emission between $q^2 + dq^2$ and q^2

$$d\mathcal{P} = \frac{\alpha_S(k_T^2)}{2\pi}\frac{dq^2}{q^2}\int_{Q_0^2/q^2}^{1-Q_0^2/q^2} dz P(z) \equiv \frac{dq^2}{q^2}\bar{P}(q^2).$$

Define probability of no emission between Q^2 and q^2 to be $\Delta(Q^2, q^2)$. Satisfies equation

$$\frac{d\Delta(Q^2,q^2)}{dq^2} = -\Delta(Q^2,q^2)\frac{d\mathcal{P}}{dq^2}$$

$$\Delta(Q^2,q^2) = \exp\left(-\int_{q^2}^{Q^2}\frac{dk^2}{k^2}\bar{P}(k^2)\right).$$

(Similar to radioactive decay. If decay constant $=\lambda$ probability of no decay before t $= \exp - \int^t \lambda dt$.

$\Delta(Q^2, q^2) \equiv \Delta(Q^2)$ is the Sudakov form factor, i.e. probability for emitting no resolvable partons.

Can use more sophisticated variables, not just virtuality.

Monte Carlo implementation.

Given parton with virtuality $Q^2 = t_1$ and momentum fraction x_1 what is (t_2, x_2) after next branching?

Probability of evolving from t_1 down to t_2 without resolvable branching is $\Delta(t_1)/\Delta(t_2)$. If we have random number ρ_1 in the interval $[0,1]$ then t_2 determined by solving

$$\frac{\Delta(t_1)}{\Delta(t_2)} = \rho_1.$$

If $t_2 \leq t_0$ then no further branching takes place, else we find scale of next branching and can repeat.

Can reverse in order to evolve up (e.g. DIS).

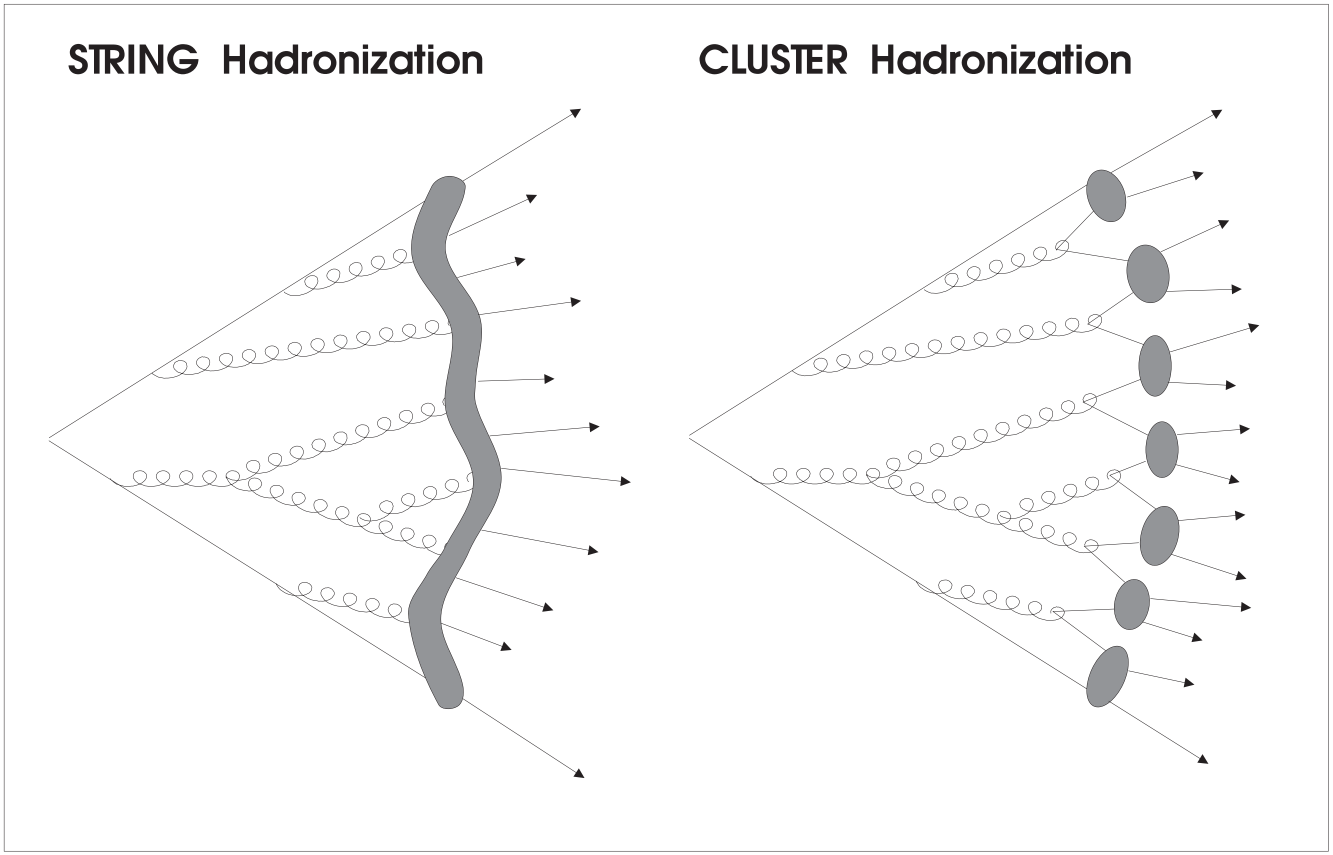
At t_2 also want momentum fraction $z = x_2/x_1$ for branching. Find this by using weight given by splitting function and solving

$$\int_{t_0/t}^{x_2/x_1} dz \frac{\alpha_S(k_T^2)}{2\pi} P(z) = \rho_2 \int_{t_0/t}^{1-t_0/t} dz \frac{\alpha_S(k_T^2)}{2\pi} P(z)$$

where ρ_2 is another random number.

Depends on various choices, particularly Q_0^2.

Main approaches (models) string hadronization and cluster hadronization. Not discussed here.



General effect, change hard quantities E, p relationship by amount $\lambda \sim \Lambda_{QCD}$.

Full machinery of final state QCD calculations works well, but requires many contributions.

Still refinements going on.

Hence, recent developments of HERWIG++ and PYTHIA 8.