Standard model notes 2013-2014

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December 23, 2013

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Chapter 1

Introduction: matter and forces

Elementary particles are divided into two big classes: fermions, the constituents of matter, and bosons, force carriers. Fermions are further divided into leptons and quarks, the difference being that the latter do feel the strong nuclear interaction, and are forced by confinement to combine with other quarks to combine hadrons. Leptons and quarks are combined into doublets. Charges and masses of the known leptons and quarks are listed in the following:

1.1 Leptons

$$\begin{array}{c} Charge = 0 \\ Charge = 1 \end{array} \begin{pmatrix} \nu_e \\ e \end{pmatrix} 0.5MeV \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} 0.1GeV \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} 1.8GeV$$

Even if leptons do not experience the strong force, they do experience the electromagnetic and weak forces.

1.2 Quarks

$$\begin{array}{c} Charge = 2/3 \\ Charge = -1/3 \end{array} \begin{pmatrix} u \\ d \end{pmatrix} \begin{array}{c} 0.3GeV \\ 0.3GeV \end{array} \begin{pmatrix} c \\ s \end{pmatrix} \begin{array}{c} 1.5GeV \\ 0.5GeV \end{array} \begin{pmatrix} t \\ b \end{pmatrix} \begin{array}{c} 180GeV \\ 5GeV \end{array}$$

The u, d, s are "constitutional" masses which one can infer from the masses of the corresponding hadronic systems. However it is thought that most if this mass is "effective" ie generated dynamically by the strong interaction. Intrinsic masses are hard to define, and thought to be $\sim 4MeV(u, d)$ and $100MeV(s)$.

- We do not understand why particles have their masses

- They are all fermions (spin$-1/2$)
- The u and d quarks combine to form the hadorns we know
- The above is believed to constitute about 4% of the mass-energy of the universe

Forces are mediated by bosons (integer spin) and can be thought to arise from local gauge invariance.

1.3 Gravity

This is by far the weakest force and is thought to be mediated by spin-2 gravitons.

It is important on an astronomical scale because of the large masses multiplying the intrinsic weakness of the force.

Gravity is universal and there are no cancellations (ie no negative causes).

Gravitational radiation should arise when masses oscillate, giving quadrupole radiation. Gravitational radiation would create a ripple in space-time. Indirect detections exist from observations by Hulse and Taylor over a seventeen year period of a binary system of two neutron stars. The change in the period of the binary system can only be accounted for by the emission of gravitational radiation, winning the Nobel Prize in 1993.

1.3.1 The Planck mass

Suppose two pointlike masses m are created by quantum fluctuations out of the vacuum. One mass can exist for a time δt given by:

$$\begin{aligned}
\delta\epsilon\delta t &\leq \hbar \\
mc^2\delta t &\sim \hbar \\
\Rightarrow \delta t &\sim \frac{\hbar}{mc^2} \\
\text{Range } r &= c\delta t \\
&= \frac{\hbar}{mc}
\end{aligned}$$

At this separation the gravitational potential V_G is given by:

$$\begin{aligned}
V_g &= \frac{Gm^2}{r} \\
&= \frac{Gm^3c}{\hbar}
\end{aligned}$$

If V_G becomes comparable with the energy mc^2 the relativistic effects become important, but there is no theory of quantum gravity.

$$
\begin{aligned}
\text{So } \frac{Gm^3c}{\hbar}\frac{1}{mc^2} &\sim 1 \\
\text{So } m_{Planck} &\simeq \sqrt{\frac{\hbar c}{G}} \\
&\simeq 10^{19}GeV
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\text{Planck length:} &\quad 2\times 10^{-35}m \\
\text{Planck time:} &\quad 10^{-44}s
\end{aligned}
$$

Without quantum gravity we cannot discuss the evolution of the universe at epochs around and before the Planck time.

1.3.2 Dark matter

Dark matter is believed to constitute about 25% of the matter-energy content of the universe. This comes from evidence from velocity rotation curves in a galaxy.

The radial velocity of a star is measured using the doppler shift and is plotted as a function of the distance from the centre of the galaxy, r.

$$
\begin{aligned}
Centripetal force &= Gravitational force \\
\text{For } r < R: & \\
\frac{\Delta m v^2(r)}{r} &= \frac{G\Delta m M_{r<R}}{r^2} \\
v^2(r) &= \frac{GM_{r<R}}{r} \\
&= \frac{G}{r}M_s\left(\frac{r}{R}\right)^2 \\
&= \frac{GM_s r^2}{R^3} \\
\text{So } v(r) &\propto r \text{ for } r < R \\
\text{For } r > R: & \\
\frac{\Delta m v^2(r)}{r} &= \frac{GM_s\Delta m}{r} \\
v^2(2) &= \frac{GM_s}{r} \\
\text{So } v(r) &\propto \frac{1}{\sqrt{r}} \text{ for } r > R
\end{aligned}
$$

The observed discrepency is hypothesised to be the result of dark matter. It is thought that there is a halo of dark matter surrounding the galaxy with a matter distribution which yields the observed discrepency.

1.3.3 Candidates for dark matter

Neutrinos There is now evidence that neutrinos have mass. However the masses are small and neutrinos are therefore relativistic particles, so this is hot dark matter which does not stay around long enough to provide gravitational binding.

Brown dwarves (Massive cool hadronic objects - MaCHOs) Brown dwarves are like small stars which are not massive enough for nuclear burning so they appear dark. There cannot be too many brown dwarves since there would then be too much baryonic matter which is incompatible with current understanding of the abundance of hydrogen, helium and deuterium in the early universe. The elements were generated in the first $\simeq 900s$. To search for brown dwarves it is necessary to look at many stars in eg a large Magellic cloud. If a brown dwarf passes across the line of sight of one of the stars then through gravitational lensing the light intensity from the star will increase and decrease in a characteristic way. Brown dwarves have been observed but not in sufficient numbers to account for the amount of dark matter.

Supersymmetric particles There are several names for such particles. In the context of dark matter a common name is Weakly Interacting Massive Particles (WIMPs). Alternatives include Lightest Symmetrical Particles (LSPs) and candidates thereof (neutralinos, gravitinos etc).

The current standard model of cosmology gives 5% of the matter observed, 25% dark matter and 70% dark energy. Where does this sub-division come from?

A survey of Type I supernovae indicates that the universe is accelerating for $z \geq 0.8 (z = \delta\lambda/\lambda)$. The type I supernovae are like "standard candles" in a binary system in which matter flows from one of the stars to another and creates a thermonuclear explosion. This constitutes the standard candle and therefore the distance can be determined. The doppler shift gives the recessional velocity.

1.3.4 Anisotropies in the cosmic microwave background and the history of the universe

The universe is now thought to be $\sim 14 \times 10^9$ years old. After about $350,000$ years the temperature had dropped to about $3000K$, at which point electrons and protons could combine to form hydrogen. At this epoch, photons decoupled from matter. Subsequently this black body spectrum cooled to $2.7K$, which is what is observed today. This was first observed in 1965 by Perzies and Wilson, receiving the Nobel Prize in 1978. The cosmic microwave background radiation is uniform tp 1 part in 10^5 in all directions.

Big bang nucleosynthesis occured in the first $900s$. Electroweak symmetry breaking took place at $10^{-12}s$ or $E \sim 100GeV$. The energy released by this symmetry breaking drives cosmic inflation which explains the flatness of the universe and the horizon problem. Grand unified theory symmetry breaking occues at $10^{15}GeV$.

Anisotropies were originally measured by the COBE sattelite and by other experiments, most recently WMAP and Boomerang.

The power spectrum gives strong evidence for:

- a flat universe
- inflation
- ratio of dark energy : dark matter of 70% : 25%

1.4 The weak force

The weak force is mediated by massive vector bosons:

$$\begin{array}{rll} W^{\pm} & M_W & \sim 80.4GeV \\ Z^0 & M_Z & \sim 91.2GeV \end{array}$$

The instrinsic strength of the force, g_W, is of the same order as the electromagnetic force, but the massive force carriers make it appear weak and short ranged.

For electromagnetism the strength is:

$$\frac{\mathrm{e}^2}{q^2}$$

where q is the propagator momentum.

For the weak force the strength is:

$$\frac{g_W^2}{q^2 + M_W^2}$$

Some examples of weak interactions include:

$$\begin{array}{cccccc} & n & \rightarrow & p & \mathrm{e}^- & \bar{\nu}_e \\ \nu_e & n & \rightarrow & p & \mathrm{e}^- & \\ \bar{\nu}_e & p & \rightarrow & n & \mathrm{e}^+ & \\ \bar{\nu}_e & u & \rightarrow & d & \mathrm{e}^+ & \end{array}$$

Pure leptonic weak processes include:

$$\begin{array}{cccccc} & \mu^- & \rightarrow & \mathrm{e}^- & \bar{\nu}_e & \nu_\mu \\ \mathrm{e}^- & \bar{\nu}_e & \rightarrow & \mu^- & \bar{\nu}_\mu & \end{array}$$

1.5 The electromagnetic interaction (QED)

The electromagnetic force is mediated by the massless photon, which has an infinite range. The coupling constant, e, is quantised $(0, \pm 1/3, \pm 2/3, \pm 1)$ and is not too strong, so perturbation theory works.

The strength of the force is characterised by α:

$$\alpha = \frac{\mathrm{e}^2}{\hbar c} = \frac{\mathrm{e}^2_{Coulomb}}{4\pi\epsilon_0 \hbar c} = \frac{\mathrm{e}^2_{HL}}{4\pi\hbar c}$$

where $e_{Coulomb}$ is the Coulomb charge and e_{HL} is the Heaviside-Lorentz charge.

In particle physics e is measured in Heaviside-Lorentz units where $\hbar = c = 1$.

$$\text{Therefore } \alpha = \frac{\mathrm{e}^2}{4\pi} \simeq \frac{1}{137.04}$$

1.5.1 The physical meaning of α

α expresses the energy of an $\mathrm{e}^+\mathrm{e}^-$ pair materialising for a short time in the vacuum as a fraction of the rest mass energy $m_e c^2$:

$$\begin{aligned}
\alpha &\sim \frac{e^2}{c\Delta t}\frac{1}{m_e c^2} \\
\Delta t \Delta E &\sim \hbar \\
\Delta t &\sim \frac{\hbar}{m_e c^2} \\
\text{So } \alpha &\sim \frac{e^2}{c}\frac{m_e c^2}{\hbar}\frac{1}{m_e c^2} \\
&= \frac{e^2}{\hbar c}
\end{aligned}$$

1.6 The strong force (QCD)

Since the 1950s, it was knwon that the $\Delta^{++}(uuu)$ particle state existed at $1238MeV$, formed in $\pi^+ p$ scattering:

$$\pi^+ \quad p \stackrel{\Delta^{++}(1238)}{\rightarrow} \pi^+ \quad p$$

In the simple quark model there are 3 up quarks, each with psin$-1/2$ in the same direction. Also in 1965 the Ω^- was discovered which consists of 3 strange quarks each with spin$-1/2$ in the same direction. These appeared to violate the Pauli exclusion principle.

The solution was introduce 3 colour charges with the strong force mediated by massless gluons which carry colour. The force is not of infinite rane because the strength increases with separation, leading to confinement.

1.7 Local gauge invariance

The weak, electromagnetic and strong forces all appear to arise from requiring local gauge invariance. This means that the Lagrangians are invariant with respect to a local gauge transformation and equations of motion derived from the Lagrangian are invariant with respect to local gauge transformation.

Quantum states can be multiplied by a phase factor:

$$\psi'(x,t) \rightarrow e^{iq\chi(x)}\psi(x,t)$$

If $\chi(x,t)$ depends on space and time then the above expression is a local gauge transformation. To allow this the electromagnetic field, A_μ must transform in a specific way.

In weak interactions this is not as simple. The Lagrangian contains four massless bosons (W^i, B). To allow for local gauge invariance, weak isospin and weak hypercharge matrices are required which combine with the massless vector fields to form the gauge transformation. Symmetry breaking occurs such that W^1_μ and W^2_μ yield $W^\pm$ and W^3_μ and B_μ mix to yield A_μ and Z^0. This then unifies the electromagnetic and weak interactions via the Higgs mechanism.

1.8 Grand unification (GUTs)

The GUT attemps to unify the electromagnetic, weak and strong forces. The coupling constants evolve with energy due to loop processes.

The couplings are expected to merge at $\sim 10^{15} GeV$ according to some models, but the couplings do not come together at this energy. The evolution of g_1, g_2, g_3 could coincide if supersymmetrical particles exist. This would then yield a grand unified group, G, or $SU(5)$ which is a combination of the three groups describing the three forces. Here gauge bosons exist which would allow quarks to turn into leptons. This would also allow protons to decay:

$$p \rightarrow \mathrm{e}^{+} \quad \pi^{0}$$

This decay has been extensively searched for but not observed. Current lower limits on the lifetime of the proton are of the order $\tau_p > 10^{30}$ years, excluding the minimal grand unified theories.

Chapter 2

Experimental concepts

2.1 Experimental possibilities

In reality there are rather few experiemental possibilities:

- Scatter one particle off another and observe the reaction
- Generate a particle and observe subsequent decay processes
- Observe neutrino oscillations
- Measure a particle's properties eg charge, lifetime

2.2 Cross sections

The cross section, σ, is an imaginary area surrounding a "target" particle through which an incident particle must pass for a particular interaction to take place.

To calculate a cross section:

- Assume N_B particles per incident beam bunch
- Assume the beam illuminates an area A of the target which is of length l and density ρ

The number of target particles that are illuminated by the beam is:

$$N_T = \frac{lA\rho N_A}{m}$$

where m is the molecular mass of the target nuclei and N_A is Avagadro's constant.

Each of the particles is imagined to have an area σ surrounding it. The probability that one beam particle interacts in the area A is:

$$\begin{aligned} P(\text{Interaction}) &= \frac{lA\rho N_A}{m}\frac{\sigma}{A} \\ &= \frac{l\rho\sigma N_A}{m} \end{aligned}$$

The total number of interactions is then:

$$N_I = \frac{l\rho N_A N_B \sigma}{m}$$

Going to infinitesimal quantities, N_I and l are replaced by $\mathrm{d}N_I$ and $\mathrm{d}l$ respectively. In order to calculate the number of interactions per second in a general collider it is necessary to make the expression symmetrical with respect to the beam and target:

- Assume the particle densities in the beam and target are ρ_B and ρ_T respectively
- Assume the interactions are occuring in a volume V with a cross-sectional area A
- Assume the beam travels with a relative speed u

The number of target particles in the volume V is $\rho_T V$, so the probability of one beam particle interacting in the area A is $\rho_T V\sigma/A$. In $1s$ there are $uA\rho_B$ particles, so the number of interacions in a fixed target experiment is:

$$\Rightarrow \frac{\mathrm{d}n}{\mathrm{d}t} = u\sigma V\rho_B\rho_T$$

In a collider there are n bunches containing N_B particles rotating at $u \sim c$. A second beam rotates in the opposite direction. The probability of an interaction is $\sigma N_B/A$ in one bunch of the second beam where A is the cross-section interaction region. In $1s$ f bunches (where f is the rotation frequency) pass through the bunch in the second beam:

$$\begin{aligned} \Rightarrow \frac{\mathrm{d}N_I}{\mathrm{d}t} &= \frac{fN_BN_Bn\sigma}{A} \\ &= L\sigma \end{aligned}$$

where the luminosity, L, is given by:

$$\begin{aligned} L &= \frac{nN_B^2 f}{A} \\ L &= \frac{1}{\sigma}\frac{\mathrm{d}N_I}{\mathrm{d}t} \qquad (2.1) \end{aligned}$$

(1) is the formal definition of the luminosty. The unit of cross-section is the barn, where $1bn = 10^{-28}m^2$ and the units of L are $bn^{-1}s^{-1}$. The instantaneous luminosity can be quoted in inverse picobarns per second and the integrated luminosity can be quoted in inverse picobarns (or femtobarns, or nanobarns).

2.2.1 Scattering pions and nucleons

Pions and nucleons interact strongly. According to an old empiric model of the strong interaction, two protons may exchange a π^0 to form a Δ^{++} which subsequently decays to a proton and π^0.

The cross sections for various kinds of interactions are:

$$\sigma_{pp} > \sigma_{np} \gg \sigma_{\gamma p} \gg \sigma_{\gamma\gamma}$$

Calorimeters are built according to the radiation lenght (for electromagnetic showers) and interction length (for hadronic showers). Both quantities are different for different materials eg high-energy electrons ($> GeV$) lose the same fraction of energy in $18cm$ of water as in $2.8mm$ of lead. Particles not producing showers still lose energy by ionisation, which is expressed as $\mathrm{d}E/\mathrm{d}x$, which is largely a function of the velocity of the particle, but is roughly constant above a given velocity.

