RELATIVISTIC SPIN-0 CASE

$$\phi = e^{i(\underline{p}\underline{x} - Et)} = e^{ip^\mu x_\mu}$$

CONSIDER RELATIVISTIC RELATION

$$E^2 = p^2 + m^2$$

TRANFORMING QUANTITIES INTO OPERATORS

$$-\frac{\partial^2 \phi}{\partial t^2} = -\nabla^2 \phi + m^2 \phi$$

KLEIN-GORDON EQ.

$$-\frac{\partial^2 \phi^*}{\partial t^2} = -\nabla^2 \phi^* + m^2 \phi^*$$

MULTIPLY ORIGINAL EQ.
BY ϕ^*

$$-\phi^* \frac{\partial^2 \phi}{\partial t^2} = -\phi^* \nabla^2 \phi + m^2 \phi^* \phi$$

MULTIPLY C.C. BY ϕ

$$-\phi \frac{\partial^2 \phi^*}{\partial t^2} = -\phi \nabla^2 \phi^* + m^2 \phi \phi^*$$

DIFFERENCE $\times -i$

$$i\left[\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2}\right] = i\left[\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^*\right]$$

$$i \frac{\partial}{\partial t}\left[\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right] - i \nabla\left[\phi^* \nabla \phi - \phi \nabla \phi^*\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$$

$$\rho = i\left\{\phi^* \frac{\partial \phi}{\partial t} - \frac{\phi \partial \phi^*}{\partial t}\right\} \qquad \underline{J} = -i\left[\phi^* \nabla \phi - \phi \nabla \phi^*\right]$$

FOR PLANAR WAVE

$$\phi = N e^{i(px - Et)}$$

$$\rho = i\left[N^* \cancel{e^{-i(px-Et)}} (-iE) N \cancel{e^{i(px-Et)}} - \right.$$

$$\left. - N \cancel{e^{i(px-Et)}} (iE) N^* \cancel{e^{-i(px-Et)}} \right] =$$

$$= 2 E N^* N$$

$$\underline{J} = 2 N N^* \underline{p}$$

BACK TO $E^2 = p^2 + m^2$

$$\rightarrow E = \pm\sqrt{p^2 + m^2}$$

SIMILAR PROBLEM WITH LORENTZ BOOST

$$d^3x \rightarrow d^3x \sqrt{1 - v^2}$$

$$\rho \rightarrow \frac{\rho}{\sqrt{1 - v^2}}$$

PAULI-WEISSKOPF APPROACH

JUST USE CHARGE DENSITIES AND CURRENTS

$$J^{\mu} = -ie\left[\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}, \phi^* \nabla \phi - \phi \nabla \phi^*\right]$$

FEYNMAN-STUCKELBERG

A PARTICLE WITH NEGATIVE ENERGY IS EQUIVALENT TO MOVING BACKWARD

EX. ELECTRON (FREE)

$$J^{\mu}(e^-) = -2eN^*N(E, \underline{p})$$

FOR NEG. ENERGY

$$J^{\mu}(e^+) = 2eN^*N(E, \underline{p}) =$$

$$= -2eN^*N(-E, -\underline{p})$$

PROPAGATOR APPROACH

$\psi(\underline{r}, t)$ INITIAL W.F.

$\psi(\underline{r}', t')$ AFTER INTERACTION

$$\psi(\underline{r}', t') = i \int G(r', t', r, t)\, \psi(\underline{r}, t)\, d^3 r$$

$t' > t$ $\qquad G = $ GREEN FUNCTION

CONSIDER SCATTERING OF WAVE ϕ (PLANAR) WITH POTENTIAL $V(\underline{r}, t)$

SCHRÖDINGER EQ.

$$(H_0 + V)\psi = i \frac{\partial \psi}{\partial t}$$

$$i \partial \psi(\underline{r}, t) - H_0 \psi(\underline{r}, t) =$$

$$= V(\underline{r}, t)\, \psi(\underline{r}, t)\, dt$$

INTEGRATE BETWEEN t_1 AND $t_1 + \Delta t_1$

$$i \int \partial \psi(\underline{r}_1, t_1) - \int_{t_1}^{t_1+\Delta t} H_0 \psi(\underline{r}_1, t_1) dt_1 =$$

$$= \int_{t_1}^{t_1+\Delta t} V(\underline{r}_1, t_1) \psi(\underline{r}_1, t_1) dt_1$$

$$\Delta \psi(\underline{r}_1, t_1) = -i V(\underline{r}_1, t_1) \psi(\underline{r}_1, t_1) \Delta t_1$$

