

# Symmetries & Conservations Laws – Exam Question – Feb 2011

SJH

**Answer all the questions. Time 1.5 hours. Total 20 marks.**

*Hint: Roughly every key step or point corresponds to 0.5 marks.*

**You should be provided with an *additional sheet* which contains Clebsch-Gordon Coefficients from the PDG Book – page 3 of this paper.**

## **1) Rotation Matrices [5 marks]**

In the lectures, we looked at how rotation matrices could be derived for rotations about the  $y$ -axis. Here we will consider what happens when we used the  $x$ -axis.

Consider a 3D representation of SU(2) with a generator:

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Evaluate the rotation matrix  $\langle jm' | \exp(i\theta J_x) | jm \rangle$  where  $j = 1$  and  $m \text{ \& } m' = -1, 0, +1$ .

How do the rotation matrix elements differ from those we saw in the lectures (and you derived in your homework) when we considered the rotation about the  $y$ -axis? Does it matter?

*(The standard rotation matrices can be seen on the attached page from the PDG Book.)*

## **2) Lie Algebra in SU(2) [8 marks]**

Using the properties of the raising and lowering operators in SU(2), derive the Lie Algebra (commutators) for the generators  $J_1, J_2, J_3$ .

The properties you should use are:

- I.  $J_{\pm} = J_1 \pm iJ_2$
- II.  $J_3 |j, m\rangle = m |j, m\rangle$
- III.  $J_{\pm} |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, (m \pm 1)\rangle$

Hints:

*Firstly, compare  $J_3 J_{\pm} |j, m\rangle$  and  $J_{\pm} J_3 |j, m\rangle$*

*Secondly, compare  $J_- J_+ |j, m\rangle$  and  $J_+ J_- |j, m\rangle$*

**PTO**

### 3) Combining Spins – Clebsch-Gordon Coefficients [2 marks]

Spin-1 states can be made from combining

- a) Spin- $1/2$  and Spin- $1/2$ , or
- b) Spin-1 and Spin-1, or
- c) Spin- $3/2$  and Spin- $1/2$

Use the Clebsch-Gordon coefficients on the attached sheet from the PDG Book to show how the Spin-1 state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ ie. } j=1, m=0$$

can be constructed from the pairs of states in each of the 3 lists (a), (b) and (c) above.

### 4) Combining Spins – Young Tableaux [5 marks]

In part (a) of the previous question, it was stated that a Spin-1 state can be constructed from the combination of two Spin- $1/2$  states for SU(2). In terms of Young Tableaux, this looks like:

	$\square$	$\otimes$	$\square$	=	$\square\square$	$\oplus$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$
Spin	$1/2$	$\otimes$	$1/2$	=	1	$\oplus$	0
Multiplicity	2	$\times$	2	=	3	+	1

The combination of the 2 Spin- $1/2$  states results in Spin-1 and Spin-0 multiplets. Show how this would look for the combinations in parts (b) and (c) of the previous question.

Hints:

*You don't need to think about the 3<sup>rd</sup> component of spin – just the multiplets.*

*For the representations for Spin-1 and Spin- $3/2$ , you will need to represent the Tableaux by 2 and 3 boxes in a row, respectively.*

*[Don't panic if you don't appear to identify all the states you might expect for the combination of Spin-1 with Spin-1 – we didn't discuss how to combine complex YT.]*



35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND  $d$  FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for  $-8/15$  read  $-\sqrt{8/15}$ .

Notation:

$J$	$J$	...
$M$	$M$	...
$m_1$	$m_2$	
$m_1$	$m_2$	Coefficients
.	.	.
.	.	.

$1/2 \times 1/2$

1	0
+1/2 +1/2	1
+1/2 -1/2	1/2 1/2
-1/2 +1/2	1/2 -1/2
-1/2 -1/2	1

$$Y_0^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$2 \times 1/2$

5/2	3/2
+5/2	1
+2 -1/2	1/5 4/5
+1 +1/2	4/5 -1/5
	5/2 3/2
	3/5 -2/5
	1/2 +1/2

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$$

$1 \times 1/2$

3/2	1/2
+3/2	1
+1 -1/2	1/3 2/3
0 +1/2	2/3 -1/3
	3/2 1/2
	1/3 -2/3
	-1/2 -1/2
	0 -1/2
	2/3 1/3
	1/3 -2/3
	-3/2

$3/2 \times 1/2$

2	1
+3/2 +1/2	1
+3/2 -1/2	1/4 3/4
+1/2 +1/2	3/4 -1/4
	2 1
	0 0
	1/2 1/2
	1/2 -1/2
	-1 -1
	-1/2 -1/2
	3/4 1/4
	1/4 -3/4
	-2 -1/2
	1

$2 \times 1$

3	2
+3	1
+2 0	1/3 2/3
+1 +1	2/3 -1/3
	3 2 1
	1 1 1
	-1 -1/2

$3/2 \times 1$

5/2	3/2
+5/2	1
+3/2 0	2/5 3/5
+1/2 +1	3/5 -2/5
	5/2 3/2 1/2
	1/2 +1/2
	3/10 -8/15
	1/6

$1 \times 1$

2	1
+2	1
+1 0	1/2 1/2
0 +1	1/2 -1/2
	2 1 0
	0 0 0
	+1 -1
	1/5 1/2 3/10
	0 0 0
	3 2 1
	0 -2/5
	-1 -1
	1/5 -1/2 3/10
	-1 -1
	0 -1
	2/5 1/2 1/10
	-1 0
	8/15 -1/6 -3/10
	1/5 -1/2 3/10
	-2 -1
	1/15 -1/3 3/5
	-2 -2
	-1 -1
	2/3 1/3 3
	-2 0
	1/3 -2/3 -3
	-2 -1
	1

$$Y_{\ell}^{-m} = (-1)^m Y_{\ell}^{m*}$$

$$d_{m,0}^{\ell} = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell}^m e^{-im\phi}$$

$$(j_1 j_2 m_1 m_2 | j_1 j_2 J M)$$

$$= (-1)^{J-j_1-j_2} (j_2 j_2 m_2 m_1 | j_2 j_1 J M)$$

$$d_{m',m}^j = (-1)^{m-m'} d_{-m,-m'}^j = d_{-m,-m'}^j$$

$3/2 \times 3/2$

3	2
+3	1
+3/2 +1/2	1/2 1/2
+1/2 +3/2	1/2 -1/2
	3 2 1
	1 1 1

$$d_{0,0}^1 = \cos \theta$$

$$d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2}$$

$$d_{1,1}^1 = \frac{1 + \cos \theta}{2}$$

$$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2}$$

$$d_{1,0}^1 = \frac{\sin \theta}{\sqrt{2}}$$

$$d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$$

$2 \times 3/2$

7/2	5/2
+7/2	1
+2 +1/2	3/7 4/7
+1 +3/2	4/7 -3/7
	7/2 5/2 3/2
	3/2 +3/2 +3/2

$2 \times 2$

4	3
+4	1
+2 +1	1/2 1/2
+1 +2	1/2 -1/2
	4 3 2
	2 2 2

$$d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{2,2}^2 = \left( \frac{1 + \cos \theta}{2} \right)^2$$

$$d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$$

$$d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$$

$$d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$$

$$d_{2,-2}^2 = \left( \frac{1 - \cos \theta}{2} \right)^2$$

$$d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$$

$$d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$$

$$d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$$

$$d_{0,0}^2 = \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.