2.3 Natural units and conversion factors

In quantum mechanics the expressions for energy and momentum can be expressed as:

$$\begin{array}{lllll} E_\nu & = & hf & = \hbar\omega & \text{Planck's law} \\ p_\nu & = & h/\lambda & = \hbar k & \text{De Broglie's relation} \end{array}$$

Travelling quantum waves are of the form:

$$\begin{aligned} \psi &= A\mathrm{e}^{i(kx-\omega t)} \\ &= A\mathrm{e}^{i/\hbar(px-Et)} \end{aligned}$$

And for light $c = f\lambda = \omega/k$.
It is conventient to set $\hbar = c = 1$. Consider the units of $c\hbar$:

$$\begin{aligned} [c\hbar] &= [L][T^{-1}][E][T] \\ &= [L][E] \\ c &= 3 \times 10^8 ms^{-1} \\ \hbar &= 6 \times 10^{-25} GeVs \\ c\hbar &= 1.97 \times 10^{-16} GeVm \end{aligned}$$

Thus setting $c = \hbar = 1$ gives $1 = 1.97 \times 10^{-16} GeVm$.
Other useful conversion factors include:

$$\begin{array}{lll} \frac{1}{(GeV)^2} & = 3.89 \times 10^{-32} m^2 & = 0.389 mb \\ \frac{1}{GeV} & = 6.6 \times 10^{-5} s & \end{array}$$

Chapter 3

Some experiments of the past 50 years

Until the 1950s particle physics was mainly studied by observing cosmic rays in cloud chambers and nuclear emulsion. After 1945 nucleon-nucleon scattering experiements were carried out at cyclotrons and energies became high enough for pions to be produced. Then pion-nucleon scattering was studied.

$$\text{In 1952 } \pi^+ \quad p \overset{\Delta^{++}}{\rightarrow} \pi^+ \quad p$$

Also, photons could be produced from electron beams:

$$\gamma \quad p \overset{\Delta^{+}}{\rightarrow} \gamma \quad \gamma \quad p \quad (\text{via } \Delta^+ \rightarrow p \quad \pi^0)$$

although the rate of production via γ processes is much lower because of the strength of the electromagnetic coupling.

Other processes were also observed:

$$\begin{array}{ccl} \pi^+ & \rightarrow & \mu^+ \quad \nu_\mu \\ \mu^+ & \rightarrow & \mathrm{e}^+ \quad \bar{\nu}_\mu \quad \nu_e \end{array}$$

where the latter process is a purely leptonic process and so provoked much theoretical interest.

In 1956 parity violation in the weak interaction was discovered, and Lee and Young received the Nobel Prize for suggesting the theoretical frameworn, and en experiement studying the β decay of polarised Cobalt 60 nuclei. An anistropy was discovered in the electron spectroscopy with respect to the ^{60}Co nuclear spin.

$$^{60}Co \rightarrow^{60} Ni * \left(\mathrm{e}^-\right)_L \left(\bar{\nu}_e\right)_R$$

By the 1960s kaon beams were generated at synchotrons and this confirmed the property of strangeness that was observed in cosmic ray experiments: some particles were produced by the strong force and decayed by the weak force. This fenomenon was attributed to a third quark, called of course "strange". It also established (in the experimentalists' eyes) the quark substructure of hadrons ie hadrons were made of $q\bar{q}$ pairs (mesons) and qqq triplets (baryons) where q is a u, d or s quark. Theorists regarded the evidence for strangeness as establishing the $SU(3)$ flavour symmetry, which is now known to have been accidental. This symmetry arises because the constituent masses of u and d quarks are about the same and the mass of the s quark is somewhat heavier, but still much smaller than that of the next quark, the charm.

3.1 Mesons and baryons

Consider $q\bar{q}$ systems consisting of u and d quarks. The strong isospin doublet is:

$$2 = \begin{pmatrix} u \\ d \end{pmatrix} \begin{array}{rcl} I_3 & = & \pm 1/2 \\ I & = & 1/2 \end{array}$$

Assume that the strong isospin is conserved (which is accidental, but works as the u and d quarks have nearly the same mass.) This doublet is combined with the antiquark doublet which is:

$$\bar{2} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \begin{array}{c} {\scriptstyle 1/2} \\ {\scriptstyle -1/2} \end{array}$$

In this form the raising and lowering operators act on the 2 doublet in the same way for the $\bar{2}$ antiquark doublet. The minus sign on $\bar{d}$ arises because of a rotation in isospin space:

$$\begin{aligned} \begin{pmatrix} u \\ d \end{pmatrix}' &= \mathrm{e}^{-i\pi\tau_z/2} \begin{pmatrix} u \\ d \end{pmatrix} \\ &= I\left(\cos\frac{\pi}{2} - i\frac{\tau}{2}\sin\frac{\pi}{2}\right) \begin{pmatrix} u \\ d \end{pmatrix} \\ \Rightarrow \begin{pmatrix} u \\ d \end{pmatrix}' &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \\ &= \begin{pmatrix} -d \\ u \end{pmatrix} \\ u' &\to -d \\ d' &\to u \end{aligned}$$

Suppose $\bar{2}$ had been defined as:

$$\bar{2} = \begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix}$$

Then the transformation would be:

$$\begin{aligned} \bar{d}' &\rightarrow -\bar{u} \\ \bar{u}' &\rightarrow \bar{d} \end{aligned}$$

The $\bar{2}$ system therefore transforms in the same way as a the 2 system:

$$\begin{aligned} \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \\ &= \begin{pmatrix} -\bar{u} \\ -\bar{d} \end{pmatrix} \\ -\bar{d}' &\rightarrow -\bar{u} \\ \bar{u}' &\rightarrow -\bar{d} \end{aligned}$$

Now 2 and $\bar{2}$ can be combined to form a representation of the π mesons:

$$\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \begin{array}{c} \begin{pmatrix} u & d \end{pmatrix} \\ \begin{pmatrix} -u\bar{d} & -d\bar{d} \\ u\bar{u} & d\bar{u} \end{pmatrix} \end{array}$$

$$\begin{aligned} \pi^+ &= -u\bar{d} \\ \pi^- &= d\bar{u} \\ \pi^0 &= \frac{1}{\sqrt{2}} \left(u\bar{u} - d\bar{d} \right) \end{aligned}$$

There is an isospin doublet to represent the antiquarks in $SU(2)$ and not in any other $SU(n)$. In $SU(3)$ flavour symmetry:

$$\begin{aligned} 3 &= \begin{pmatrix} u \\ d \\ s \end{pmatrix} \\ \bar{3} &= \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \end{aligned}$$

To construct the $q\bar{q}$ states in $SU(3)$:

$$\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \begin{array}{c} \begin{pmatrix} u & d & s \end{pmatrix} \\ \begin{pmatrix} u\bar{u} & d\bar{u} & s\bar{u} \\ u\bar{d} & d\bar{d} & s\bar{d} \\ u\bar{s} & d\bar{s} & s\bar{s} \end{pmatrix} \end{array}$$

The non diagonal elements are easily identified as the following particles:

$$
\begin{aligned}
s\bar{u} &= K^- & u\bar{s} &= K^+ \\
d\bar{s} &= \bar{K}^0 & s\bar{d} &= K^0 \\
u\bar{d} &= \pi^+ & d\bar{u} &= \pi^-
\end{aligned}
$$

There is also a symmetric state:

$$\eta_1 : \quad \frac{1}{\sqrt{3}} \left(u\bar{u} + d\bar{d} + s\bar{s}\right)$$

This is called the $SU(3)$ singlet state. The final neutral state is:

$$\eta_8 : \quad \frac{1}{\sqrt{6}} \left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$$

This is constructed to be orthogonal to the π^0 and the singlet state.

3.2 Neutrino experiments

3.2.1 The Gargamelle experiment

Using the CERN Proton Synchrotron, protons were extracted from the accelerator and impinged on a thin Beryllium target within a neutrino horn. In the targets πs and Ks were created and the horn partially selected either positive or negative charges. The partially focussed π^+ beam decayed to $\mu^+\nu_\mu$. An iron shield filtered out the remaining hadrons and muons. Measurements of the muon tracks enabled the neutrino spectrum to be determined. The neutrios then passed into the large heavy liquid bubble chamber, Gargamelle. Scattering via the exchange of the $W^\pm$ was expected, but scattering via the Z^0 was also observed, in the form of nfinal states without production of muons. That result means that weak interactions can also proceed via neutral currents, and it was the first stronf proof of the validity of the electroweak unification.

3.2.2 Underground experiments

Solar neutrinos are produced primarily by the following reaction:

$$p \quad p \rightarrow d \quad e^{+} \quad \nu_e$$

even if the energy of these neutrinos is very small, and they are only detectable using very sensitive detectors where the single atom conversions are counted. Other nuclear reactions in the sun can produce more energetic neutrinos, observable by more traditional high-energy physics techniques. Atmospheric neutrinos are produced primarily by proton bombardment of the upper atmosphere:

$$\begin{array}{lllllll} p \quad N & \rightarrow & \pi^{+}/K^{+} \quad H & & & & \\ & & \quad \hookrightarrow & \mu^{+} \quad \nu_{\mu} & & & \\ & & & \quad \hookrightarrow & e^{+} & \bar{\nu}_{\mu} & \nu_e \end{array}$$

Naively, it is expected that the production rate would be:

$$\frac{\nu_e}{\nu_\mu} \sim \frac{1}{2}$$

The ratio has been measured by Superkamikande to be closer to 1, demonstrating that ν_μ neutrinos were missing and also measured an azimuthal variation. ie the experiment measured the number of neutrinos $N(\nu_\mu)$ and $N(\nu_e)$ from the atmosphere (above) and the other side of the Earth (below.) About half from the other side of the Earth were lost, suggesting that that neutrinos oscillated into ν_τ. The oscillations imply that neurinos have mass and as such must have a velocity $\beta < 1$.

A large water Cerenkov detector was used to look for the processes:

$$\begin{array}{cccccc} \nu_\mu & N & \rightarrow & \mu^- & H \\ \nu_e & N & \rightarrow & \mathrm{e}^- & H \end{array}$$

Both muons and electrons were detected by $\sim 5,000$ phototubes by considering their characteristic signals for Cerenkov light. The muon signal rings are sharp whereas those for electrons are diffuse.

3.2.3 Solar neutrinos

In the experiment by Ray Davis mainly "high" energy ($14 MeV$) neutrinos were used from a process:

$$\begin{array}{cccccccc} p & {}^7Be & \rightarrow & {}^8Be & \gamma & & & \\ & & & \hookrightarrow & & {}^8Be & \mathrm{e}^+ & \nu_e \end{array}$$

The reaction considered was:

$$\nu_e^{37}Cl \rightarrow \mathrm{e}^- \quad {}^{37}Ar$$

The neutrinos were impinging on a tank of C_2Cl_4. There were not as many such reactions as expected according to the Standard Model.

To detect low energy neutrinos, tanks of Gallium were used:

$$\nu_e \quad Ga \rightarrow Ge \quad \mathrm{e}^-$$

These processes were also observed at a lower than expected rate.

In the Sudbury Neutrino Observatory (SNO) a tank of heavy water (D_20) was used. The following reaction was detected:

$$\nu_e \quad n \rightarrow \mathrm{e}^- \quad p$$

Again, a deficit of electron neutrinos was observed. Around 1/3 of the expected signal was observed. Combined with the results from Superkamiokande this explains the solar neutrino problem where $\sim 1/3\nu_e$ are observed and $\sim 2/3\nu_e$ oscillate into ν_μ and ν_τ.

The results at SNO were further confirmed when salt ($NaCl$) was added to the water, increasing the sensitivity to ν_μ and ν_τ:

$$\begin{array}{rcccl} \nu_{e/\mu/\tau} & n & \rightarrow & \nu_{e/\mu/\tau} & n_{scattered} \\ \text{Then } {}^{37}Cl & n & \rightarrow & {}^{38}Be & \gamma \end{array}$$

This was then consistent with the expected solar flux.

Further neutrino oscillation experiments are ongoing at reactors (a good source of copious low energy neutrinos from b decay) and at accelerators such as MINOS.

3.3 Colliding beam and some fixed target experiments

There are various different types of colliding beams which have different properties and can probe different phenomena. They can be classified into three types:

e^+e^- Purely leptonic beams give rise to "clean" output and also have a controled centre of mass energy. There is a large discovery potential, however there are limits due to synchrotron radiation, so future developments will lead to linear colliders.

$NN(pp)$ Purely hardonic beams are not as clean and do not have a well defined centre of mass energy. However there is a large discovery potential due to the possibility of much higher energies.

lN A mixed pair of beams allows probing of the partons.

3.3.1 Lepton-nucleon colliders

In the late 1960s and the early 1970s deep inelastic scattering experiments using lepton beams of electrons, neutrinos and also muons were used to probe the structure of the proton and neutron. It looked as if scattering occured on pointlike objects in the nucleon and around 50% of the nucleon interacted in this way. The remaining 50% was made up of gluons. This was the beginnings of quantum chromodynamics (QCD).

At HERA, this has been advanced further in *ep* collisions. Electrons (or positrons) of energies at $27.5GeV$ collide with protons at $920GeV$, yielding a centre of mass energy of around $320GeV$. There are two multipurpose colliding beam experiments which measure a wide range of phenomena such as proton and photon structure; many other aspects of QCD; electroweak physics and searches for effects beyond the Standard Model (eg leptoquarks).

The structure of the proton has been measured over a vast kinematic range compared to the first measurements in the 1960s.

x is the proton's momentum fraction carried by the struck quark. Q^2 is the four momentum transfer, essentially related to the wavelength of the probing photon. A high value for Q^2 implies a high resolving power.

This gives us precise knowledge of the structure of matter which is one of the fundamental goals of physics. Also practically many colliders use protons (eg LHC) so it is useful for understand the structure of what is being collided.

Demonstration of the unification of the electroweak force is shown by:

The processes became the "same" at $M_{W,Z}^2 \sim 10^4 GeV^2$.

3.3.2 e$^+$e$^-$ colliders

There have been a multitude of e$^+$e$^-$ experiments with a centre of mass enery of a few GeV to over $200GeV$. There is planning for a linear e$^+$e$^-$ collider up to $1TeV$.

The charm quark was discovered in 1974 at SLAC (and in a $p \quad Be$ experiment at BNL) via the detection of the decay of the bound state, the J/ψ meson. $m_{J/\psi} \simeq 3.1GeV$.

In 1979 the gluon was discovered by the experiments at the PETRA collider in DESY at $\sqrt{s} = 35GeV$. Although e$^+$e$^-$ is a clean leptonic enviornment, it can provide a powerful probe of QCD eg discovery of the gluon through the oserbvation of 3−jet events

Most simply one would expect a srtaightforward process with back to back jets of equal energy. However, in the detector three jets were seen (one of the quarks radiated a gluon.)

In 1989 the Large Electron-Positron (LEP) collider turned on, embarking on a new era of precision physocs (running at the mass of the Z^0 boson.) Initial LEP running was at the Z^0 peak $\simeq 91GeV$, and then moved through $2M_W \simeq 160GeV$ and finally to just over $200GeV$, looking for the Higgs boson. There were four multipurpose experiments. The experiments were most famous for the precision measurements of electroweak parameters such as M_Z and M_W. The measurement of the cross-section as a function of $\sqrt{s}$ was fundamental in measuring M_Z and constraining the number of light neutrinos.

$$\begin{aligned} M_Z &= 91.1876 && \pm 0.0024GeV \\ M_W &= 80.403 && \pm 0.029GeV \end{aligned}$$

In the absence of direct measurements, precise determination of known parameters constrain new physics phenomena eg Higgs boson. In its final throws LEP also searched for the Higgs boson via Higgstrahlung, where a virtual Z^0 results in a real Z^0 and Higgs boson.

The centre of mass energy was constantly increased as the necessary energy, E had to satisfy $E > M_H + M_Z$, but the Higgs boson was not observed.

Limits of circular e^+e^- machines are being reached due to the rate of energy loss due to synchrotron radiation.

The next planned major collider is the International Linear Collider (ILC). This would complement the LHC because of the cleanliness of the signal. This can act as a "factory" for eg $t\bar{t}$ production, Higgstrahlung etc.

3.3.3 Hadron-hadron colliders

Due to the hadronic structure and multitude of final states, these colliders are generally more complicated than e^+e^- colliders. However, they are usually the energy frontier and thereby produce discoveries and measurements of known phenomena over a large kinematic range.

The discovery of the b quark took place in 1977 by obvserving the production and decay of the Υ mesons via $\mu^+\mu^-$ in $p \quad Be$ collisions at Fermilab.

(picture of cross-section)

The invariant mass of the $\mu^+\mu^-$ pair was was observed and a resonance was apparent.

The $W^\pm$ and Z^0 bosons were discovered in 1984 at the SppS collider at CERN. Leptonic decays of the $W^\pm$ and Z^0 were searched for, as they give a lower background relative to hadronic decays ($\sqrt{s} = 540GeV$).

The t quark was discovered by CDF and D0 at $\sqrt{s} = 1800GeV$ in 1995.

