DIRAC EQUATION

$$E\psi = (\alpha_i P_i + \beta m)\psi$$

$$E^2\psi = (\alpha_i P_i + \beta m)(\alpha_i P_i + \beta m)\psi$$

$$(\alpha_i \alpha_J P_i P_J + (\alpha_i \beta + \beta \alpha_J) P_i m + \beta^2 m^2)\psi$$

BUT $E^2 = P^2 + m^2$

$$\rightarrow \quad \alpha_i \alpha_J + \alpha_J \alpha_i = 2\delta_{iJ} \underline{\mathbb{1}}$$

$$\alpha_i \beta + \beta \alpha_J = 0$$

$$\beta^2 = \underline{\mathbb{1}}$$

ALSO WE WANT

α AND β ARE HERMITIAN

$$\alpha = \alpha^\dagger \qquad \beta = \beta^\dagger$$

$$\alpha^2 = \underline{\mathbb{1}}$$

FROM $\alpha_i \beta + \beta \alpha_J = 0$, THEY MUST BE TRACELESS

$$\alpha_i \beta = -\beta \alpha_j$$

MULTIPLY BY β

$$\alpha_i \beta^2 = -\beta \alpha_j \beta$$

$$\alpha_i = -\beta \alpha_j \beta$$

$$\mathrm{Tr}(\alpha_i) = -\mathrm{Tr}(\beta \alpha_j \beta)$$

BUT $\mathrm{Tr}(\alpha\beta\gamma\delta \ldots) = \mathrm{Tr}(\delta\alpha\beta\gamma)$

$$\rightarrow \mathrm{Tr}(\alpha_i) = -\mathrm{Tr}(\beta^2 \alpha_j)$$

$$= + \mathrm{Tr}(\alpha_j) = -\mathrm{Tr}(\alpha_i)$$

FOR $i = j$

$$\rightarrow \mathrm{Tr}(\alpha_i) = 0$$

DEFINE

$$\underline{\alpha}_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

COVARIANT FORM

$$E\psi = (\alpha_i \cdot \underline{p} + \beta m)\psi$$

REPLACE ENERGY AND MOMENTUM BY OPERATORS

$$p \to -i\underline{\nabla} \qquad E \to i\frac{\partial}{\partial t}$$

$$\Rightarrow \quad i\frac{\partial \psi}{\partial t} = -i\alpha \cdot \nabla\psi + \beta m \psi$$

MULTIPLY BY β

$$i\beta\frac{\partial \psi}{\partial t} = -i\beta\underline{\alpha} \cdot \underline{\nabla}\psi + m\psi$$

DEFINE $\quad \gamma^0 = \beta \quad \gamma^k = \beta\alpha^k$

$$i\gamma^0\frac{\partial \psi}{\partial t} + i\gamma^k\nabla\psi - m\psi = 0$$

$$i\gamma^0\frac{\partial \psi}{\partial x_0} + i\gamma^k\frac{\partial \psi}{\partial x_k} - m\psi = 0$$

$$\longrightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\gamma^\mu = (\gamma^0, \gamma^k)$$

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \underline{\nabla}\right)$$

β IS HERMITIAN $\longrightarrow \gamma^0$ IS

BUT $\quad (\gamma^k)^\dagger = -\gamma^k$

BECAUSE: $\quad \gamma^k = \beta\alpha^k$

$$\gamma^\dagger = (\alpha\alpha^k)^\dagger = \alpha^{k\dagger}\beta^\dagger$$

$$= \alpha^k\beta = -\beta\alpha^k = -\gamma^k$$

– $(\gamma^0)^2 = \mathbb{1}$ SINCE $\beta^2 = \mathbb{1}$

$$(\gamma^k)^2 = \gamma^k\gamma^k = \beta\alpha^k\beta\alpha^k = -\beta\alpha^k\alpha^k\beta =$$

$$= -\beta^2 = -\mathbb{1}$$

TAKE NOW HERMITIAN CONJUGATE OF DIRAC EQ.