WE ASSUMED THAT DURING Δt_1 $V >> H_0$

BEFORE INTERACTION, WAVEFUNCTION WAS PLANAL

$$\psi(\underline{r}, t) = \phi(\underline{r}, t)$$

SO

$$\Delta \psi(\underline{r}_1, t_1) = -i V(\underline{r}_1, t_1) \phi(\underline{r}_1, t_1) \Delta t_1$$

AFTER INTERACTION

$$\psi(\underline{r}_1, t_1) = \phi(\underline{r}_1, t_1) + \Delta\psi(\underline{r}_1, t_1)$$

USE EQUATION OF PROPAGATOR

$$\Delta\psi(\underline{r}', t') = i\int d^3 r_1\, G_0(r', t'; r_1, t_1)\,\Delta\psi(\underline{r}_1, t_1)$$

IT IS THE DIFFERENCE OF WAVE FUNCTION FROM A PLANAR WAVE $(\Delta\psi)$ THAT IS AFFECTED BY THE INTERACTION (PROPAGATOR)

- USE RELATION BETWEEN $\Delta\psi$ AND ϕ

$$\Delta\psi(\underline{r}_1, t_1) = -i\,V(r_1 t_1)\,\phi(\underline{r}_1, t_1)\,\Delta t_1$$

$$\Rightarrow \Delta\psi(\underline{r}', t') = i\int d^3 r_1\, G_0(r' t'; r_1 t_1)(-i)\,V(r_1 t_1)\,\phi(r_1 t_1)\,\Delta t_1$$

GOING BACK TO DEFINITION OF ψ

$$\psi(r' t') = \phi(r' t') + \int d^3 r_1\, G_0(r' t'; r_1 t_1)\,V(r_1 t_1)\,\phi(r_1 t_1)\,\Delta t_1$$

IN SHORT FORM. $\Delta t_1 \to dt_1$

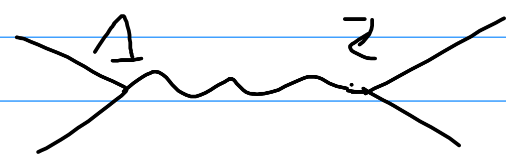
$$\psi(r't') = \phi(r't') + \int d^4x_1\, G_0(x_1', 1) V(1) \phi(1)$$

$V(1) \equiv V(r_1 t_1)$ x IS A 4-VECTOR

$\phi(1) \doteq \phi(r_1 t_1)$

IN A FEYNMAN DIAGRAM, 2 SCATTERING POINTS

E.G.



NEED A GREEN'S FUNCTION TO GO FROM POINT 1 TO 2

$$\psi(r't') = \phi(r', t') + \int d^4x\, G_0(x_1', 1) V(1) \phi(1) +$$

$$+ \int d^3r_2\, G_0(x_1', 2) V(2) \phi(2) \Delta t_2$$

$$+ \iint d^3r_1\, d^3r_2\, G_0(x_1', 2) V(2) G_0(2, 1) V(1) \phi(1) \Delta t_1 \Delta t_2$$

BY DEFINITION

$$\psi(r',t') = i\int_{t'>t} d^3r\, G(r',r)\,\psi(r,t)$$

EXPRESS CONDITION $t'>t$ USING WAVE FUNCTION

$$\theta(t'-t)\,\psi(x') = i\int d^3r\, G(x',x)\,\psi(r,t)$$

APPLY SCHRODINGER EQ

LHS : $\left[i\frac{\partial}{\partial t} - H(x')\right] \theta(t'-t)\,\psi(x') =$

$$= i\delta(t'-t)\psi(x') + \cancel{\theta(t'-t)\, i\frac{\partial\psi(x')}{\partial t}} - \cancel{H\theta\psi(x')}$$

$$= i\delta(t'-t)\,\psi(x')$$

RHS : $i\int d^3r \left[i\frac{\partial}{\partial t} - H(x')\right] G_0(x',x)\,\psi(x)$