There are high jet energies measures over nine orders of magnitude in the cross-section. This allows physicsts to

- verify and understand QCD
- look for new physics at the highest energies (eg quark substructure)

Chapter 4

Review of non-relativistic quantum mechanics

4.1 Schroedinger picture and probability current

For a free particle of mass m the classical energy-momentum relation is:

$$E = \frac{p^2}{2m}$$

In quantum mechanics, p and E become differential operators:

$$\begin{aligned}
E &\rightarrow i\frac{\partial}{\partial t} \\
p &\rightarrow -i\nabla \\
(\hbar &= 1)
\end{aligned}$$

These operate on the wavefunction:

$$\begin{aligned}
\frac{(-i)^2}{2m}\nabla^2\psi &= i\frac{\partial}{\partial t}\psi \\
\frac{-1}{2m}\nabla^2\psi &= i\frac{\partial\psi}{\partial t} \qquad (4.1) \\
\psi^\star \times (2): \quad \frac{-1}{2m}\psi^\star\nabla^2\psi &= i\psi^\star\frac{\partial\psi}{\partial t} \qquad (4.2) \\
\text{Complex conjugate of (2):} \quad \frac{-1}{2m}\nabla^2\psi^\star &= -i\frac{\partial\psi^\star}{\partial t} \qquad (4.3) \\
\psi \times (4): \quad \frac{-1}{2m}\psi\nabla^2\psi^\star &= -i\psi\frac{\partial\psi^\star}{\partial t} \qquad (4.4)
\end{aligned}$$

$$\begin{aligned}
(3)-(5) &= i\left(\psi^{\star}\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^{\star}}{\partial t}\right)\\
&= \frac{-1}{2m}\left(\psi^{\star}\nabla^{2}\psi-\psi\nabla^{2}\psi^{\star}\right)\\
\text{So } i\left(\psi^{\star}\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^{\star}}{\partial t}\right) &= \frac{-1}{2m}\left(\psi^{\star}\nabla^{2}\psi-\psi\nabla^{2}\psi^{\star}\right)\\
\text{or } \frac{\partial}{\partial t}\left(\psi^{\star}\psi\right)+\frac{i}{2m}\left(\psi\nabla^{2}\psi^{\star}-\psi^{\star}\nabla^{2}\psi\right) &= 0\\
\frac{\partial}{\partial t}\left(\psi^{\star}\psi\right)+\frac{i}{2m}\underline{\nabla}\cdot\left(\psi\underline{\nabla}\psi^{\star}-\psi^{\star}\underline{\nabla}\psi\right) &= 0\\
\frac{\partial}{\partial t}\int_{V}\left(\psi^{\star}\psi\right)\mathrm{d}V+\frac{i}{2m}\int_{V}\underline{\nabla}\cdot\left(\psi\underline{\nabla}\psi^{\star}-\psi\underline{\nabla}\psi\right)\mathrm{d}V &= 0 \qquad (4.5)
\end{aligned}$$

(6) resembles a conservation equation of the form:

$$\frac{\partial\rho}{\partial t}+\underline{\nabla}\cdot\underline{J}=0$$

where:

$$\begin{aligned}
\underline{J} &= \frac{i}{2m}\left(\psi\nabla\underline{\psi}^{\star}-\psi^{\star}\underline{\nabla}\psi\right)\\
\rho &= |\psi|^{2}
\end{aligned}$$

The divergence theorem then gives:

$$\frac{i}{2m}\int_{S}\left(\psi\underline{\nabla}\psi^{\star}-\psi^{\star}\underline{\nabla}\psi\right)\cdot\underline{\hat{n}}\mathrm{d}S=\frac{i}{2m}\int_{V}\left(\psi\underline{\nabla}\psi^{\star}-\psi^{\star}\underline{\nabla}\psi\right)\mathrm{d}V$$

As an example, consider ρ and $\underline{J}$ for a plane quantum wave:

$$\begin{aligned}
\psi &= N\mathrm{e}^{i(px-Et)}\\
\rho &= |\psi|^{2}\\
&= \psi^{\star}\psi\\
&= N\mathrm{e}^{i(px-Et)}N^{\star}\mathrm{e}^{-i(px-Et)}\\
&= NN^{\star}\\
&= |N|^{2}\\
\text{So: } \rho &= |N|^{2}\\
\underline{J} &= \frac{i}{2m}\left(\psi\underline{\nabla}\psi^{\star}-\psi\underline{\nabla}\psi\right)\\
&= \frac{i}{2m}\{N\mathrm{e}^{i(px-Et)}\underline{\nabla}N^{\star}\mathrm{e}^{-i(px-Et)}-N^{\star}\mathrm{e}^{-i(px-Et)}\underline{\nabla}N\mathrm{e}^{i(px-Et)}\}\\
&= \frac{i}{2m}\{NN^{\star}\mathrm{e}^{i(px-Et)}\mathrm{e}^{iEt}\left(-ip\right)\mathrm{e}^{-ipx}-N^{\star}\mathrm{e}^{-i(px-Et)}N\mathrm{e}^{-iEt}\left(ip\right)\mathrm{e}^{ipx}\}
\end{aligned}$$

$$\begin{aligned} &= \frac{i}{2m}\{NN^\star(-ip) - N^\star N(ip)\} \\ &= \frac{|N|^2}{2m}p \times 2 \\ &= \frac{p}{m}|N|^2 \end{aligned}$$

In the Schroedinger picture the operators are time independent whereas the wavefunctions are time dependent. In classical mechanics the "operators" (momentum and energy) are time dependent. The Heisenberg picture gives a formulation which gives time dependent operators.

4.2 The Heisenberg picture

Starting from the Schroedinger equation:

$$i\frac{\partial}{\partial t}\psi(\underline{r},t) = H\psi(\underline{r},t)$$

Also recall that the expectation of an observable, A, represented by the operator $\hat{A}$ is given by:

$$\langle\hat{A}\rangle = \int \psi^\star(\underline{r},t)A\psi(\underline{r},t)\mathrm{d}^3r$$

Solving the Schroedinger equation:

$$\begin{aligned} i\frac{\partial\psi(\underline{r},t)}{\partial t} &= H\psi(\underline{r},t) \\ i\frac{\partial\psi(\underline{r},t)}{\psi(\underline{r},t)} &= H\partial t \\ i\int_0^t \frac{\partial\psi(\underline{r},t)}{\psi(\underline{r},t)} &= \int_0^t H\partial t' \\ \Rightarrow \ln[\psi(\underline{r},t)] - \ln[\psi(\underline{r},0)] &= -iHt \\ \text{So } \frac{\psi(\underline{r},t)}{\psi(\underline{r},0)} &= \mathrm{e}^{-iHt} \\ \Rightarrow \psi(\underline{r},t) &= \psi(\underline{r},0)\mathrm{e}^{-iHt} \end{aligned}$$

Condsider the expectation of A:

$$\begin{aligned} \langle\hat{A}\rangle &= \int \psi^\star(\underline{r},0)\mathrm{e}^{iHt}\hat{A}\psi(\underline{r},0)\mathrm{e}^{-iHt}\mathrm{d}^3r \\ \text{Define } A_H &= \mathrm{e}^{iHt}\hat{A}\mathrm{e}^{-iHt} \\ \frac{\mathrm{d}A_H}{\mathrm{d}t} &= iH\mathrm{e}^{iHt}\bar{A}\mathrm{e}^{-iHt} - \mathrm{e}^{iHt}\hat{A}(iH)\mathrm{e}^{-iHt} \\ &= i(H\hat{A} - \bar{A}H) \\ &= i[H,\hat{A}] \end{aligned}$$

In the interaction picture the time dependent perturbations are given by:

$$H = H_0 + H'$$

where H_0 is unperturbed and H' is an interaction perturbation.

$$\text{Define } H_I^0 = \mathrm{e}^{iH_0 t} H' \mathrm{e}^{-H_0 t}$$

This gives:

$$\frac{\mathrm{d}H_I'}{\mathrm{d}t} = i[H_0, H']$$

4.3 The harmonic oscillator

This is a mechanical system which can be used to introduce the concept of creation annihilation operators.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

The potential energy arises because of a force proportonal to q:

$$\begin{aligned}
\text{Recall } F &= -kq \\
&= m\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \\
\omega &= \sqrt{\frac{k}{m}} \\
V &= -\int F \mathrm{d}q \\
&= \int kq \mathrm{d}q \\
&= \frac{kq^2}{2} \\
k &= m\omega^2 \\
\text{So } V &= \frac{m\omega^2 q^2}{2}
\end{aligned}$$

Consider the creation and annihilation operators:

$$\begin{aligned}
\text{Let } \hat{a} &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\hat{p}\right) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p}\right)
\end{aligned}$$

Consider the commutator given by $[\hat{q},\hat{p}] = i$:

$$\begin{aligned}
[\hat{a},\hat{a}^\dagger] &= \frac{1}{\sqrt{2}\sqrt{2}}\left(m\omega[\hat{q},\hat{q}] + \frac{i}{\sqrt{m\omega}}[\hat{p},\hat{q}]\sqrt{m\omega}\right. \\
&\quad \left. -i\sqrt{m\omega}[\hat{q},\hat{p}]\frac{1}{\sqrt{m\omega}} + \frac{1}{m\omega}[\hat{p},\hat{p}]\right) \\
&= \frac{1}{2}\left(i(-i) - i(i)\right) \\
&= 1 \\
\text{So } [\hat{a},\hat{a}^\dagger] &= 1
\end{aligned}$$

The Hamiltonian can be written as:

$$H = \frac{\left(\hat{a}^\dagger\hat{a} + \bar{a}\bar{a}^\dagger\right)}{2}\omega \tag{4.6}$$

To demonstrate (7):

$$\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p}\right)\frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\hat{p}\right) \\
&= \frac{1}{2}\left(m\omega\hat{q}^2 + \frac{\hat{p}^2}{m\omega} + 2i\left(\hat{q}\hat{p} - \hat{q}\hat{q}\right)\right) \\
\hat{a}\hat{a}^\dagger &= \frac{1}{2}\left(m\omega\hat{q}^2 + \frac{\hat{p}^2}{m\omega} + i\left(\hat{p}\hat{q} - \hat{q}\hat{p}\right)\right) \\
\text{So } \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger &= m\omega\hat{q}^2 + \frac{\hat{p}^2}{m\omega}
\end{aligned}$$

$$\begin{aligned}
\text{Recall } H &= \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \\
\text{then } H &= \left(\frac{p^2}{m\omega} + m\omega q^2\right)\frac{\omega}{2}
\end{aligned}$$

which leads to (7).
(7) may also be written as:

$$\begin{aligned}
H &= \left(\hat{a}^\dagger\hat{a} + 1/2\right)\omega \\
\text{using } [\hat{a},\hat{a}^\dagger] &= 1
\end{aligned}$$

$\hat{a}^\dagger$ can be considered as a creation operator:

$$[H,\hat{a}] = [\left(\hat{a}^\dagger\hat{a} + 1/2\right)\omega, \hat{a}]$$

$$\begin{aligned}
&= \left(\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger\hat{a}\right)\omega \\
&= [\hat{a}^\dagger, \hat{a}]\hat{a}\omega \\
&= -\hat{a}\omega \\
\text{Similarly } [H, \hat{a}^\dagger] &= \hat{a}^\dagger\omega
\end{aligned}$$

Consider a state $|n\rangle$ such that:

$$H|n\rangle = E_n|n\rangle$$

The energy of $\hat{a}^\dagger|n\rangle$ is given by:

$$\begin{aligned}
H\hat{a}^\dagger|n\rangle &= \left(\hat{a}^\dagger\omega + \hat{a}^\dagger H\right)|n\rangle \\
&= \left(\hat{a}^\dagger\omega + \hat{a}^\dagger E_n\right)|n\rangle \\
&= \left(E_n + \omega\right)\hat{a}^\dagger|n\rangle \\
\text{Similarly } H\hat{a}|n\rangle &= \left(E_n - \omega\right)\hat{a}|n\rangle
\end{aligned}$$

So the energy of the state $\hat{a}^\dagger|n\rangle$ is ω greater then that of state $|n\rangle$. Similarly, the energy of the state $\hat{a}|n\rangle$ is ω smaller than that of state $|n\rangle$.

There must be a state containing no quantum mechanical oscillations such that:

$$\hat{a}|0\rangle = 0$$

Applying the creation and annihilation operators to the ground state gives:

$$\begin{aligned}
\hat{a}^\dagger|0\rangle &= |1\rangle \\
\frac{\left(\hat{a}^\dagger\right)^2}{\sqrt{2}}|0\rangle &= |2\rangle \\
\text{Generally } |n\rangle &= \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle
\end{aligned}$$

Applying the Hamiltonian operator to the ground state gives:

$$\begin{aligned}
H|0\rangle &= \left(\hat{a}^\dagger\hat{a} + 1/2\right)\omega|0\rangle \\
&= \frac{\omega}{2}|0\rangle
\end{aligned}$$

Applying H to $\hat{a}^\dagger|0\rangle$ gives:

$$\begin{aligned}
H\hat{a}^\dagger|0\rangle &= \left(\hat{a}^\dagger\omega + \hat{a}^\dagger H\right)|0\rangle \\
&= \hat{a}^\dagger\left(\omega + H\right)|0\rangle \\
&= \hat{a}^\dagger\left(\omega + \frac{\omega}{2}\right)|0\rangle \\
&= \left(1 + 1/2\right)\omega\hat{a}^\dagger|0\rangle
\end{aligned}$$

Therefore the energy of the system is ω greater than that of the state $|0\rangle$. In general:

$$
\begin{aligned}
H_n \hat{a}^\dagger |0\rangle &= (n+1/2)\,\omega \hat{a}^\dagger |0\rangle \\
\text{But } H &= \left(\hat{a}^\dagger \hat{a} + 1/2\right)\omega \\
\Rightarrow \hat{a}^\dagger \hat{a} |0\rangle &= n|0\rangle = |n\rangle
\end{aligned}
$$

So $\hat{a}^\dagger \hat{a}$ is the number operator and the energy is given by $E_n = (n+1/2)\omega$. The number operator then acts as:

$$
\begin{aligned}
|n\rangle &= \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle \\
|n+1\rangle &= \frac{1}{\sqrt{(n+1)!}} \left(\hat{a}^\dagger\right)^{n+1} |0\rangle \\
&= \frac{1}{\sqrt{(n+1)!}} \hat{a}^\dagger \left(\hat{a}\right)^n |0\rangle \\
&= \frac{1}{\sqrt{(n+1)!}} \hat{a}^\dagger \sqrt{n!} |n\rangle \\
\Rightarrow \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\
\hat{a}|n\rangle &= \sqrt{n} |n-1\rangle
\end{aligned}
$$

It is now possible to conceive of a primitive field theory which can create or annihilate quanta of the appropriate field eg electromagnetism.

4.4 The anharmonic oscillator

This model will describe how to put interactions into a field theory. An interaction will move quanta from a state to higher or lower states.

For the anharmonic oscillator:

$$
\begin{aligned}
H &= \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^3 \\
&= H_0 + \lambda H'
\end{aligned}
$$

4.4.1 Rayleigh-Schroedinger perturbation theory

$$
\begin{aligned}
H &= H_0 + \lambda H' \\
\text{where } H_0 |n\rangle^0 &= E_n^0 |n\rangle^0 \\
E_n &= E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \cdots \\
\text{and } |n\rangle &= |n\rangle^0 + \lambda |n\rangle^1 + \lambda |n\rangle^2 + \cdots
\end{aligned}
$$

Note that $|n\rangle^1$, $|n\rangle^2$ etc are orthogonal to $|n\rangle^0$.

$$\begin{aligned}
\text{Since } H &= H_0 + \lambda H' : \\
H|n\rangle &= (H_0 + \lambda H_1)\left(|n\rangle^0 + \lambda|n\rangle^1 + \lambda^2|n\rangle^2\right) \\
&= E|n\rangle \\
&= \left(E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2\right)\left(|n\rangle^0 + \lambda|n\rangle^1 + \lambda^2|n\rangle^2\right)
\end{aligned}$$

$$\begin{array}{lcl}
\lambda^0 \text{ terms:} & H_0|n\rangle^0 & = E_n^0|n\rangle \\
\lambda^1 \text{ terms:} & H_0|n\rangle^1 - E_n^0|n\rangle^1 + H_1|n\rangle^0 - E_n^1|n\rangle^0 & = 0
\end{array}$$

$$\text{ie } \left(H_0 - E_n^0\right)|n\rangle^1 + \left(H^1 - E_n^1\right)|n\rangle^0 = 0$$

Premultiplying by ${}^0\langle n|$:

$$\begin{aligned}
{}^0\langle n|H_0 - E_n^0|n\rangle^1 +^0 \langle n|H^1 - E_n^1|n\rangle^0 &= 0 \\
\Rightarrow^0 \langle n|H^1 - E_n^0|n\rangle &= 0 \\
\Rightarrow \langle E_n^1\rangle &= {}^0\langle n|H^1|n\rangle^0
\end{aligned}$$

Premultiplying the expression for λ^1 terms by ${}^0\langle m|$ (orthogonal to $|n\rangle^0$):

$$\begin{aligned}
{}^0\langle m|H_0|n\rangle^1 -^0 \langle m|E_n^0|n\rangle^1 +^0 \langle m|H^1|n\rangle^0 -^0 \langle m|E_n^1|n|\rangle^0 &= 0 \\
\Rightarrow^0 \langle m|E_m^0 - E_n^1|n\rangle^1 +^0 \langle m|H^1|n\rangle^0 - {}^0\langle m|n\rangle^0 E_n^1 &= 0 \\
\text{So } {}^0\langle m|E_m^0 - E_n^0|n\rangle^1 +^0 \langle m|H^2|n\rangle^0 &= 0 \\
\text{So } {}^0\langle n\rangle^1 &= \frac{{}^0\langle m|H^1|n\rangle^0}{E_m^0 - E_n^0}
\end{aligned}$$

$$\begin{aligned}
\text{Therefore } |n\rangle &= |n\rangle^0 + \lambda|n\rangle^1 + \cdots \\
&= |n\rangle^0 + \lambda \sum_m |m\rangle^0 \frac{{}^0\langle m|H^1|n\rangle^0}{E_m^0 - E_n^0}
\end{aligned}$$

using the identity:

$$|n\rangle^1 = \sum_m |m\rangle^{00}\langle m|n\rangle^1$$

Calculating the value of λx^3:

$$
\begin{aligned}
\hat{a} &= \sqrt{\frac{m\omega}{2}}\hat{q} - \frac{i}{\sqrt{2m\omega}}\hat{p} \\
\hat{a}^\dagger &= \sqrt{\frac{m\omega}{2}}\hat{q} + \frac{i}{\sqrt{2m\omega}}\hat{p} \\
\left(\hat{a} + \hat{a}^\dagger\right) &= \sqrt{2m\omega}\hat{q} \\
\text{So } \hat{q} &= \frac{1}{\sqrt{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \\
\hat{q}^3 &= \frac{1}{(2m\omega)^{3/2}}\left(\hat{a} + \hat{a}^\dagger\right)^3
\end{aligned}
$$

So the term becomes:

$$\frac{\lambda}{(2m\omega)^{3/2}} \sum_m |m\rangle^0 \frac{{}^0\langle m| \left(\hat{a} + \hat{a}^\dagger\right)^3 |n\rangle^0}{E_m^0 - E_n^0}$$

Looking at ${}^0\langle m| \left(\hat{a} + \hat{a}^\dagger\right) |n\rangle^0$:

$$
\begin{aligned}
{}^0\langle m| \left(\hat{a} + \hat{a}^\dagger\right) |n\rangle^0 &= {}^0\langle m| \left(\hat{a} + \hat{a}^\dagger\right)^2 \left(\hat{a} + \hat{a}^\dagger\right) |n\rangle^0 \\
&= {}^0\langle m| \left(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}\right) \left(\hat{a} + \hat{a}^\dagger\right) |n\rangle^0
\end{aligned}
$$

using $[\bar{a}, \bar{a}^\dagger] = 1$ and $\bar{a}^\dagger\bar{a} = n$

$$
\begin{aligned}
\text{Also: } \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\
&= \sqrt{n}|n-1\rangle
\end{aligned}
$$

$$
\begin{aligned}
\lambda x^3 \quad =^0 \langle m|[\quad & \sqrt{n(n-1)(n-2)}|n-3\rangle^0 \\
& + \sqrt{(n+1)(n+2)(n+3)}|n+3\rangle^0 \\
& + (3+2n)\sqrt{n}|n-1\rangle^0 \\
& + (1+3n)\sqrt{n+1}|n+1\rangle^0]
\end{aligned}
$$

4.5 The Lagrangian

Physical systems evolve such that the action:

$$s = \int L(q, \dot{q})\mathrm{d}t$$

is extremised (usually minimised.) ie the system follows a path from t_1 to t_2 such that the integral of the Lagrangian over t is minimised.