$$-i\frac{\partial \psi^{\dagger}}{\partial t}\gamma^{0} + i\frac{\partial}{\partial x^{k}}(-\gamma^{k})\psi^{\dagger} - m\psi^{\dagger} = 0$$

MULTPLY BY γ^{0}

$$-i\frac{\partial \psi^{\dagger}}{\partial t} - i\frac{\partial}{\partial x^{k}}\gamma^{k}\gamma^{0}\psi^{\dagger} - m\psi^{\dagger}\gamma^{0} = 0$$

$$\gamma^{0}\gamma^{k} = -\gamma^{k}\gamma^{0}$$

$$+i\frac{\partial}{\partial x^{k}}\gamma^{0}\gamma^{k}\psi^{\dagger}$$

DEFINE ADJOINT $\bar{\psi} = \psi^{\dagger}\gamma^{0}$

$$-i\frac{\partial \bar{\psi}}{\partial t}\gamma^{0} - i\frac{\partial \bar{\psi}}{\partial x^{k}}\gamma^{k} - m\bar{\psi} = 0$$

$$i\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0$$

ADJOINT FORM OF DIRAC EQ

MULTIPLY "STANDARD" EQUATION BY $\bar{\psi}$ AND ADJOINT ONE BY ψ

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi = 0$$

$$i\partial_\mu\bar{\psi}\gamma^\mu\psi - \bar{\psi}m\psi = 0$$

– TAKE DIFFERENCE

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\partial_\mu\bar{\psi}\gamma^\mu\psi = 0$$

$$= \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

SO $J^\mu = \bar{\psi}\gamma^\mu\psi$

SATISFIES CONTINUITY EQN

$$\rho = J^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$$
$$= |\psi|^2$$

E.M. CURRENT DENSITY

$$J^\mu = -e\bar{\psi}\gamma^\mu\psi$$

FREE PARTICLE SOLUTION

$$\psi = u(p)\, e^{-ipx}$$

$\longrightarrow$ NEEDS TO HAVE DIMENTION 4

$$(i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ipx} = 0$$

$$(i\gamma^\mu(-ip_\mu) - m)\, u(p)\, e^{-ipx} = 0$$

$$(\gamma^\mu p_\mu - m)\, u(p)\, e^{-ipx} = 0$$

$$\Rightarrow \quad (\gamma^\mu p_\mu - m)\, u(p) = 0$$

DEFINE $\not{p} = \gamma^\mu p_\mu$

$$(\not{p} - m)\, u(p) = 0$$

$$(\gamma^0 E - \gamma^k p_k - m)\, u(p) = 0$$

MULTIPLY BY γ^0

$$((\gamma^0)^2 E - \gamma^0 \gamma^k p_k - \gamma^0 m)\, u(p) = 0$$

$$(E - \alpha^k p_k - \underbrace{\gamma^0}_{=\beta} m)\, u(p) = 0$$

FOR A PARTICLE AT REST $\underline{p} = \underline{0}$

$$(E \underline{\mathbb{1}} - \beta m)\, u(p) = 0$$

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} E\, u(p) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} m\, u(p)$$

SOLUTIONS

$$\begin{vmatrix} (m-E)\mathbb{1} & 0 \\ 0 & -(m+E)\mathbb{1} \end{vmatrix} = 0$$

OR IN 4-D

$$0 = \begin{vmatrix} m-E & & & 0 \\ & m-E & & \\ & & -m-E & \\ 0 & & & -m-E \end{vmatrix}$$

$$\left[(m-E)(m+E)\right]^2 = 0$$

SOLUTIONS: $E = \pm m$

IF PARTICLE NOT AT REST $\underline{P} \neq \underline{0}$

$$\left[(\underline{\alpha} \cdot \underline{P}) + \beta m \right] U = 0$$

$$\left[\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot P + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m \right] \times$$

$$\times \begin{pmatrix} U_A \\ U_B \end{pmatrix} = E \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$

$$\begin{cases} \underline{\sigma} \cdot \underline{P} \, U_A - m \, U_B = E \, U_B \\ \underline{\sigma} \cdot \underline{P} \, U_B + m \, U_A = E \, U_A \end{cases}$$

$$\begin{cases} U_A = \dfrac{\sigma \cdot P}{E - M} U_B \\ U_B = \dfrac{\sigma \cdot P}{E + M} U_A \end{cases}$$

