Symmetries & Conservations Laws – Exam Question – Feb 2011

SJH

Answer all the questions. Time 1.5 hours. Total 20 marks.
Hint: Roughly every key step or point corresponds to 0.5 marks.

*You should be provided with an **additional sheet** which contains Clebsch-Gordon Coefficients from the PDG Book – page 3 of this paper.*

1) Rotation Matrices [5 marks]

In the lectures, we looked at how rotation matrices could be derived for rotations about the *y*-axis. Here we will consider what happens when we used the *x*-axis.

Consider a 3D representation of SU(2) with a generator:

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Evaluate the rotation matrix $< jm' \,|\, \exp(i\theta J_x) \,|\, j\, m>$ where $j = 1$ and m & $m' = -1,0,+1$.

How do the rotation matrix elements differ from those we saw in the lectures (and you derived in your homework) when we considered the rotation about the *y*-axis ? Does it matter ?

(*The standard rotation matrices can be seen on the attached page from the PDG Book.*)

2) Lie Algebra in SU(2) [8 marks]

Using the properties of the raising and lowering operators in SU(2), derive the Lie Algebra (commutators) for the generators J_1, J_2, J_3.

The properties you should use are:

I. $J_\pm = J_1 \pm iJ_2$

II. $J_3 \,|\, j,m> = m \,|\, j,m>$

III. $J_\pm \,|\, j,m> = \sqrt{(j \mp m)(j \pm m + 1)} \,|\, j,(m \pm 1)>$

Hints:

Firstly, compare $J_3 J_\pm \,|\, j,m>$ *and* $J_\pm J_3 \,|\, j,m>$

Secondly, compare $J_- J_+ \,|\, j,m>$ *and* $J_+ J_- \,|\, j,m>$

PTO

3) Combining Spins – Clebsch-Gordon Coefficients [2 marks]

Spin-1 states can be made from combining

a) Spin-$^{1}/_{2}$ and Spin-$^{1}/_{2}$, or
b) Spin-1 and Spin-1, or
c) Spin-$^{3}/_{2}$ and Spin-$^{1}/_{2}$

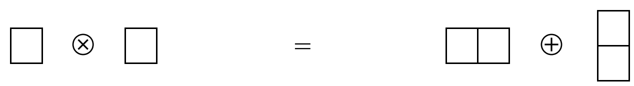
Use the Clebsch-Gordon coefficients on the attached sheet from the PDG Book to show how the Spin-1 state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ ie. } j\text{=1, } m\text{=0}$$

can be constructed from the pairs of states in each of the 3 lists (a), (b) and (c) above.

4) Combining Spins – Young Tableaux [5 marks]

In part (a) of the previous question, it was stated that a Spin-1 state can be constructed from the combination of two Spin-$^{1}/_{2}$ states for SU(2). In terms of Young Tableaux, this looks like:



Spin	½	⊗	½	=	1	⊕	0
Multiplicity	2	×	2	=	3	+	1

The combination of the 2 Spin-$^{1}/_{2}$ states results in Spin-1 and Spin-0 multiplets.
Show how this would look for the combinations in parts (b) and (c) of the previous question.

Hints:
You don't need to think about the 3rd component of spin – just the multiplets.
For the representations for Spin-1 and Spin-$^{3}/_{2}$, you will need to represent the Tableaux by 2 and 3 boxes in a row, respectively.
[*Don't panic if you don't appear to identify all the states you might expect for the combination of Spin-1 with Spin-1 – we didn't discuss how to combine complex YT.*]

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

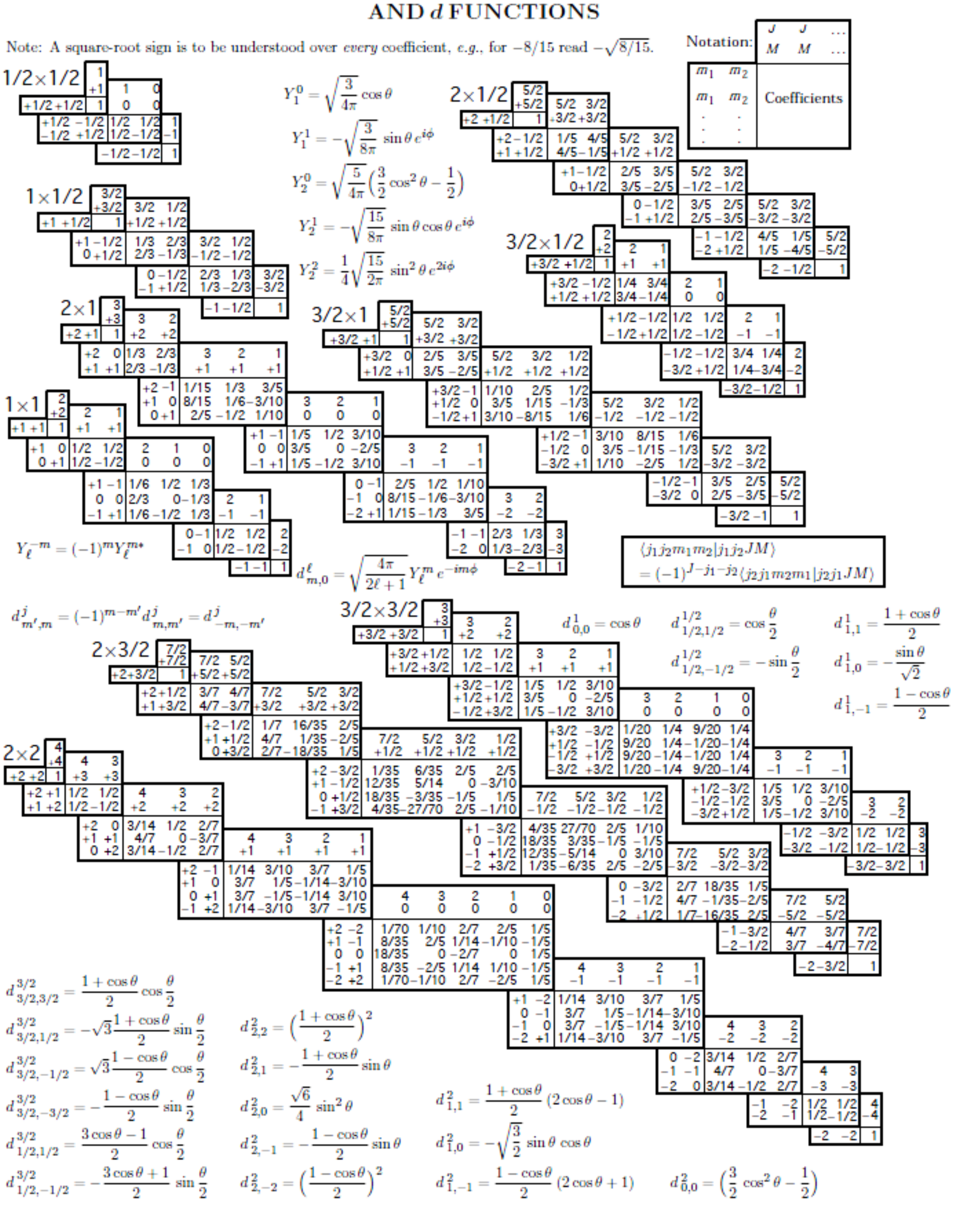


Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.