This is an alternative formulation of classical mechanics eg Newton's law:

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv)$$

(Picture of action graph)

$$\begin{aligned}
\text{If } s &= \int_{t_1}^{t_2} L(q,\dot{q})\mathrm{d}t \\
\text{then } \delta s &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right)\mathrm{d}t \\
\text{Now } \delta \dot{q} &= \delta \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta q) \\
\text{So } \delta s &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q)\right]\mathrm{d}t
\end{aligned}$$

Integration by parts gives:

$$\begin{aligned}
\delta s &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\delta q\right]\mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}}\right]_{t_1}^{t_2} \\
&= 0 \\
\text{But } \delta q(t_2) = \delta q(t_1) &= 0 \\
\Rightarrow \frac{\partial L}{\partial q}\delta q - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\delta q &= 0 \\
\text{or } \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} &= 0 \text{ (The Euler-Lagrange equation)}
\end{aligned}$$

For the harmonic oscillator:

$$\begin{aligned}
L &= \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2} \\
\frac{\partial L}{\partial \dot{q}} &= m\dot{q} \\
\frac{\partial L}{\partial q} &= m\omega^2 q \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} & \\
\Rightarrow m\ddot{q} &= m\omega^2 q
\end{aligned}$$

giving Newton's second law.

In quantum mechanics, dynamical variables (eg momentum) are operators, and in general do not commute.

$$\text{Specifically} \quad [\hat{q}, \hat{p}] = i$$

where $\hat{p}$ is the generalisation:

$$\hat{p} = \frac{\partial L}{\partial \dot{q}}$$

The Heisenberg equation of motion for an operator $\hat{A}$ is:

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = i[H, \hat{A}]$$

The Hamiltonian, defined in terms of the Lagrangian is:

$$H = \hat{p} \cdot \hat{\dot{q}} - L$$

In the classical operator:

$$\begin{aligned}
L &= \frac{m\hat{\dot{q}}^2}{2} - \frac{m\omega^2\hat{q}^2}{2} \\
\frac{\partial L}{\partial \hat{\dot{q}}} &= \hat{p} \\
\text{So } H &= \hat{p}\hat{q} - L \\
&= \hat{p}\hat{q} - \frac{m\hat{\dot{q}}^2}{2} + \frac{m\omega^2\hat{q}^2}{2} \\
&= \hat{p}\frac{\hat{p}}{m} - \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 \\
\text{So } H &= \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2
\end{aligned}$$

4.5.1 The Dirac δ function

The Dirac δ function may be thought of as a function of height $1/\Delta x$ and width Δx around a value of $x = x_0$ in the limit $\Delta x \to 0$.

(Picture of delta function)

$$Area = \int \delta(x - x_0)\mathrm{d}x = 1$$

$$\delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ 1 & x = 0 \end{cases}$$

Consider the function $f(x)$ to be split into elements Δx wide:

$$\begin{aligned}\int f(x)\mathrm{d}x &= \sum_i f(x_i)\Delta x \\ \int f(x)\delta(x-x_0)\mathrm{d}x &= \sum_i f(x_i)\delta(x_i-x_0)\end{aligned}$$

For the bin $x_i \neq x_0$ it is clear that $\delta(x_i - x_0) = 0$, but when $x_i = x_0$, $\delta(x_i - x_0) = 1/\Delta x$.

$$\begin{aligned}\Rightarrow \int f(x)\delta(x-x_0)\mathrm{d}x &= \lim_{x\to 0} f(x_0)\frac{1}{\Delta x}\Delta x \\ \Rightarrow \int f(x)\delta(x-x_0)\mathrm{d}x &= f(x_0)\end{aligned}$$

Some useful expressions and functions limit to the Dirac δ function:

$$\begin{aligned}\delta(x) = &\quad \lim_{\epsilon\to 0}\frac{1}{\epsilon} &&\text{for } -\epsilon/2 < x < \epsilon/2 \\ \delta(x) = &\quad \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} &&\text{(Breit-Wigner resonance)}\end{aligned}$$

Consider the Breit-Wigner resonance:

$$\begin{aligned}I &= \int_{-\infty}^{\infty}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}\mathrm{d}x \\ x &= \epsilon\tan\theta \\ \mathrm{d}x &= \epsilon\sec^2\theta \\ I &= \frac{1}{\pi}\int_{\pi/2}^{\pi/2}\frac{\epsilon\epsilon\sec^2\theta}{\epsilon^2(1+\tan^2\theta)}\mathrm{d}\theta \\ &= 1\end{aligned}$$

So it appears that the Breit-Wigner function approaches the Dirac δ function in the appropriate limit. At $x = 0$ the function has a value $1/\pi\epsilon$. At $x = \pm\epsilon$ the function reaches the half maximum, $1/2\pi\epsilon$. $\Gamma/2 = \epsilon$ where Γ is the full width at half height and is more commonly used in the Breit-Wigner function.

Another form of the Dirac δ function is:

$$\delta(x-x_0) = \int_{-\infty}^{\infty}\frac{1}{2\pi}\mathrm{e}^{ik(x-x_0)}\mathrm{d}k$$

Consider Fourier analysis of the above form. Recall that for a well behaved function of x between $-L/2$ and $L/2$ the function can be expressed as a Fourier series with wavelengths $\lambda = L,\ \frac{L}{2},\ \frac{L}{3}\cdots$

$$f(x) = \sum_{x=-\infty}^{\infty} a_n \mathrm{e}^{i2\pi nx/L}$$

To find a_n integrate both sides:

$$\begin{aligned}
\int_{-L/2}^{L/2} \mathrm{d}x f(x)\mathrm{e}^{-12\pi nx/L} &= \int_{L/2}^{L/2} a_n \mathrm{d}x \\
\Rightarrow a_n &= \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x f(x)\mathrm{e}^{-12\pi nx/L} \\
\text{Consider} \lim L \to \infty & \\
f(x) &= \sum_{-\infty}^{\infty} a_n \mathrm{e}^{i2\pi nx/L}\Delta n
\end{aligned}$$

where Δn is the interval and $\Delta n = 1$

$$\begin{aligned}
\text{Define } k &= \frac{2\pi n}{L} \\
\mathrm{d}k &= 2\pi\frac{\Delta n}{L} \\
f(x) &= \sum_{-\infty}^{\infty} a_n \mathrm{e}^{ikx}\frac{L}{2\pi}\mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ikx} g(k)\mathrm{d}k \\
\text{where } g(k) &= La_n \\
g(k) &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ikx}\mathrm{d}x \\
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)\mathrm{e}^{ikx}\mathrm{d}k
\end{aligned}$$

Substituting $g(k)$ into the expression for $f(x)$:

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')\mathrm{e}^{-ikx'}\mathrm{e}^{ikx}\mathrm{d}k\mathrm{d}x' \\
&= \int_{-\infty}^{\infty} f(x')\mathrm{d}x' \int_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{ik(x-x')}\mathrm{d}k \\
\text{So } \delta(x-x') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ik(x-x')}\mathrm{d}k
\end{aligned}$$

Some properties of the Dirac δ function:

1. $\delta(ax) = \delta(x)/a$ Proof:

$$\begin{aligned}
\text{Let } ax &= y \\
\Rightarrow a\mathrm{d}x &= \mathrm{d}y \\
\text{So } \int_{-\infty}^{\infty} \delta(ax)\mathrm{d}x &= \int_{-\infty}^{\infty} \delta(y)\frac{\mathrm{d}y}{a} \\
&= \frac{1}{a}
\end{aligned}$$

2. $\delta(x) = \delta(-x) \Rightarrow \delta(x)$ is an even function.

3. $f(x)$ and $f'(x)$ For a function $f(x)$:

$$\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=a_i}}$$

where a_i satisfy $f(a_i) = 0$
At each place where $f(x) = 0$, then:

$$\begin{aligned}
f(x) &= f(a_i) + (x-a_i)\frac{\mathrm{d}f}{\mathrm{d}x} + \cdots \\
f(a_i) &= 0
\end{aligned}$$

So the δ function has non-zero contributions from each of the roots a_i of the form:

$$\begin{aligned}
\delta(f(x)) &= \sum_i \delta\left((x-a_i)\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=a_i}\right) \\
&= \sum_i \frac{\delta(x-a_i)}{\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=a_i}}
\end{aligned}$$

4.5.2 The Heaviside step function

$$\begin{aligned}
\theta(\tau) &= \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases} \\
\frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= \delta(\tau) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega\tau}\mathrm{d}\omega
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \theta &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}\infty e^{-i\omega\tau}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}}{-i\omega}d\omega \\
\theta &= \frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-\omega\tau}d\omega}{\omega+i\epsilon}
\end{aligned}
$$

Using Cauchy's theorem gives:

$$
\begin{aligned}
\oint_C \frac{f(z)dz}{z-a} &= 2\pi i f(a) \\
\text{So } \theta &= \frac{-1}{2\pi i}\left(-2ni e^{(-i\tau)(-i\epsilon)}\right) \\
&= e^{-\epsilon\tau} \\
\text{For } \epsilon \to 0, \theta &= 1
\end{aligned}
$$

Chapter 5

Relativity

An understanding of special relativity is necessary to properly understand much of particle physics. Four vectors are defined by:

$$\begin{array}{rclcrcl} x^\mu & = & (t, \underline{r}) & \qquad & x_\mu & = & (t, -\underline{r}) \\ p^\mu & = & (E, \underline{p}) & \qquad & p_\mu & = & (E, -\underline{p}) \end{array}$$

Scalar products are given by:

$$\begin{array}{rclclcl} \underline{x} \cdot \underline{x} & = & x^\mu x_\mu & = & t^2 - x^2 & & \\ \underline{p} \cdot \underline{p} & = & p^\mu p_\mu & = & E^2 - p^2 & = & m^2 \end{array}$$

Also $x_\mu = g_{\mu\nu} x^\nu$, $x^\nu = g^{\nu\mu} x_\mu$.
where:

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ g_{\mu\nu} g^{\mu\nu} &= \sum_\mu \sum_\nu g_{\mu\nu} g^{\mu\nu} \\ &= \sum_\mu g_{\mu\mu} g^{\mu\mu} \\ &= \sum_\mu g^2_{\mu\mu} \\ &= 4 \end{aligned}$$

Recall that from quantum mechanics:

$$E = i\frac{\partial}{\partial t} \qquad \underline{p} = -i\underline{\nabla}$$

$$
\begin{aligned}
\text{But } p^\mu &= (E, \underline{p}) \\
&= i\left(\frac{\partial}{\partial t}, -\underline{\nabla}\right) \\
\text{So } p^\mu &= i\partial^\mu \\
\text{where } \partial^\mu &= \left(\frac{\partial}{\partial t}, -\underline{\nabla}\right) \\
&= \frac{\partial}{\partial x_\mu} \\
\text{and } \partial_\mu &= \left(\frac{\partial}{\partial t}, \underline{\nabla}\right) \\
&= \frac{\partial}{\partial x^\mu}
\end{aligned}
$$

Forming the De L'ambertian operator:

$$\partial^\mu \partial_\mu = \Box^2 = \left(\frac{\partial^2}{\partial t^2}, -\nabla^2\right)$$

5.0.3 Lorentz transformations

A Lorentz transformation relates the coordinates in one frame to the coordinates of another frame. By convention the lorentz transformation takes place along the (mutually parallel) x axis, with velocity v. x' tranforms as:

$$
\begin{aligned}
t' &= \gamma(t - vx_1) \\
x_1' &= \gamma(-vt + x_1) \\
x_2' &= x_2 \\
x_3' &= x_3 \\
\gamma &= \frac{1}{\sqrt{1-v^2}}
\end{aligned}
$$

The product $p^\mu q_\mu$ is invariant under a Lorentz transformation.

5.0.4 The light cone

Let x^μ denote the four vector $(x^0, \underline{x})$ and suppose light is emitted at $y^\mu = (y^0, \underline{y})$. Consider the difference between x^μ and y^μ:

$$s^2 = (x^\mu - y^\mu)^2 = (x_0 - y_0)^2 - (\underline{x} - \underline{y})^2$$

If the above is zero then:

$$(x^0 - y^0) = (\underline{x} - \underline{y})^2$$

This is the equation of a light beam and defines a light cone. If $s^2 > 0$ then the separation is time-like and the events are in the forward light cone and causally related. If $s^2 < 0$ then the separation is space-like and the events have no causal connection.

(Picture of light cone)

5.0.5 Relativistic kinematics

Usually, one of two processes is considered, either $A \rightarrow B+C$ (decay) or $A+B \rightarrow C+D$ (scattering). For a decay the centre of mass energy is the mass of the particle A. For scattering:

$$s = (p_A^\mu + p_B^\mu)^2 = (p_c^\mu + p_D^\mu)^2$$

For a fixed target experiment B is at rest:

$$\begin{aligned}
s &= (E_A + E_B)^2 - (\underline{p}_A + \underline{p}_B)^2 \\
&= (E_A + m_B)^2 - P_A^2 \\
&= m_A^2 + m_B^2 + 2m_B E_A \\
\text{If } E_A &\gg m_A, m_B \text{ then:} \\
\sqrt{s} &\simeq \sqrt{2E_A m_B}
\end{aligned}$$

5.0.6 Centre of mass frame

$$\begin{aligned}
s &= (p_A^\mu + p_B^\mu)^2 \\
&= \left(E_A^{CMS} + E_B^{CMS}\right)^2 - \left(p_A^{CMS} + p_B^{CMS}\right)^2 \\
&= \left(E_A^{CMS}\right)^2 + 2E_A^{CMS} E_B^{CMS} + \left(E_B^{CMS}\right)^2 \\
\text{So } \sqrt{s} &= E_A^{CMS} + E_B^{CMS}
\end{aligned}$$

At Tevatron $E_P = E_{\bar{p}} = 980GeV$, so $\sqrt{s} \simeq 2TeV$.

At HERA $E_p = 920GeV$, $E_e = 27.5GeV$, so $\sqrt{s} = 318GeV$.

