Symmetries & Conservations Laws – Exam Question – 8 Feb 2013

SJH

Answer all the questions. Time 1.5 hours. Total 20 marks.
Hint: Roughly every key step or point corresponds to 0.5 marks.

1) Exponentials of Operators [6 marks]

$F(\alpha) = \exp(i\alpha A)\, B\, \exp(-i\alpha A)$, where A and B are operators, satisfying $[A,B] = A$

- Find a simplified expression for $F(\alpha)$ by
a) Expanding one of the exponentials as a power series.
b) Considering derivatives.

2) SU(2) Singlet [4 marks]

- Write down an $SU(2)_{spin}$ singlet state for a quark-quark pair.

Consider the rotation of the state about the y-axis through an angle θ.

- Find the new state and comment.

Formulae:

$$\exp(i\alpha \mathbf{n} \bullet \boldsymbol{\sigma}) = \cos\alpha + i\sin\alpha\ \mathbf{n} \bullet \boldsymbol{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3) A Group of Transformations [7 marks]

Consider a set of transformations in the x-y plane {I, X, Y, R}:

I	Identity
X	Reflection in x=0
Y	Reflection in y=0
R	Rotation by 180^{o}

- Identify the 2×2 matrices which correspond to these transformations.
- Show that this set is a group under the operation “follows”.
- When addressing the issues of Associativity, prove that this is valid for all square matrices of the same order.
- Prove that all diagonal matrices of the same order commute, and hence that the group is Abelian.
- Identify all subgroups.

PTO

4) Product of Groups [3 marks]

This question looks scary, but actually it is pretty straightforward. It is effectively a simplified version of the studies we made of general SU(2) transformations on spin states for two spin-½ particles (see also Question 2) – but expressed more formally. When we looked at SU(2) we didn't explicitly refer to a product of groups, but that is effectively what we had.

Consider the *product* of two groups: $Z_2 \otimes Z_2$, where Z_2 is the cyclic group of order 2 (this subscript "2" relates to the order of the group).
Z_2 could be represented by two transformations $\{g = i, f\}$ operating on spin states χ: i leaves the spin unchanged; f flips the spin.
If we have two spin states (labelled with subscripts 1 and 2), we can imagine a product operator $g_1 \times g_2$ belonging to $Z_2 \otimes Z_2$, operating on the combined spin state $\chi_1\chi_2$, where g_1 only operates on χ_1 and g_2 only operates on χ_2.

Now consider the set of these products of two transformations. We could indicate this set by $G = \{i{\times}i, f{\times}i, i{\times}f, f{\times}f\}$, where × denotes the *product* of the members of the two groups.
If • denotes the combination of two members of a group, with the *operation* "follows", then transformations associated with the first particle are combined separately, and likewise for the transformations associated with the second particle:

$$(g_1' \times g_2') \bullet (g_1 \times g_2) \text{ corresponds to } (g_1' \bullet g_1) \times (g_2' \bullet g_2),$$
$$\text{which in turn can be written } g_1'' \times g_2''.$$

So for example $(i{\times}i) \bullet (i{\times}f) = (i \bullet i) \times (i \bullet f) = i{\times}f$.

- Find the 4×4 combination table for the members of the set G.

There is an obvious Isomorphism between this group and the group in Question 3.

- Identify the isomorphism and give some simple rationale as to how this can be understood.

(In this question, distinguish carefully between the product *of two groups, and the* combination *with an* operation *of members of a given group.)*