SIMPLEST CASE : $V=0$

THE HAMILTONIAN IS $P^2/2M$

RHS : $i \int d^3x \left[E - \frac{P^2}{2m} \right] G_0(x', x)\, \psi(x)$

APPLY FOURIER TRANSFORM

RHS : $i \int \frac{d^3P\, dE}{(2\pi)^4} \left(E - \frac{P^2}{2m} \right) G_0(E, P)\, e^{iP(x'-x)}\, e^{iE(t-t')}\, \psi(x)$

LHS : $i\delta(t'-t)\,\psi(x') \Longrightarrow i\delta(t'-t) \int d^3x\, \psi(x)\, \delta^3(x'-x)$

$= i\delta^4(x'-x)\,\psi(x)$

TWO SIDES ARE EQUAL

$$i\delta^4(x'-x)\,\psi(x) = i \int \frac{d^3P\, dE}{(2\pi)^4} \left(E - \frac{P^2}{2m} \right) G_0(E, P) \cdot e^{iP(x'-x)}\, e^{-iE(t'-t)}\, \psi(x)$$

FROM THE FOURIER DEFINITION OF δ

$$\delta^4(x'-x)\,\psi(x) = \int \frac{d^4P}{(2\pi)^4} e^{iP(x-x)}\, e^{iE(t-t')}\, \psi$$

$$\Longrightarrow G_0 = \frac{1}{E - P^2/2m} \quad \text{FOR } E \neq \frac{P^2}{2m}$$

TO AVOID INFINITIES,

$$G_0 \to G_0 = \lim_{\varepsilon \to 0} \frac{1}{E - \frac{p^2}{2m} + i\varepsilon}$$

ELECTROMAGNETIC SCATTERING EXAMPLE $e^- \mu^-$

FOR ELECTROMAGNETISM

$$p^\mu \longrightarrow p^\mu + eA^\mu$$

$$A^\mu = (A^0, \underline{A})$$

$$p_\mu p^\mu = m^2$$

$$\longrightarrow (p_\mu + eA_\mu)(p^\mu + eA^\mu)$$

IN Q.M. $\quad p_\mu \longrightarrow i\partial_\mu$

$$(i\partial_\mu + eA_\mu)(i\partial^\mu + eA^\mu)\,\phi = m^2\phi$$

$$-\partial_\mu\partial^\mu + ie\partial_\mu A^\mu\phi + ieA_\mu\partial^\mu\phi + e^2A_\mu A^\mu\phi = m^2\phi$$

$$(\partial_\mu\partial^\mu + m^2)\,\phi = (ie\partial_\mu A^\mu + ieA_\mu\partial^\mu + e^2A_\mu A^\mu)\,\phi$$

$$(E - \cancel{m^2} + \cancel{m^2})\,\phi = -V\,\phi$$

WRITE A TRANSITION AMPL.

$$T_{fi} = -i\int d^4x\, \phi_f^*(x)\, V(x)\, \phi_i(x) =$$

$$= -i\int d^4x\, \phi_f^*(x)(-ie)(\partial_\mu A^\mu + A_\mu \partial^\mu)\phi_i(x)$$

NEGLECTING $e^2 A_\mu A^\mu$

CONSIDER FIRST PART OF SUM

$$I = \int \phi_f^* \partial_\mu A^\mu \phi_i \, d^4x$$

INTEGRATE BY PARTS

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$u = \phi_f^* \qquad \frac{dv}{dx} = \partial_\mu (A^\mu \phi_i)$$

$$I = \cancel{\left[\phi_f^* A^\mu \phi_i\right]_{-\infty}^{\infty}} - \int d^4x (\partial_\mu \phi_f^*) A^\mu \phi_i$$