To transform between the centre of mass frame and the laboratory frame it is necessary to determine β and γ:

$$\begin{aligned}
\beta &= \frac{\text{Three momentum part of four vector}}{\text{Energy part of four momentum}} \\
&= \frac{|\underline{p}_A| + |\underline{p}_B|}{E_A + E_B} \\
\gamma &= \frac{\text{Energy part of four momentum}}{\text{Centre of mass energy}} \\
&= \frac{|\underline{p}_A| + |\underline{p}_B|}{\sqrt{s}}
\end{aligned}$$

The energy and momentum of a particle in the centre of mass frame can be determined using invariance rather than Lorentz transformations:

$$
\begin{aligned}
p_A^\mu + p_B^\mu &= \left(\sqrt{s}, 0\right) \text{ in the centre of mass frame} \\
p_{A\mu}\left(p_A^\mu + p_B^\mu\right) &= \left(E_A^{CMS}, \underline{p}_A^{CMS}\right) \cdot \left(\sqrt{s}, 0\right) \\
\Rightarrow p_{A\mu}\left(p_A^\mu + p_B^\mu\right) &= E_A^{CMS}\sqrt{s} \\
m_a^2 + p_{A\mu}p_B^\mu &= E_A^{CMS}\sqrt{s} \\
\text{Recall } s &= \left(p_A^\mu + p_B^\mu\right)\left(p_{A\mu} + p_{B\mu}\right) \\
&= p_A \cdot p_A + p_B \cdot p_B + 2p_A^\mu \cdot p_{B\mu} \\
\text{So } p_{A\mu}p_B^\mu &= \frac{s - m_A^2 - m_B^2}{2} \\
\text{So } E_A^{CMS} &= \frac{2m_A^2 + s - m_A^2 - m_B^2}{2\sqrt{s}} \\
\Rightarrow E_A^{CMS} &= \frac{s + m_A^2 - m_B^2}{2\sqrt{s}}
\end{aligned}
$$

And similarly for the other energies.

$$
\begin{aligned}
\left(p_A^{CMS}\right)^2 &= \left(E_A^{CMS}\right)^2 - m_A^2 \\
\text{So } \left(p_A^{CMS}\right)^2 &= \left(\frac{s + m_A^2 - m_B^2}{2\sqrt{s}}\right)^2 - m_A^2 \\
&= \frac{\left(s + m_A^2 - m_B^2\right)^2}{4s} - m_A^2 \\
&= \frac{s^2 + \left(m_A^2 - m_B^2\right)^2 + 2sm_A^2 - 2sm_B^2 - 4sm_A^2}{4s} \\
&= \frac{s^2 + \left(m_A^2 - m_B^2\right)^2 - 2s\left(m_A^2 + m_B^2\right)}{4s} \\
&= \frac{[s - (m_A + m_B)^2][s - (m_A - m_B)^2]}{4s}
\end{aligned}
$$

5.0.7 Mandelstan variables

$$
\begin{aligned}
s &= (p_A + p_B)^2 \\
t &= (p_A - p_C)^2 \\
&= (p_B - p_D)^2 \\
&= -Q^2 = q^2 \\
u &= (p_A - p_D)^2 \\
&= (p_C - p_B)^2 \\
s + t + u &= m_A^2 + m_B^2 + m_C^2 + m_D^2 \\
\text{Proof: } s + t + u &= (p_A + p_B)^2 + (p_A - p_C)^2 + (p_A - p_D)^2 \\
&= p_A^2 + P_B^2 + 2p_B p_A + p_A^2 + p_C^2 - 2p_A p_C + p_A^2 + p_D^2 - 2p_A p_D \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2\left(p_B p_A - p_A p_C - p_A p_B\right) \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2p_A\left(p_B - p_C - p_D\right) \\
\text{But } p_A + p_B &= p_C + p_D \\
\Rightarrow p_B - p_C - p_D &= -p_A \\
\text{So } s + t + u &= m_A^2 + m_B^2 + m_C^2 + m_D^2
\end{aligned}
$$

5.1 Relativistic spin−0 particles

5.1.1 The Klein-Gordon equation

The quantum wavefunction for a free scalar particle propagating in the $x-$direction is:

$$
\begin{aligned}
\phi &\sim \mathrm{e}^{i(px - Et)} \\
&\sim \mathrm{e}^{ip^\mu x_\mu}
\end{aligned}
$$

Repeating the procedure that yielded the Schroedinger equation, but using:

$$E^2 = p^2 + m^2 \text{ (rather than } E = p^2/2m) \tag{5.1}$$

$$\left.\begin{array}{ccc} E & \rightarrow & i\frac{\partial}{\partial t} \\ \underline{p} & \rightarrow & -i\underline{\nabla} \end{array}\right\} \tag{5.2}$$

Combining (8) and (9) and transforming in an equation acting on ϕ gives:

$$\begin{aligned} -\frac{\partial^2\phi}{\partial t^2} &= -\nabla^2\phi + m^2\phi \\ \text{Or } \Box^2\phi &= m^2\phi \end{aligned} \tag{5.3}$$

(10) is the Klein-Gordon equation (or the relativistic Schroedinger equation. The complex conjugate of (10) is:

$$-\frac{\partial^2\phi^\star}{\partial t^2} = -\nabla^2\phi^\star + m^2\phi^\star \tag{5.4}$$

$$\phi^\star \times (10) = -\phi^\star\frac{\partial^2\phi}{\partial t^2} = -\phi^\star\nabla^2\phi + m^2\phi^\star\phi \tag{5.5}$$

$$\phi \times (11) = -\phi\frac{\partial^2\phi^\star}{\partial t^2} = -\phi\nabla^2\phi^\star + m^2\phi\phi^\star \tag{5.6}$$

$$-i\left((12) - (13)\right) = i\left(\phi^\star\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi}{\partial t^2}\right) = i\left(\phi^\star\nabla^2\phi - \phi\nabla^2\phi^\star\right)$$

$$\begin{aligned} \Rightarrow i\frac{\partial}{\partial t}\left[\phi^\star\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^\star}{\partial t}\right] - i\underline{\nabla}[\phi^\star\underline{\nabla}\phi - \phi\underline{\nabla}\phi_\star] &= 0 \\ \text{or } \frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{j} &= 0 \end{aligned}$$

$$\begin{aligned} \rho &= i\left(\phi^\star\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^\star}{\partial t}\right) \\ \underline{J} &= i\left(\phi^\star\underline{\nabla}\phi - \phi\underline{\nabla}\phi^\star\right) \end{aligned}$$

Consider the form of ρ for $\phi = N\mathrm{e}^{i\underline{p}\cdot\underline{x}}$:

$$\begin{aligned} \phi &= N\mathrm{e}^{i(px-Et)} \\ \rho &= i\left(\phi^\star\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^\star}{\partial t}\right) \\ &= i\left(N^\star\mathrm{e}^{-i(px-Et)}\left(-iE\right)N\mathrm{e}^{i(px-Et)} - N\mathrm{e}^{i(px-Et)}\left(iE\right)N^\star\mathrm{e}^{-1(px-Et)}\right) \\ &= i[N^\star N\left(-iE\right) - NN^\star\left(ie\right)] \\ &= 2N^\star NE \\ \text{So } \rho &= 2EN^\star N \end{aligned}$$

Similarly $\underline{j} = 2N^\star N\underline{p}$.

This appeared disastrous because $E^2 = p^2 + m^2$ lead to negative energy solutions (ie $E_- = -\sqrt{p^2+m^2}$) which lead nagative probability densities, $\rho < 0$. Note that ρ, being proportional to E may have been antiticipated. Under a Lorentz boost of speed v the volume element undergoes a contraction:

$$\mathrm{d}^3 r \rightarrow \gamma \mathrm{d}^3 r$$

Therefore to keep $\rho \mathrm{d}^3 r$ invariant ρ must transform as a time-like component:

$$\rho \rightarrow \gamma \rho$$

5.1.2 The perceived distaster of the Klein-Gordon equation

The problem faced is that the Klein-Gordon equation can give $\rho < 0$. There are two steps to remove this problem which works for scalar particles. In 1934 Pauli and Weisskopf derived the Klein-Gordon equiation by multiplying $\underline{j} = \left(\rho, \underline{j}\right)$ by the charge of the particle, so $q\underline{j}^\mu$ becomes $\underline{j}^\mu_{em}$:

$$\underline{j}^\mu_{em} = -i\mathrm{e}\left(\phi^\star \partial^\mu \phi - \phi \partial^\mu \phi^\star\right)$$

where $-$e is the charge on the scalar electron. Now $\rho = j^0_{em}$ is a charge densiity and not a probability density, so ρ can be negative.

5.1.3 The Feynmann-Stueckelberg interpretation of $E < 0$ solutions

The approach taken by Feynmann and Stueckelberg is that the negative energy solutions describe a negative energy particle propagating backwards in time or equivalently, a positive energy antiparticle propagating forwards in time. Consider an electron of energy E, momentum $\underline{p}$ and charge $-$e:

$$\underline{j}_{em}(\mathrm{e}^-) = -2eN^\star N(E, \underline{p})$$

For a positron the charge is e:

$$\underline{j}_{em}(\mathrm{e}^+) = 2\mathrm{e}N^\star N(E, \underline{p})$$

which is equivalent to:

$$\underline{j}_{em}(\mathrm{e}^-) = -2\mathrm{e}N^\star N(-E, -\underline{p})$$

which is the same as $\underline{j}_{em}$ but with $(-E, -\underline{p})$ so the emission of a positron of energy E is the same as the absorption of an electron with energy $-E$. These ideas can describe many particle interactions. Consider double scattering in an interaction volume:

At time t_1 and position $\underline{r}_1$ the electron scatters at I and then at a later time t_2 and position $\underline{r}_2$ scatters at II. All particles go forwards in time.

(Second diagram of double scattering process)

At the earlier time t_1 and position $\underline{r}_1$ an electron-positron pair is created. The electron leaves the volume and the positron propagates forwards in time to t_2 and $\underline{r}_1$ where it annihilates with an electron.

Both of the above diagrams have the same initial and final state, but the first involves just one electron while the second involves three particles. They would both have to be included when calculating the probability of that process occuring.

Chapter 6

Calculating amplitudes

6.1 Possible approaches

- Make the Born approximation relativistic, which is the Feynmass-Stueckelberg approach. This is not so easy to make systematic and is rather ad hoc. This method motivates the propagator.
- Make non-relativistic perturbation theory relativistic, as in Halzen and Martin. This is not systematic and does not motivate the propagator.
- Canonical field theory enables a more systematic approach, but it takes more time.
- The path integral approach is systematic but mathematically more challenging.

6.2 Propagator approach

The basic idea behind the propagator approach is to know the quantum wave (which is called $\psi(\underline{r}', t'))$ given the wavefunction at initial coordinates $\psi(\underline{r}, t)$.

$$\psi(\underline{r}', t') = i \int G(\underline{r}', t', \underline{r}, t)\psi(\underline{r}, t)\mathrm{d}^3 r \text{ for } t' > t$$

where G is a Green's function.

The wave at $\underline{r}$ has been propagated by G to $\underline{r}'$. Consdier the scattering process. An incident particle described by the plane quantum wave $\phi(\underline{r}, t)$ is incident on a potential $V(\underline{r}, t)$. Schroedinger's equation should describe what happens. Recall:

$$\begin{aligned} (H_0 + V)\,\psi &= i\frac{\partial\psi}{\partial t} \\ i\partial\psi(\underline{r}, t) - H_0\psi(\underline{r}, t)\mathrm{d}t &= V(\underline{r}, t)\psi(\underline{r}, t)\mathrm{d}t \end{aligned}$$

Suppose the potential acts at $\underline{r}_1$ and t_1 for a short time interval Δt_1:

$$\begin{aligned}\Rightarrow i\int \partial\psi(\underline{r}_1,t_1) - \int_{t_1}^{t_1+\Delta t_1} H_0\psi(\underline{r},t_1)\mathrm{d}t_1 &= \int_{t_1}^{t_1+\Delta t_1} V(\underline{r}_1,t_1)\psi(\underline{r}_1,t_1)\mathrm{d}t_1 \\ \Delta\psi(\underline{r}_1,t_1) &= -iV(\underline{r}_1,t_1)\psi(\underline{r}_1,t_1)\Delta t_1\end{aligned}$$

where H_0 does not contribute significantly.

$$\begin{aligned}\Delta\psi(\underline{r}_1,t_1) &\sim -iV(\underline{r}_1,t_1)\phi(\underline{r}_1,t_1)\Delta t_1 \\ \text{where } \psi(\underline{r}_1,t_1) &= \phi(\underline{r}_1,t_1)+\Delta\psi(\underline{r}_1,t_1) \\ \Delta\psi(\underline{r}',t') &= i\int \mathrm{d}^3r_1 G_0(\underline{r}',t';\underline{r}_1,t_1)\Delta\psi(\underline{r}_1,t_1) \\ &= i\int \mathrm{d}^3r_1 G_0(\underline{r}',t';\underline{r}_1,t_1)(-i)V(\underline{r}_1,t_1)\phi(\underline{r}_1,t_1)\Delta t_1 \\ \text{ie } \psi(\underline{r}',t') &= \phi(\underline{r}',t') + \int \mathrm{d}^3r_1 G_0(\underline{r}',t';\underline{r}_1,t_1)V(\underline{r}_1,t_1)\phi(\underline{r}_1,t_1)\Delta t_1 \\ \psi(\underline{r}',t') &= \phi(\underline{r}',t') + \int \mathrm{d}^4x_1 G_0(x';1)V(1)\phi(1) \\ \text{where } V(1) &= V(\underline{r}_1,t_1) \text{ etc and } x' \text{ is now a 4 vector.}\end{aligned}$$

Now applying a potential at $(\underline{r}_2,t_2)$ for Δt_2 generates the wavefunction:

$$\begin{aligned}\psi(\underline{r}',t') = {} & \phi(\underline{r}',t') + \int \mathrm{d}^3r_1 G_0(x';1)V(1)\phi(1)\Delta t_1 \\ & + \int \mathrm{d}^3r_2 G_0(x';2)V(2)\phi(2)\Delta t_2 \\ & + \int\int \mathrm{d}^3r_1\mathrm{d}^3r_2 G_0(x';2)V(2)G_0(2;1)V(1)\phi(1)\Delta t_1\Delta t_2\end{aligned}$$

Integrating over Δt_1, Δt_2 gives:

$$\psi(\underline{r}',t') = \phi(\underline{r}',t') + \int \mathrm{d}^4x_1 G_0(x';1)V(1)\phi(1) + \int\int \mathrm{d}^4x_1\mathrm{d}^4x_2 G_0(x';2)V(2)G_0(2;1)V(1)\phi(1)$$

Now it is necessary to evaluate G_0. In particular $G_0(2;1)$ ie the G_0 for the intermediate states:

$$\psi(\underline{r}',t') = i\int_{t'>t} \mathrm{d}^3r G(x';x)\psi(\underline{r},t)$$

This can be written in a form valid for all times (using the Heaviside step function centred at $\tau = t'$):

$$\theta(t'-t)\psi(x') = i\int \mathrm{d}^3 r G(x';x)\psi(x)$$

$$\theta(t'-t) = \begin{cases} 1 & t' > t \\ 0 & \text{otherwise} \end{cases}$$

Applying the Schroedinger equation to both sides:

$$\begin{aligned}
LHS &: \left[i\frac{\partial}{\partial t} - H(x')\right]\theta(t'-t)\psi(x') \\
&= i\delta(t'-t)\psi(x') + \theta(t'-t)i\frac{\delta}{\delta t}\psi(x') - H(x')\theta(t'-t)\psi(x') \\
&= i\delta(t'-t)\psi(x') \\
RHS &: i\int \mathrm{d}^3 r\left[i\frac{\partial}{\partial t'} - H(x')\right]G(x';x)\psi(x)
\end{aligned}$$

Consider a particle in the absence of a potential ie $V = 0$, then solve explicitly for the free particle propagator.

$$RHS : i\int \mathrm{d}^3 r\left[E - \frac{p^2}{2m}\right]G_0(x';x)\psi(x)$$

Transforming to four-momentum space via a Fourier transform:

$$\begin{aligned}
RHS &: i\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\mathrm{d}E}{2\pi}\left(E - \frac{p^2}{2m}\right)G_0(E;\underline{p})\mathrm{e}^{i\underline{p}(\underline{r}'-\underline{x})}\mathrm{e}^{-iE(t'-t)}\psi(x) \\
LHS &: i\delta(t'-t)\psi(x') \\
&= i\delta(t'-t)\int \mathrm{d}^3 r\psi(x)\delta^3(\underline{r}'-\underline{r}) \\
&= i\delta^4(x'-x)\psi(x) \\
LHS &= RHS \text{ so:} \\
i\delta^4(x'-x)\psi(x) &= i\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\mathrm{d}E}{2\pi}\left(E - \frac{p^2}{2m}\right)G_0(E;p)\mathrm{e}^{ip(x'-x)}\mathrm{e}^{-E(t'-t)}\psi(x) \\
\Rightarrow \delta^4(x'-x)\psi(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{ip(x'-x)}\mathrm{e}^{-iE(t'-t)}\psi(x) \\
\text{where } G_0 &= \frac{1}{E - \frac{p^2}{2m}} \text{ for } E \neq \frac{p^2}{2m}
\end{aligned}$$

As $E = p^2/2m$ the value of G_0 becomes undefined. The simple model cannot account for the singularity, so a new term in introduced:

$$G_0 \to G_0 = \frac{1}{E - \frac{p^2}{2m} + i\epsilon}$$

The free particle propagator for real or virtual particle in momentum space is the inverse of the free particle Schroedinger equation. Assume that in momentum space the free propagators of the Klein-Gordon equation, Dirac equation and Proca equation are all obtained by inverting the appropriate equation.

Chapter 7

Spinless $e^-\mu^-$ scattering

7.1 Electrodynamics of spinless particles

(Picture of scattering process)

Consider spinless electrons scattering off spinless muons. Starting with the Klein-Gordon equation and including the electromagnetic interaction will lead to a simple model of scattering. In electrodynamics the motion of a particle of charge $-\mathrm{e}$ in an electromagnetic potential $A^\mu (= (A^0, \underline{A}))$ is obtained by the substitution:

$$\begin{aligned} p^\mu &\rightarrow p^\mu + \mathrm{e}A^\mu \\ \text{So } p_\mu p^\mu = m^2 &\rightarrow (p_\mu + \mathrm{e}A_\mu)(p^\mu + \mathrm{e}A^\mu) = m^2 \end{aligned}$$

In quantum mechanics this is:

$$\begin{aligned} (i\partial_\mu + \mathrm{e}A_\mu)(i\partial^\mu + \mathrm{e}A^\mu)\phi &= m^2\phi \\ -\partial_\mu\partial^\mu\phi + i\mathrm{e}\partial_\mu A^\mu\phi + i\mathrm{e}A_\mu\partial^\mu\phi + \mathrm{e}^2 A_\mu A^\mu\phi &= m^2\phi \\ \Rightarrow (\partial_\mu\partial^\mu + m^2)\phi &= i\mathrm{e}\partial_\mu A^\mu\phi + i\mathrm{e}A_\mu\partial^\mu\phi + \mathrm{e}^2 A_\mu A^\mu\phi \\ &= -V\phi \end{aligned}$$

where V is the electromagnetic perturbation and the minus sign is required to that there is the same relateive sign to the $p^2/2m$ term in the Schroedinger equation.

The transition amplitude is:

$$\begin{aligned} T_{fi} &= -i\int \mathrm{d}^4x\,\phi_f^\star(x)V(x)\phi_i(x) \\ &= -i\int \mathrm{d}^4x\,\phi_f^\star(x)(-i\mathrm{e})(\partial_\mu A^\mu + A_\mu\partial^\mu)\phi_i(x) \end{aligned}$$

where the second order term is neglected.