$U_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ SPIN UP

$U_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ SPIN DOWN

FOR $E > 0$

$$U_1 = \begin{pmatrix} 1 \\ 0 \\ \dfrac{\sigma \cdot P}{E + M} \\ 0 \end{pmatrix} N$$

$$U_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{\sigma \cdot P}{E + M} \end{pmatrix} N$$

FOR $E<0$

$$u_A = \frac{\sigma \cdot p}{E-m} u_B =$$

$$= \frac{\sigma \cdot p}{-|E|-m} u_B =$$

$$= -\frac{\sigma \cdot p}{|E|+m} u_B$$

$$u_3 = N \begin{pmatrix} -\frac{\sigma \cdot p}{|E|+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_4 = N \begin{pmatrix} 0 \\ -\frac{\sigma \cdot p}{|E|+m} \\ 0 \\ 1 \end{pmatrix}$$

REDEFINE BASES

U_3, U_4 TO MAKE EXPLICIT THAT THEY ARE NEGATIVE ENERGY SOLUTIONS

$$U^{(3,4)}(-p)\,e^{-i(-p)x} = N^{(2,1)}(p)\,e^{ipx}$$

DEFINES $N^{2,1}$

$$N_2 = N \begin{pmatrix} \frac{\sigma \cdot p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$ SPIN DOWN

$$N_1 = N \begin{pmatrix} 0 \\ \frac{\sigma \cdot p}{E+m} \\ 0 \\ 1 \end{pmatrix}$$ SPIN UP

CALCULATE N

$$\psi_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{\sigma \cdot p}{E+m} \\ 0 \end{pmatrix} e^{-ipx}$$

$$\psi_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma \cdot p}{E+m} \end{pmatrix} e^{-ipx}$$

ORTHOGONALITY

$$\int \psi_1^{\dagger} \psi_2 \, d^3x = 0$$

$$N^{*} N \left(1, 0, \frac{(\sigma \cdot p)^{\dagger}}{E+m}, 0\right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma \cdot p}{E+m} \end{pmatrix} = 0$$

IS SATISFIED

NORMALISATION

$$\int \psi_1^\dagger \psi_1 \, d^3x = 2E$$

$$2E = \int N^* N \left(1, 0, \left(\frac{\sigma \cdot p}{E+m}\right)^\dagger, 0\right) \begin{pmatrix} 1 \\ 0 \\ \frac{\sigma \cdot p}{E+m} \\ 0 \end{pmatrix} d^3x$$

$$2E = N^* N \int \left[1 + \left(\frac{\sigma \cdot p}{E+m}\right)^2\right] d^3x$$

FOR A PARTICLE MOVING ALONG y

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma_y p_y)^2 = \begin{pmatrix} 0 & -ip_y \\ ip_y & 0 \end{pmatrix}^2 = \begin{pmatrix} p_y^2 & 0 \\ 0 & p_y^2 \end{pmatrix} =$$

$$= p_y^2 \, \mathbb{1}$$

$$2E = N^\dagger N \int \left(1 + \frac{p^2}{(E+m)^2}\right) d^3x$$

$$= N^* N \int \frac{E^2 + 2Em + \cancel{m^2} + E^2 - \cancel{m^2}}{(E+m)^2} d^3x$$

$$= 2E N^* N \int \frac{\cancel{E+m}}{(E+m)^{\cancel{2}}} d^3x$$

$$\cancel{2E} = \cancel{2E} N^* N \frac{V}{E+m}$$

$$N = \sqrt{\frac{E+m}{V}}$$