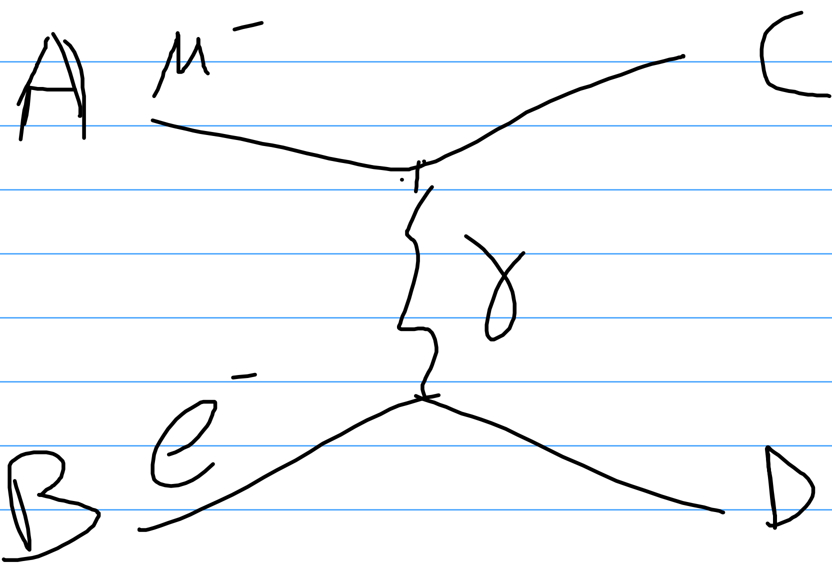
ADDING THE SECOND TERM TO T_{fi}, ALSO INTEGRATING BY PARTS

$$T_{fi} = -e \int d^4x \left(\phi_f^* A_\mu \partial^\mu \phi_i - \partial_\mu \phi_f^* A^\mu \phi_i \right)$$

DEFINE TRANSITION CURRENT

$$J_\mu^{fi} = -ie \left(\phi_f^* \partial_\mu \phi_i - \partial_\mu \phi_f^* \phi_i \right)$$

$$T_{fi} = -i \int d^4x \, A^\mu J_\mu^{fi}$$



$$\phi_A(x) = N_A e^{-ip_A x} \qquad \phi_C(x) = N_C e^{-ip_C x}$$

$$J^{CA}_\mu(x) = -ie\left[N_C^* e^{iP_C x} N_A e^{-iP_A x}(-iP_A)_\mu - \right.$$

$$\left. - N_C^* e^{iP_C x}(iP_C)_\mu N_A e^{-iP_A x}\right] =$$

$$= -ie N_C^* N_A e^{i(P_C - P_A)x}(-iP_A - iP_C)_\mu$$

$$= e N_C^* N_A (P_A + P_C)_\mu e^{i(P_C - P_A)x}$$

FROM COMPACT FORM OF MAXWELL'S EQNS:

$$\square^2 A^\mu(x) = J^\mu(x)$$

$$A^\mu = \frac{-g^{\mu\nu} J_\nu^{AC}}{q^2} \qquad q = P_C - P_A$$

CHECK:

$$\partial_\nu \partial^\nu \left(-\frac{1}{q^2}\right) J_{AC}^\mu =$$

FROM THE EXPRESSION

OF $J_{AC} = N \, e^{i(P_C - P_A)x}$

$$\partial_\nu \partial^\nu J_{AC} = N \left[-i(P_C - P_A)\right]^2 \underbrace{}_{exp} =$$

$$= N \left(-q^2\right) e^{i(P_C - P_A)}$$

SO

$$\partial_\nu \partial^\nu \left(-\frac{1}{q^2}\right) J_{AC}^\mu = J_{AC}^\mu$$

$$T_{fi} = -i \int d^4x \; J_\mu^{(A}(x) A^\mu =$$

$$T_{fi} = -i\int d^4x\,(-e)\,N_C^* N_A\,(p_A+p_C)_\mu\, e^{i(p_C-p_A)x}\left(-\frac{1}{q^2}\right)$$

$$(-e)\,N_D^* N_B\,(p_B+p_D)^\mu\, e^{i(p_D-p_B)x} =$$

$$= \frac{ie^2}{q^2}\int N_C^* N_A N_D^* N_B\,(p_A+p_C)_\mu\,(p_D+p_B)^\mu$$

$$e^{i(p_C+p_D-p_A-p_B)x}\, d^4x$$