Transforming the first term so that the derivative acts on $\phi_f^\star$:

$$\begin{aligned}
T_{fi} &= \int \phi_f^\star \partial_\mu \left(A^\mu \phi_2\right) \mathrm{d}^4 x + \cdots \\
I &= \int \phi_f^\star \partial_\mu \left(A^\mu \phi_2\right) \mathrm{d}^4 x \\
\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x &= uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x \\
\text{So let } u &= \phi_f^\star \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \partial_\mu \left(A^\mu \phi_i\right) \\
\text{So } I &= \left[\phi_f^\star A^\mu \phi_i\right]_{-\infty}^{\infty} - \int \mathrm{d}^4 x \left(\partial_\mu \phi_f^\star\right) A^\mu \phi_i \\
\text{So } T_{fi} &= -\mathrm{e} \int \mathrm{d}^4 x \left(\phi_f^\star A_\mu \partial^\mu \phi_i - \partial_\mu \phi_f^\star A^\mu \phi_i\right) \\
&= -i \int \mathrm{d}^4 x A^\mu j_\mu^{fi} \\
\text{where } j_\mu^{fi} &= -ie \left(\phi_f^\star \left(\partial_\mu \phi_i\right) - \left(\partial_\mu \phi_f^\star\right) \phi_i\right)
\end{aligned}$$

At the top vertex the particle A is described by:

$$\phi_A(x) = N_A \mathrm{e}^{-ip_A x}$$

Similarly, particle C is described by:

$$\begin{aligned}
\phi_C(x) &= N_C \mathrm{e}^{-ip_C x} \\
\Rightarrow j_\mu^{CA}(x) &= -i\mathrm{e}\Big[N_C^\star \mathrm{e}^{ip_C x} N_A \mathrm{e}^{-ip_A x} \left(-ip_A\right)_\mu - N_C^\star \mathrm{e}^{ip_C x} \left(ip_C\right)_\mu N_A \mathrm{e}^{-ip_A x}\Big] \\
&= -i\mathrm{e} N_C^\star N_A \mathrm{e}^{i(p_C - p_A)x} \left(-ip_A - ip_C\right)_\mu \\
&= -\mathrm{e} N_C^\star N_A \left(p_A + p_C\right)_\mu \mathrm{e}^{i(p_C - p_A)x}
\end{aligned}$$

and similarly for j_μ^{BD}

For A^μ, using Maxwell's equations:

$$\Box^2 A^\mu(x) = j^\mu(x) \tag{7.1}$$

The solution is found by inspection and the field A^μ arises because of the field from the current at the lower vertex:

$$j_{DB}^\mu = -\mathrm{e} N_D^\star N_B \left(p_B + p_D\right)^\mu \mathrm{e}^{i(p_D - p_B)x}$$

By inspection the substitution

$$A^\mu = \frac{-g^{\mu\nu} j_\nu^{DB}}{q^2}$$

satisfies (14) where $q = p_D - p_B$.
Substituting this into (14) gives:

$$\begin{aligned}
\partial_\mu \partial^\mu \left(\frac{-1}{q^2}\right) j^\mu_{DB} &= \frac{-1}{q^2} j^\mu_{DB} \Big[- i\,(p_D - p_B)\Big]^2 \\
&= \frac{1}{q^2} j^\mu_{DB} q^2
\end{aligned}$$

So the expression for A^μ satisfies (14).

$$\begin{aligned}
T_{fi} &= -i \int \mathrm{d}^4 x j_\mu^{CA}(x) A^\mu \\
&= -i \int \mathrm{d}^4 x \,(-\mathrm{e})\, N_C^\star N_A \,(p_A + p_C)_\mu\, \mathrm{e}^{i(p_C - p_A)x} \left(\frac{-1}{q^2}\right) (-\mathrm{e})\, N_D^\star N_B\, (p_B + p_D)^\mu\, \mathrm{e}^{i(p_D - p_B)x} \\
&= \frac{i\mathrm{e}^2}{q^2} \int N_C^\star N_A N_D^\star N_B\, (p_A + p_C)_\mu\, (p_D + p_B)^\mu\, \mathrm{e}^{i(p_C + p_D - p_A - p_B)x} \mathrm{d}^4 x
\end{aligned}$$

7.2 Definition of the cross section

The cross section is imagined to take place in an intercation volume V and the normalisation is such that there are $2E$ particles of each kind (A, B, C, D) in this volume.

$$\begin{aligned}
\text{If } \phi_A &= N_A \mathrm{e}^{ip_A x} \\
\text{then } \int \rho \mathrm{d}^3 r &= \int 2E \phi_A^\star \phi_A \mathrm{d}^3 r \\
&= 2E N_A^\star N_A V \\
&= 2E \\
\text{where } N_A &= \frac{1}{\sqrt{V}}
\end{aligned}$$

T_{fi} is the amplitude of transmission from an initial state i to a final state f. The number of transitions per unit time per unit volume, W_{fi} is given by:

$$W_{fi} = \frac{T_{fi} T_{fi}^\star}{\text{Unit time and volume}}$$

The cross section is then given by:

$$
\begin{aligned}
\sigma &= W_{fi}\left(\frac{\text{number of final states}}{\text{initial flux}}\right)\\
T_{fi}T^{\star}_{fi} &= \frac{\mathrm{e}^4}{q^4}\int\int\frac{1}{V^4}\Big[\left(p_A+p_C\right)_\mu\left(p_B+p_D\right)^\mu\Big]^2 \mathrm{e}^{i(p_C+p_D-p_A-p_B)x}\mathrm{e}^{-i(p_C+p_D-p_A-p_B)x'}\mathrm{d}^4x\mathrm{d}^4x'\\
W_{fi} &= \frac{\mathrm{e}^4}{q^4V^4}\Big[\left(p_A+p_C\right)_\mu\left(p_B+p_D\right)^\mu\Big]^2\delta^4\left(p_C+p_D-p_A-p_B\right)\frac{2\pi^4TV}{TV}\\
&= \frac{\mathrm{e}^4}{q^4V^4}\Big[\left(p_A+p_C\right)_\mu\left(p_B+p_D\right)^\mu\Big]^2\delta^4\left(p_C+p_D-p_A-p_B\right)2\pi^4\\
&= \frac{\mathrm{e}^4}{q^4}\frac{2\pi^4}{V^4}\Big[\left(p_A+p_C\right)_\mu\left(p_B+p_D\right)^\mu\Big]^2\delta^4\left(p_C+p_D-p_A-p_B\right)
\end{aligned}
$$

Consider the number of final states. Each particle in the final state has a three momentum between $\underline{p}_C$ and $\underline{p}_C+\mathrm{d}^3\underline{p}_C$, and $\underline{p}_D$ and $\underline{p}_D+\mathrm{d}^3\underline{p}_D$ and energies between E_C and $E_C+\mathrm{d}E_C$, and E_D and $E_D+\mathrm{d}E_D$. The number of final states per unit volume is determined by constraints imposed by the δ function. These particles are imagined to be travelling in waves, which enter and leave the interaction volume, V. The propagator particle only exists within the interaction volume, so it must have the wavefunction of a particle in an infinite square well, leading to quantisation of the transfer momentum.

Suppose the entrance is at $x=0$ and the exit is as $x=L_x$, then:

$$p_xL_x = 2\pi n_x \quad n_x\in N$$

and similarly for y and z.

The momentum separation, Δp_x is given by:

$$
\begin{aligned}
\left(p_x+\Delta p_x\right)L_x &= 2\pi\left(n_x+1\right)\\
\Rightarrow \Delta p_x &= \frac{2\pi}{L_x}
\end{aligned}
$$

Therefore the number of final states is:

$$
\begin{aligned}
N_f &= \frac{\mathrm{d}p_x}{2\pi}\frac{\mathrm{d}p_y}{2\pi}\frac{\mathrm{d}p_z}{2\pi}L_xL_yL_z\\
&= \frac{V}{\left(2\pi\right)^3}\mathrm{d}^3p
\end{aligned}
$$

There are $2E$ particles in V, so the normalised number of states is:

$$N_f = \frac{V}{\left(2\pi\right)^3 2E}\mathrm{d}^3p$$

Summing over states in C and D:

$$N_f^{CD} = \frac{V\mathrm{d}^3 p_C}{(2\pi)^3 E_C} \frac{V\mathrm{d}^3 p_D}{(2\pi)^2 E_D}$$

The flux factor is $\rho_A \rho_B u_{AB}$ where ρ_i is the desnity of particles in V and u_{AB} is the relative velocity of particles A and B.

$$\begin{aligned}
u_{AB} &= u_A - u_B \\
\Rightarrow flux &= \frac{2E_A}{V}\frac{2E_B}{V} u_{AB} \\
&= \frac{2E_A}{V}\frac{2E_B}{V}\left(\frac{p_A}{E_A} - \frac{P_B}{E_B}\right) \\
&= \frac{4E_A E_B}{V^2}\left(\frac{p_A E_B - E_A p_B}{E_A E_B}\right) \\
&= \frac{4}{V^2}\left(p_A E_B - p_B E_A\right)
\end{aligned}$$

In the centre of mass system $p_A = p_B$, so:

$$\begin{aligned}
flux &= \frac{4}{V^2} p_A \left(E_A + E_B\right) \\
&= \frac{4}{V^2} p_A \sqrt{s}
\end{aligned}$$

Combining all the terms into the cross section formula gives:

$$\begin{aligned}
\mathrm{d}\sigma &= \frac{\mathrm{e}^4}{q^4}\frac{(2\pi)^4}{V^4}\left[\left(p_A + p_C\right)_\mu \left(p_B + p_D\right)^\mu\right]^2 \delta^4\left(p_C + p_D - p_A - p_B\right) \\
&\quad \times \frac{V\mathrm{d}^3 p_C}{2E_C}\frac{1}{(2\pi)^3}\frac{V\mathrm{d}^3 p_D}{2E_D}\frac{1}{(2\pi)^3}\frac{V^2}{4p_A\sqrt{s}}
\end{aligned}$$

The Lorentz invariant phase space, $\mathrm{d}Q$ is given by:

$$\mathrm{d}Q = \frac{V}{2E_C}\frac{\mathrm{d}^3 p_C}{(2\pi)^3}\frac{V}{2E_D}\frac{\mathrm{d}^3 p_D}{(2\pi)^3}(2\pi)^4 \delta\left(p_C + P_D - p_A - p_B\right)$$

In the centre of mass system:

$$\mathrm{d}Q = (2\pi)^4 \delta\left(\sqrt{s} - (E_C + E_D)\right) \delta^3\left(p_C + p_D\right) \frac{\mathrm{d}^3 p_C}{2E_C}\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3 p_D}{2E_D}\frac{1}{(2\pi)^3} V^2$$

The degree of the differential can be reduced by integrating over $\mathrm{d}^3 p_D$:

$$\mathrm{d}Q = (2\pi)^4 \delta\left(\sqrt{s} - (E_C + E_D)\right) \frac{\mathrm{d}^3 p_C}{2E\,(2\pi)^3}\frac{1}{2E_D\,(2\pi)^3} V^2$$

The integration over $\mathrm{d}^3 p_C$ is given by:

$$\begin{aligned}
\int \mathrm{d}^3 p_C &= \int 2\pi p_C \sin\theta p_C \mathrm{d}\theta \mathrm{d}p_C \\
&= \int p_C^2 \mathrm{d}p_C \left(2\pi \sin\theta \mathrm{d}\theta\right) \\
&= \int p_C^2 \mathrm{d}p_C \mathrm{d}\Omega \\
\text{so } \mathrm{d}Q &= \frac{V^2}{(4\pi)^2}\delta\left(\sqrt{s}-(E_C+E_D)\right)\frac{1}{4E_C E_D} p_C^2 \mathrm{d}p_C \mathrm{d}\Omega
\end{aligned}$$

In the centre of mass system:

$$\begin{aligned}
\sqrt{s} &= (E_C+E_D) \\
&= \sqrt{p_C^2+m_C^2}+\sqrt{p_D^2+m_D^2} \\
&= \sqrt{p_C^2+m_C^2}+\sqrt{p_C^2+m_D^2} \\
\Rightarrow \mathrm{d}\sqrt{s} &= \frac{2p_C \mathrm{d}p_C}{2\sqrt{p_C^2+m_C^2}}+\frac{2p_C \mathrm{d}p_C}{2\sqrt{p_C^2+m_D^2}} \\
&= p_C \mathrm{d}p_C\left(\frac{1}{\sqrt{p_C^2+m_C^2}}+\frac{1}{\sqrt{p_D^2+m_D^2}}\right) \\
&= p_C \mathrm{d}p_C\left(\frac{1}{E_C}+\frac{1}{E_D}\right) \\
&= p_C\left(\frac{E_D+E_C}{E_D E_C}\right)\mathrm{d}p_C \\
&= \frac{p_C\sqrt{s}\mathrm{d}p_C}{E_C E_D} \\
\Rightarrow \mathrm{d}Q &= \frac{V^2}{4\pi^2}\delta(\sqrt{s}-(E_C+E_D))\frac{1}{4}\mathrm{d}\Omega\frac{1}{E_C E_D}p_C^2 \mathrm{d}p_C \\
&= \frac{V^2}{4\pi^2}\delta\left(\sqrt{s}-(E_C+E_D)\right)\frac{1}{4}\mathrm{d}\Omega\frac{p_C}{\sqrt{s}}\sqrt{s} \\
&= \frac{V^2}{16\pi^2}\mathrm{d}\Omega\frac{\mathrm{d}p_C}{\sqrt{s}} \\
\Rightarrow \mathrm{d}\sigma &= \frac{\mathrm{e}^4}{q^4}\frac{1}{V^4}\left[(p_A+p_C)_\mu(p_B+p_D)^\mu\right]^2 \mathrm{d}Q\frac{V^2}{4p_A\sqrt{s}} \\
&= \frac{\mathrm{e}^4}{q^4}\frac{1}{V^4}\left[(p_A+p_C)_\mu(p_B+p_D)^\mu\right]\frac{V^2}{4p_A\sqrt{s}}\frac{V^2}{16\pi^2}\mathrm{d}\Omega\frac{p_C}{\sqrt{s}} \\
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\mathrm{e}^4}{q^4}\frac{1}{64\pi^2}\frac{\left[(p_A+p_C)_\mu(p_B+p_D)^\mu\right]^2 p_C}{p_A s}
\end{aligned}$$

The number of final states divided by the flux factor in two body processes is used in many calculations and is:

$$\frac{1}{64\pi^2 s}\frac{p_C}{p_A}$$

Consider the cross-section for the massless limit, $m_i \to 0$ or $E_i \gg m_i$.

$$\begin{aligned}
q^2 &= (p_A - p_C)^2 \\
&= \left(E_A - E_C, \underline{p}_A - \underline{p}_C\right)^2 \\
\text{For } m \to 0 \quad |\underline{p}_A| &\simeq E_A \text{ etc} \\
\Rightarrow q^2 &= (E_A - E_C)^2 - \left(\underline{p}_A - \underline{p}_C\right)^2 \\
&= E_A^2 + E_C^2 - 2E_A E_C - \left(\underline{p}_A - \underline{p}_C\right)^2 \\
&= -2E_A E_C + 2\underline{p}_A \cdot \underline{p}_C \\
&= -2E_A E_C \left(1 - \cos\theta\right) \\
\text{So } q^4 &= 4E_A^2 E_C^2 \left(1 - \cos\theta\right)^2 \\
\Big[\left(p_A + p_C\right)_\mu \left(p_B + p_D\right)^\mu\Big]^2 &= \Big[p_A \cdot p_B + p_A \cdot p_D + p_B \cdot p_C + p_C \cdot p_D\Big]^2 \\
\text{Using } p_A &= \left(|\underline{p}|, \underline{p}\right) \\
p_B &= \left(|\underline{p}|, -\underline{p}\right) \\
p_C &= \left(|\underline{p}|, \underline{p}'\right) \\
p_D &= \left(|\underline{p}|, -\underline{p}'\right) \\
\text{With } |\underline{p}| &= |\underline{p}'| \\
\Rightarrow \Big[\left(p_A + p_C\right)_\mu \left(p_B + p_D\right)^\mu\Big]^2 &= \left(6p^2 + 2p^2\cos\theta\right)^2 \\
\Rightarrow \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{e\mu} &= \frac{\mathrm{e}^4}{64\pi^4 s}\left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2
\end{aligned}$$

7.2.1 Note on decay

Consider the number of states per flux factor for the decay $A \to B \quad C$.

The number of states is as before:

$$\frac{1}{16\pi^2}\frac{p_C V^2}{m_A}\mathrm{d}\Omega$$

Due to the choice of normalisation there are $2E_m$ particles per state in the centre of mass frame in a volume V:

$$\rho = \frac{2m_A}{V}$$

The flux factor is:

$$\begin{aligned} F &= \frac{1}{16\pi^2}\frac{p_C V^2}{m_A}\mathrm{d}\Omega\frac{V}{2m_A} \\ &= \frac{1}{32\pi^2}\frac{p_C}{m_A^2}V^3\mathrm{d}\Omega \end{aligned}$$

The decay rate is:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = |T_{fi}|^2\frac{1}{32\pi^2}\frac{p_C}{m_A^2}V^3\mathrm{d}\Omega$$

This factor is universal for decays if the decay is istropic:

$$\begin{aligned} \mathrm{d}\Omega &\to 4\pi \\ \mathrm{d}\Gamma &\to \Gamma \end{aligned}$$

For the decay of A:

$$\frac{\mathrm{d}N_A}{\mathrm{d}t} = \Gamma N_A$$

ie $N_A(t) = N_A(0)e^{-\Gamma t}$ and Γ^{-1} is the mean lifetime.

Chapter 8

Relativistic spin$-\frac{1}{2}$ particles (Dirac equation)

8.1 Non-relativistic description

There are two spin states, up ($\frac{1}{2}$) and down ($-\frac{1}{2}$). In spin space there are spin operators given by:

$$\bar{s} = \frac{\hbar}{2}\bar{\sigma}$$

where $\bar{\sigma}$ are the so-called Pauli matrices. The spin algebra is the same as that of orbital angular momentum.:

$$L^2 = l(l+1)\hbar^2$$

In angular spin momentum:

$$S^2 = s(s+1)\hbar^2$$

The Pauli matrices are:

$$\begin{aligned}
\bar{\sigma}_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\bar{\sigma}_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\bar{\sigma}_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\text{where } \bar{I} &= \bar{\sigma}_x^2 = \bar{\sigma}_y^2 = \bar{\sigma}_z^2
\end{aligned}$$

The communtation relations are:

$$\begin{aligned}
[\bar{s}_x, \bar{s}_y] &= i\hbar\bar{s}_z \\
\text{ie}\left[\frac{\hbar}{2}\bar{\sigma}_x, \frac{\hbar}{2}\bar{\sigma}_y\right] &= i\left(\frac{\hbar}{2}\right)^2 \bar{\sigma}_z
\end{aligned}$$

Or generally:

$$\left[\bar{\sigma}_i, \bar{\sigma}_j\right] = 2i\epsilon_{ijk}\bar{\sigma}_k$$

where ϵ_{ijk} is the antisymmetric tensor.

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any index is repeated} \\ 1 & \text{for } 123, 231, 312 \\ -1 & \text{for } 132, 321, 213 \end{cases}$$

Also: $\bar{\sigma}_i\bar{\sigma}_j + \bar{\sigma}_j\bar{\sigma}_i = 2\delta_{ij}\bar{I}$. This is the anticommutation relation.

$$\Rightarrow \bar{\sigma}_i\bar{\sigma}_j = \delta_{ij}\bar{I} + i\epsilon_{ijk}\bar{\sigma}_k$$

$$\begin{aligned}
\text{Consider } \bar{\sigma}_i\underline{A}\bar{\sigma}_j\underline{B}_j &= \underline{A}_i\underline{B}_j\delta_{ij} + i\epsilon_{ijk}\bar{\sigma}_k\underline{A}_i\underline{B}_j \\
\text{then } (\bar{\sigma}\cdot\underline{A})(\bar{\sigma}\cdot\underline{B}) &= \underline{A}\cdot\underline{B} + i\bar{\sigma}\cdot(\underline{A}\times\underline{B}) \\
&= \underline{A}\cdot\underline{B} + i\begin{vmatrix} \bar{\sigma}_x & \bar{\sigma}_y & \bar{\sigma}_z \\ \underline{A}_x & \underline{A}_y & \underline{A}_z \\ \underline{B}_x & \underline{B}_y & \underline{B}_z \end{vmatrix}
\end{aligned}$$

Suppose $\underline{A} = \underline{B} = \underline{p}$, then:

$$\left(\bar{\sigma}\cdot\underline{p}\right)\left(\bar{\sigma}\cdot\underline{p}\right) = |\underline{p}|^2$$

The gyromagnetic ratio of a particle can be (correctly) determined if the expression for the energy is:

$$E = V + \frac{\left(\bar{\sigma}\cdot\underline{p}\right)\left(\bar{\sigma}\cdot\underline{p}\right)}{2m}$$

as opposed to:

$$E = V + \frac{p^2}{2m}$$

and the electromagnetic coupling is included.

Experimentally the gyromagnetically ratio is $g = 2.00232$ and the Dirac equation gives $g = 2$.

8.2 The Dirac equation

To avoid the problem of negative probability in the negative energy if the Klein-Gordon equation, Dirac proposed an equation linear in $\frac{\partial}{\partial t}$:

$$H\psi = \left(\bar{\alpha} \cdot \underline{p} + \beta m\right) \psi$$

where $\bar{\alpha}$ and β are 4×4 matrices and solutions for ψ are multi-component objects. The formulation must be consistent with $E^2\psi = \left(p^2 + m^2\right) \psi$.

$$\begin{aligned}
\text{If } E\psi &= \left(\alpha_i p_i + \beta m\right) \psi \\
\text{then } E^2\psi &= \left(\alpha_i p_i + \beta m\right) \left(\alpha_j p_j + \beta m\right) \psi \\
&= \left(\alpha_i \alpha_j p_i p_j + \left(\alpha_i \beta + \beta \alpha_j\right) p_i m + \beta^2 m^2\right) \psi \\
&= \left(\left(\frac{\alpha_i \alpha_j + \alpha_j \alpha_i}{2} p_i p_j + \alpha \left(\alpha_i \beta \beta \alpha_j\right) p_i m + \beta^2 m^2\right)\right) \psi \\
\Rightarrow \bar{\alpha}_i \bar{\alpha}_j + \bar{\alpha}_j \bar{\alpha}_i &= 2\delta_{ij} \bar{I} \\
\bar{\alpha}\bar{\beta} + \bar{\beta}\bar{\alpha} &= \bar{0} \\
\bar{\beta}^2 &= \bar{I}
\end{aligned}$$

- α and β are Hermitian matrices: $\bar{\alpha} = \bar{\alpha}^\dagger$, $\bar{\beta} = \bar{\beta}^\dagger$.
- $\bar{\alpha}_i^2 = \bar{I}$
- α and β are traceless

The proof that α and β are traceless is as follows:

$$\bar{\alpha}_i \bar{\beta} = -\bar{\beta} \bar{\alpha}_i$$

Postmultiplying by $\bar{\beta}$ gives:

$$\begin{aligned}
\bar{\alpha}_i \bar{\beta}^2 &= -\bar{\beta} \bar{\alpha}_i \bar{\beta} \\
\bar{\alpha}_i \bar{I} &= -\bar{\beta} \bar{\alpha}_i \bar{\beta}
\end{aligned}$$

Taking the trace:

$$Tr\left(\bar{\alpha}_i\right) = -Tr\left(\bar{\beta} \bar{\alpha}_i \bar{\beta}\right)$$

Moving the elements cyclically:

$$\begin{aligned}
Tr\left(\bar{\alpha}_i\right) &= -Tr\left(\bar{\beta}^2 \bar{\alpha}_i\right) \\
&= -Tr\left(\bar{\alpha}_i\right) \\
\text{So } Tr\left(\bar{\alpha}_i\right) &= 0
\end{aligned}$$

A common choice for $\bar{\alpha}$ and $\bar{\beta}$ is:

$$\begin{aligned}
\bar{\alpha}_i &= \begin{pmatrix} 0 & \bar{\sigma}_i \\ \bar{\sigma}_i & 0 \end{pmatrix} \\
\bar{\beta} &= \begin{pmatrix} \bar{I} & 0 \\ 0 & -\bar{I} \end{pmatrix}
\end{aligned}$$

where $\bar{\sigma}_i$ are the Pauli matrices and $\bar{I}$ is the identity matrix.

8.2.1 The covariant form of the Dirac equation

$$\begin{aligned}
E\psi &= \left(\bar{\alpha}_i \cdot \underline{p} + \beta m\right)\psi \\
E &\rightarrow i\frac{\partial}{\partial t} \\
\underline{p} &\rightarrow -i\underline{\nabla} \\
\Rightarrow i\frac{\partial}{\partial t}\psi &= -i\bar{\alpha} \cdot \underline{\nabla}\psi + \beta m\psi
\end{aligned}$$

Premultiplying by β:

$$\begin{aligned}
i\beta\frac{\partial\psi}{\partial t} &= -i\beta\bar{\alpha} \cdot \underline{\nabla}\psi + \beta^2 m\psi \\
\text{So } i\beta\frac{\partial\psi}{\partial t} &= -i\beta\bar{\alpha} \cdot \underline{\nabla}\psi + m\psi \\
\text{Let } \gamma^0 &= \beta \\
\gamma^k &= \beta\alpha \\
i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^k\underline{\nabla}\psi - m\psi &= 0 \\
\left(i\gamma^0\frac{\partial}{\partial x^0} + i\gamma^k\frac{\partial}{\partial x^k} - m\right) &= 0 \\
\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0 \\
\text{where } \gamma^\mu &= \left(\gamma^0, \underline{\gamma}^k\right) \\
\partial_\mu &= \left(\frac{\partial}{\partial x^0}, \underline{\nabla}\right)
\end{aligned}$$

γ^0 is Hermitian, since β is Hermitian. However, γ^k is not Hermitian:

$$\begin{aligned}
\left(\gamma^k\right)^\dagger &= -\gamma^k \\
\gamma^k &= \beta\alpha^k \\
(\gamma)^\dagger &= \left(\beta\alpha^k\right)^\dagger
\end{aligned}$$

$$\begin{aligned}
&= \left(\alpha^k\right)^\dagger \beta^\dagger \\
&= \alpha^k \beta \\
&= -\beta\alpha^k \\
&= -\gamma^k
\end{aligned}$$

$$\left(\gamma^0\right)^2 = \bar{I} \text{ as } \beta^2 = \bar{I}$$

$$\begin{aligned}
\left(\gamma^k\right)^2 &= \gamma^k\gamma^k \\
&= \beta\alpha^k\beta\alpha^k \\
&= -\beta\alpha^k\alpha^k\beta \\
&= -\beta\beta \\
&= -I
\end{aligned}$$

8.2.2 Adjoint Dirac equation and conserved current

Since the Dirac equation is a matrix equation, to obtain the adjoint equation it is necessary to take the Hermitian conjugate and not the complex conjugate.

$$\begin{aligned}
\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0 \qquad (8.1) \\
\text{or } \left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^k\frac{\partial}{\partial x^k} - m\right)\psi &= 0
\end{aligned}$$

Taking the Hermitian conjugate:

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x^k}\left(-\gamma^k\right) - m\psi^\dagger = 0$$

Postmultiply by γ^0:

$$\begin{aligned}
-i\frac{\partial\psi^\dagger}{\partial t} + i\frac{\partial\psi^\dagger}{\partial x^k}\gamma^k\gamma^0 - m\psi^\dagger\gamma^0 &= 0 \\
\gamma^0\gamma^k &= -\gamma^k\gamma^0 \\
\Rightarrow -i\frac{\partial\psi^\dagger}{\partial t} - i\frac{\partial\psi^\dagger}{\partial x^k}\gamma^0\gamma^k - m\psi^\dagger\gamma^0 &= 0
\end{aligned}$$

Defining the adjoint as:

$$\bar{\psi} = \psi^\dagger\gamma^0$$

gives:

$$\begin{aligned}
-i\frac{\partial\bar{\psi}}{\partial t}\gamma^0 - i\frac{\partial\bar{\psi}}{\partial x^k}\gamma^k - m\bar{\psi} &= 0 \\
\text{or } i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} &= 0 \qquad (8.2)
\end{aligned}$$

Consider $\bar{\psi}\times(15)+(16)\times\psi$:

$$\begin{aligned}\bar{\psi}\times(15) &= \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi = 0\\ (16)\times\psi &= i\partial_\mu\bar{\psi}\gamma^\mu\psi + m\bar{\psi}\psi = 0\\ \text{So } i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\partial_\mu\bar{\psi}\gamma^\mu\psi &= 0\\ \Rightarrow \partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) &= 0\end{aligned}$$

So the expression for j^μ:

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

satisfies the continuity equation.
j^μ is identified as the probability and flux densities ρ and $\underline{j}$.
The probability density is:

$$\begin{aligned}\rho &= j^0\\ &= \bar{\psi}\psi\\ &= \psi^\dagger\gamma^0\gamma^0\psi\\ &= \psi^\dagger\psi\\ &= |\psi|^2\end{aligned}$$

Hence ρ is always positive. For the electromagnetic interaction the charge current density is:

$$j^\mu = -\mathrm{e}\bar{\psi}\gamma^\mu\psi$$

8.2.3 Free particle solutions of the Dirac equation

Consider solutions of the form:

$$\psi = u(p)\mathrm{e}^{-ipx}$$

Substitute ψ into the Dirac equation:

$$\begin{aligned}\left(i\gamma^\mu\partial_\mu - m\right)u(p)\mathrm{e}^{-ipx} &= 0\\ \left(i\gamma^\mu\left(-ip_\mu\right) - m\right)u(p)\mathrm{e}^{-ipx} &= 0\\ \left(\gamma^\mu p_\mu - m\right)u(p)\mathrm{e}^{-ipx} &= 0\\ \text{So } \left(\gamma^\mu p_\mu - m\right)u(p) &= 0\\ \text{Define } \gamma^\mu p_\mu &= \not{p}\\ \text{So } \left(\not{p} - m\right)u(p) &= 0\end{aligned}$$

To obtain solutions for $u(p)$, write the above in terms of the α and β matrices:

$$\left(\gamma^0 E - \gamma^k p_k - m\right) u(p) = 0$$

Premultiplying by γ^0:

$$\begin{aligned}
\left(\left(\gamma^0\right)^2 E - \gamma^0\gamma^k p_k - \gamma^0 m\right) u(p) &= 0 \\
\left(IE - \alpha^k p_k - \beta m\right) u(p) &= 0
\end{aligned}$$

For a particle at rest, $\underline{p} = \underline{0}$:

$$\begin{aligned}
(IE - \beta m)\, u(p) &= 0 \\
\Rightarrow m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} u(p) &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} Eu(p)
\end{aligned}$$

Solutions exist if:

$$\begin{vmatrix} (m-E)I & 0 \\ 0 & -(m+E)I \end{vmatrix} = 0$$

or in longhand:

$$\begin{aligned}
\begin{vmatrix} m-E & 0 & 0 & 0 \\ 0 & m-E & 0 & 0 \\ 0 & 0 & -m-E & 0 \\ 0 & 0 & 0 & -m-e \end{vmatrix} &= 0 \\
\Big[(m-E)\,(m+E)\Big]^2 &= 0
\end{aligned}$$

So there are four eigenvalues corresponding to $E = \pm m$ in two coincident pairs. This means that negative energy solutions still exist. u_1, u_2 are associated with positive energy solutions and u_3, u_4 are associated with negative energy solutions.

Consider the solutions when $\underline{p} \neq \underline{0}$. From $\left(\bar{\alpha} \cdot \underline{p} + \beta m\right) u = 0$:

$$\left[\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot \underline{p} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m\right] \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

This yields:

$$\bar{\sigma} \cdot \underline{p} u_B + m u_A = E u_A \tag{8.3}$$

$$\bar{\sigma} \cdot \underline{p} u_A - m u_B = E u_B \tag{8.4}$$

$$\begin{aligned} \text{From (17)} \quad u_A &= \frac{\bar{\sigma}\cdot \underline{p} u_B}{E-m} \\ \text{From (18)} \quad u_A &= \frac{\bar{\sigma}\cdot \underline{p} u_A}{E+m} \end{aligned}$$

$$\begin{aligned} \text{For } E > 0: & \\ u_A &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(spin up)} \\ \text{or} & \\ u_A &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(spin down)} \end{aligned}$$

$$\begin{aligned} \text{So } u_1 &= N \begin{pmatrix} 1 \\ 0 \\ \frac{\bar{\sigma}\cdot \underline{p}}{E+m} \\ 0 \end{pmatrix} \\ u_2 &= N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\bar{\sigma}\cdot \underline{p}}{E+m} \end{pmatrix} \end{aligned}$$

where N is a normalisation constant.
For $E < 0$:

$$\begin{aligned} u_A &= \frac{\bar{\sigma}\cdot \underline{p}}{E-m} u_B \\ &= \frac{\bar{\sigma}\cdot \underline{p}}{-|E|-m} u_B \\ &= -\frac{\bar{\sigma}\cdot \underline{p}}{|E|+m} u_B \end{aligned}$$

$$\begin{aligned} \text{So } u_3 &= N \begin{pmatrix} -\frac{\bar{\sigma}\cdot \underline{p}}{|E|+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ u_4 &= N \begin{pmatrix} 0 \\ -\frac{\bar{\sigma}\cdot \underline{p}}{E+m} \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

In summary all the above comes from:

$$(\not{p} - m) = 0$$

with the propagation factor e^{-ipx}.

Now associate negative energy solutions (u_3, u_4) such that they describe positron solutions propagating backwards in time with the propagation factor e^{ipx}:

$$\begin{aligned} u^{(3,4)}(-p)\mathrm{e}^{-i(-p)x} &= v^{(2,1)}(p)\mathrm{e}^{ipx} \\ \Rightarrow v_2 &= \begin{pmatrix} \frac{\bar{\sigma}\cdot\underline{p}}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{(spin down)} \\ v_1 &= \begin{pmatrix} 0 \\ \frac{\bar{\sigma}\cdot\underline{p}}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad \text{(spin up)} \end{aligned}$$

The original equation for an electron of energy $-E$ and momentum $-\underline{p}$ is:

$$\begin{aligned} (-\not{p} - m)\, u(-p) &= 0 \\ \Rightarrow (\not{p} + m)\, v(p) &= 0 \end{aligned}$$

8.2.4 Orthogonality and normalisation of spinors

$$\begin{aligned} \psi_1 &= N \begin{pmatrix} 1 \\ 0 \\ \frac{\bar{\sigma}\cdot\underline{p}}{E+m} \\ 0 \end{pmatrix} \mathrm{e}^{-ipx} \\ \psi_2 &= N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\bar{\sigma}\cdot\underline{p}}{E+m} \end{pmatrix} \mathrm{e}^{-ipx} \end{aligned}$$

For orthogonality:

$$\begin{aligned} \int \psi_1^\dagger \psi_2 \mathrm{d}^3x &= 0 \\ \Rightarrow N^\star N \left(10\frac{(\bar{\sigma}\cdot\underline{p})^\dagger}{E+m}0 \right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\bar{\sigma}\cdot\underline{p}}{E+m} \end{pmatrix} &= 0 \end{aligned}$$

So ψ_1 and ψ_2 are orthogonal to each other. Similarly ψ_3 and ψ_4 are orthogonal to each other.

In order to normalise to $2E$ particles in a volume V:

$$\begin{aligned}
\int \psi_1^\dagger \psi_1 \mathrm{d}^3x &= 2E \\
\Rightarrow \int N^\star N \begin{pmatrix} 1 & 0 & \left(\frac{\bar{\sigma}\cdot\underline{p}}{E+m}\right)^\dagger & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{\bar{\sigma}\cdot\underline{p}}{E+m} \\ 0 \end{pmatrix} \mathrm{d}^3x &= 2E \\
\Rightarrow \int N^\star N \left[1 + \left(\frac{\bar{\sigma}\cdot\underline{p}}{E+m}\right)^2\right] \mathrm{d}^3x &= 2E \\
\text{where } (\bar{\sigma}\cdot\underline{p})^\dagger &= \bar{\sigma}\cdot\underline{p}
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\Rightarrow \left(\bar{\sigma}_y \cdot \underline{p}_y\right)^2 &= \begin{pmatrix} 0 & -ip_y \\ ip_y & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} p_y^2 & 0 \\ 0 & p_y^2 \end{pmatrix} \\
&= p_y^2 \bar{I}
\end{aligned}$$

$$\begin{aligned}
\text{So } \int N^\star N \left(1 + \frac{p^2}{(E+m)^2}\right) \mathrm{d}^3x &= 2E \\
\Rightarrow N^\star N \left(\frac{(E+m)^2 + E^2 - m^2}{(E+m)^2}\right) \mathrm{d}^3x &= 2E \\
&= \int N^\star N \frac{2E}{E+m} \mathrm{d}^3x \\
\text{So } \frac{N^\star N V}{E+m} &= 1 \\
\Rightarrow N &= \sqrt{\frac{E+m}{V}}
\end{aligned}$$

8.2.5 Spin

Neither the orbital angular momentum nor the spin angular momentum commute with the Dirac Hamiltonian, but $J = L + S$ does commute. To find an operator other than J that commutes with the Dirac Hamiltonian recall the Dirac equation:

$$H\begin{pmatrix} u_A \\ u_B \end{pmatrix} = (\bar{\alpha}\cdot\underline{p} + \beta m)\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$\bar{\alpha}\cdot\underline{p} + \beta m$ is the Dirac Hamiltonian.

$$\begin{aligned} H\begin{pmatrix} u_A \\ u_B \end{pmatrix} &= E\begin{pmatrix} u_A \\ u_B \end{pmatrix} \\ &= \begin{pmatrix} m\bar{I} & \bar{\sigma}\cdot\underline{p}\bar{I} \\ \bar{\sigma}\cdot\underline{p}\bar{I} & -m\bar{I} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} \end{aligned}$$

By inspection, the Dirac Hamiltonian commutes with $\bar{\sigma}\cdot\underline{p}\bar{I}$:

$$\begin{pmatrix} m\bar{I} & \bar{\sigma}\cdot\underline{p}\bar{I} \\ \bar{\sigma}\cdot\underline{p}\bar{I} & -m\bar{I} \end{pmatrix}\begin{pmatrix} \bar{\sigma}\cdot\underline{p}\bar{I} & 0 \\ 0 & \bar{\sigma}\cdot\underline{p}\bar{I} \end{pmatrix} - \begin{pmatrix} \bar{\sigma}\cdot\underline{p}\bar{I} & 0 \\ 0 & \bar{\sigma}\cdot\underline{p}\bar{I} \end{pmatrix}\begin{pmatrix} m\bar{I} & \bar{\sigma}\cdot\underline{p}\bar{I} \\ \bar{\sigma}\cdot\underline{p}\bar{I} & m\bar{I} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Define the helicity operator as:

$$H = \frac{1}{2}\bar{\sigma}\cdot\underline{\hat{p}} = \frac{1}{2}\frac{\bar{\sigma}\cdot\underline{p}}{|\underline{p}|}$$

The helicity is the projection of the spin along the direction of motion and its eigenvalues are $\pm\frac{1}{2}$. $H = \frac{1}{2}$ corresponds to positive helicity and $H = \frac{-1}{2}$ corresponds to negative helicity.

Suppose a particle has a momentum $\underline{p}$ where:

$$\begin{aligned} \underline{\hat{p}} &= \sin\theta\cos\phi\underline{\hat{i}} + \sin\theta\sin\phi\underline{\hat{j}} + \cos\theta\underline{\hat{k}} \\ \bar{\sigma}\cdot\underline{\hat{p}} &= \bar{\sigma}_x\cdot\underline{\hat{p}}_x + \bar{\sigma}_y\cdot\underline{\hat{p}}_y + \bar{\sigma}_z\cdot\underline{\hat{p}}_z \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta \\ \text{So } \frac{1}{2}\bar{\sigma}\cdot\underline{\hat{p}}\begin{pmatrix} u_A \\ u_B \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} \end{aligned}$$

So the eigenvalue equation is:

$$\begin{aligned}
\frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} &= \lambda\begin{pmatrix} u_A \\ u_B \end{pmatrix} \\
\begin{vmatrix} \cos\theta - 2\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - 2\lambda \end{vmatrix} &= 0 \\
(-\cos\theta - 2\lambda)(\cos\theta + 2\lambda) - \sin^2\theta &= 0 \\
-\cos^2\theta + 4\lambda^2 - \sin^2\theta &= 0 \\
\Rightarrow \lambda &= \pm\frac{1}{2}
\end{aligned}$$

8.2.6 The γ^5 matrix

The γ^5 matrix is used to simplify notation:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

γ^5 has many properties:

$$\begin{aligned}
\left(\gamma^5\right)^\dagger &= \gamma^5 \\
\left(\gamma^5\right)^2 &= \bar{I} \\
\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 &= 0
\end{aligned}$$

n Dirac-Pauli representation:

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} \bar{I} & \bar{0} \\ \bar{0} & \bar{I} \end{pmatrix} \\
\gamma^k &= \begin{pmatrix} \bar{0} & \bar{\sigma}_k \\ \bar{\sigma}_k & \bar{0} \end{pmatrix} \\
\gamma^5 &= \begin{pmatrix} \bar{0} & \bar{I} \\ \bar{I} & \bar{0} \end{pmatrix}
\end{aligned}$$

Consider the effect of γ^5 operating upon the Dirac equation (dropping the bars that signify matrices):

$$\begin{aligned}
\gamma^5\begin{pmatrix} u_A \\ u_B \end{pmatrix} &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \left(\frac{\sigma\cdot p}{E+m}\right)\chi \end{pmatrix} \\
\text{where } \chi &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\gamma^5\begin{pmatrix} u_A \\ u_B \end{pmatrix} &= \begin{pmatrix} \left(\frac{\sigma\cdot p}{E+m}\right)\chi \\ \chi \end{pmatrix}
\end{aligned}$$

For high energies, or in the limit $m \to 0$, $E \sim p$:

$$\begin{aligned}
\gamma^5 \begin{pmatrix} u_A \\ u_B \end{pmatrix} &\simeq \begin{pmatrix} \sigma \cdot \hat{p}\chi \\ \chi \end{pmatrix} \\
&= \begin{pmatrix} \sigma \cdot \hat{p}\chi \\ I\chi \end{pmatrix} \\
&= \begin{pmatrix} \sigma \cdot \hat{p}\chi \\ (\sigma \cdot \hat{p})^2 \chi \end{pmatrix} \\
&= \sigma \cdot \hat{p} \begin{pmatrix} \chi \\ \sigma \cdot \hat{p}\chi \end{pmatrix} \\
&= \sigma \cdot \hat{p} \begin{pmatrix} \chi \\ \left(\frac{\sigma \cdot p}{E+m}\right) \chi \end{pmatrix} \\
&= \sigma \cdot \hat{p} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \\
\text{So } \gamma^5 \begin{pmatrix} u_A \\ u_B \end{pmatrix} &= \begin{pmatrix} \sigma \cdot \hat{p} & 0 \\ 0 & \sigma \cdot \hat{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}
\end{aligned}$$

So in the limit $m \to 0$, γ^5 becomes the helicity operator.
Operators can then be defined as follows:

$$\begin{aligned}
P_R &= \frac{1}{2}\left(1+\gamma^5\right) \\
P_L &= \frac{1}{2}\left(1-\gamma^5\right)
\end{aligned}$$

These operators are then respectively the left and right-handed projection operators of helicity.

Where $m \neq 0$ (which is generally the case) the operator $\frac{1}{2}\left(1+\gamma^5\right)$ is the right-handed chirality state and if m is small then the state can be approximated as a right-handed helicity state, but will also contain a small fraction of the left-handed helicity component.

8.2.7 Completeness relation

The completeness relations are used extensively in the evaluation of Feynmann diagram calculations.

Consider the summation over all spin states:

$$\begin{aligned}
\sum_{S=1,2} u_s(p)\bar{u}_s(p) &= \sum_{S=1,2} N^\star N \begin{pmatrix} \chi_S \\ \frac{\sigma \cdot p}{E+m}\chi_S \end{pmatrix} \left(\chi_S^\dagger, -\frac{\sigma \cdot p}{E+m}\chi_S^\dagger\right) \\
&= \sum_{S=1,2} N^\star N \begin{pmatrix} \chi_S^\dagger \chi_S & -\frac{(\sigma \cdot p)^\dagger}{E+m}\chi_S^\dagger \chi_S \\ \frac{\sigma \cdot p}{E+m}\chi_S^\dagger \chi_S & -\frac{E^2-m^2}{(E+m)^2}\chi_S^\dagger \chi_S \end{pmatrix}
\end{aligned}$$

$$= \sum_{S=1,2} N^\star N \begin{pmatrix} I & -\frac{(\sigma\cdot p)^\dagger}{E+m} \\ \frac{\sigma\cdot p}{E+m} & -\frac{E^2-m^2}{(E+m)^2} \end{pmatrix} \chi_S^\dagger \chi$$

The summation over states is:

$$\begin{aligned} \sum_{S=1,2} \chi_S^\dagger \chi_S &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (10) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (01) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$N^\star N = E + m$$

$$\begin{aligned} \sum_{S=1,2} u_S \bar{u}_S &= \begin{pmatrix} I & -\frac{\sigma\cdot p}{E+m} \\ \frac{\sigma\cdot p}{E+m} & -\frac{E^2-m^2}{(E+m)^2} I \end{pmatrix} (E+m) \\ &= \begin{pmatrix} (E+m)\,I & -\sigma\cdot p \\ \sigma\cdot p & -I\,(E-m) \end{pmatrix} \\ &= \begin{pmatrix} (E+m)\,I & -\sigma\cdot p \\ \sigma\cdot p & (m-E)\,I \end{pmatrix} \qquad (8.5) \\ \text{However} \quad \not{p} + m &= \gamma^\mu p_\mu + mI \\ &= \gamma^0 E - \gamma^k p_k + mI \\ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} p_k + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} m \\ &= \begin{pmatrix} (E+m)\,I & -\sigma\cdot p \\ \sigma\cdot p & (m-E)\,I \end{pmatrix} \qquad (8.6) \end{aligned}$$

So (19)=(20).

$$\begin{aligned} \sum_{S=1,2} u_S \bar{u}_S &= \not{p} + m \\ \sum_{S=1,2} v_S \bar{v}_S &= \not{p} - m \end{aligned}$$

8.2.8 Possible forms of interaction

An exhaustive set of possibilities of interaction is:

Scalar interactions $\bar{u}u$ (even parity)

Vector interactions $\bar{u}\gamma^\mu u$ (odd parity) This interaction causes allowed Fermi transitions in β decay and in μ decay.

Tensor interactions $\bar{u}\sigma^{\mu\nu}u$ As far as has been confirmed by experiment this interaction does not exist, except for anomalous magnetic moments.

Axial vector interactions $\bar{u}\gamma^5\gamma^\mu u$ (even parity) The weak interaction is a mixture of the vector and axial vector interactions. In some nuclei, either the vector or axial vector process cannot take place. In many β decays of nuclei both of the processes take place.

Pseudoscalar interactions $\bar{u}\gamma^5 u$ (odd parity)

8.3 Trace theorems

8.3.1 $Tr[I]$

$$Tr[I] = 4$$

8.3.2 $Tr[\not{a}\ \not{b}]$

$$\begin{aligned}
Tr[\not{a}\ \not{b}] &= Tr[\not{b}\ \not{a}] \\
\text{So } Tr[\not{a}\ \not{b}] &= \frac{1}{2}Tr[\not{a}\ \not{b} + \not{b}\ \not{a}] \\
&= \frac{1}{2}Tr[\gamma^\mu\gamma^\nu a_\mu b_\nu + \gamma^\mu\gamma^\nu b_\mu a_\nu] \\
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}I \\
&= \frac{1}{2}a\cdot b 2Tr[I] \\
&= 4a\cdot b
\end{aligned}$$

8.3.3 $Tr[\not{a}\ \not{b}\ \not{c}\ \not{d}]$

$$\begin{aligned}
Tr[\not{a}\ \not{b}\ \not{c}\ \not{d}] &= Tr[\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma a_\mu b_\nu c_\delta d_\sigma] \\
\text{But } Tr[\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma] &= -Tr[\gamma^\nu\gamma^\mu\gamma^\delta\gamma^\sigma] + Tr[2g^{\mu\nu}\gamma^\delta\gamma^\sigma] \\
&= Tr[\gamma^\nu\gamma^\delta\gamma^\mu\gamma^\sigma] - Tr[2g^{\mu\delta}\gamma^\nu\gamma^\sigma] + Tr[2g^{\mu\nu}\gamma^\delta\gamma^\sigma] \\
&= -Tr[\gamma^\nu\gamma^\delta\gamma^\sigma\gamma^\mu] + Tr[2g^{\mu\sigma}\gamma^\nu\gamma^\delta] - Tr[2g^{\mu\delta}\gamma^\nu\gamma^\sigma] + Tr[2g^{\mu\nu}\gamma^\delta\gamma^\sigma] \\
\Rightarrow Tr[\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma] &= Tr[g^{\mu\sigma}\gamma^\nu\gamma^\delta] - Tr[g^{\mu\delta}\gamma^\nu\gamma^\sigma] + Tr[g^{\mu\nu}\gamma^\delta\gamma^\sigma] \\
Tr[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\
Tr[\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma] &= 4\left(g^{\mu\sigma}g^{\nu\delta} - g^{\mu\delta}g^{\nu\sigma} + g^{\mu\nu}g^{\delta\sigma}\right) \\
\Rightarrow Tr[\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma]a_\mu b_\nu c_\delta d_\sigma &= 4\Big[(a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d) + (a\cdot b)(c\cdot d)\Big]
\end{aligned}$$

Some other identities, which shall be used in the calculation of the cross-section for Compton scattering include:

8.3.4 $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$

$$\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\mu &= -\gamma_\mu\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\gamma_\mu \\
&= -4\gamma^\nu + 2\gamma^\nu \\
&= -2\gamma^\nu \\
\text{or } \gamma_\mu \not{a}\gamma^\mu &= -2 \not{a}
\end{aligned}$$

8.3.5 $\gamma_\mu\gamma^\delta\gamma^\sigma\gamma^\mu = 4g^{\delta\sigma}$

$$\begin{aligned}
\gamma_\mu\gamma^\delta\gamma^\sigma\gamma^\mu &= -\gamma^\delta\gamma_\mu\gamma^\sigma\gamma^\mu + 2g^\delta_\mu\gamma^\sigma\gamma^\mu \\
&= 2\gamma^\delta\gamma^\sigma + 2\gamma^\sigma\gamma^\delta \\
&= 4g^{\delta\sigma} \\
\text{or } \gamma_\mu \not{a} \not{b}\gamma^\mu &= 4a\cdot b
\end{aligned}$$

8.3.6 $\gamma_\mu \not{a} \not{b} \not{c}\gamma^\mu = -2 \not{c} \not{b} \not{a}$

$$\begin{aligned}
\gamma_\mu \not{a} \not{b}\gamma^\nu\gamma^\mu c_\nu &= -\gamma_\mu \not{a} \not{b}\gamma^\mu\gamma^\nu c_\nu + \gamma_\mu \not{a} \not{b}2g^{\nu\mu}c_\nu \\
&= -4(a\cdot b) \not{c} + 2 \not{c} \not{a} \not{b} \\
&= -4(a\cdot b) \not{c} + 2 \not{c}\gamma^\alpha\gamma^\beta a_\alpha b_\beta \\
&= -4(a\cdot b) \not{c} - 2 \not{c}\gamma^\beta\gamma^\alpha a_\alpha b_\beta + 2 \not{c}2g^{\alpha\beta}a_\alpha b_\beta \\
&= -2 \not{c} \not{b} \not{a}
\end{aligned}$$

8.3.7 $Tr[\gamma^5\gamma^\mu] = 0$

$$\begin{aligned}
\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 &= 0 \\
\Rightarrow Tr[\gamma^5\gamma^\mu] &= -Tr[\gamma^\mu\gamma^5] \\
&= -Tr[\gamma^5\gamma^\mu] \\
\text{So } Tr[\gamma^5\gamma^\mu] &= 0
\end{aligned}$$

8.3.8 $Tr[\gamma^5\gamma^\mu\gamma^\nu] = 0$

Assume that $\gamma^\mu = \gamma^\nu$

$$\begin{aligned}
Tr[\gamma^5\,(\gamma^\mu)^2] &= Tr(\gamma^5 I) \\
&= Tr\left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] \\
&= 0
\end{aligned}$$

Now assume $\mu = 1$, $\nu = 2$

$$\begin{aligned}
Tr[i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2] &= Tr[i\gamma^0\left(\gamma^1\right)^2\gamma^2\gamma^3\gamma^2] \\
&= -Tr[i\gamma^0\left(\gamma^2\right)^2\gamma^3] \\
&= -iTr[\gamma^0\gamma^3] \\
&= -iTr[\gamma^3\gamma^0] \\
\text{and } Tr[\gamma^0\gamma^3+\gamma^3\gamma^0] &= 2Tr[g^{03}I] \\
&= 0 \\
\Rightarrow Tr[\gamma^3\gamma^0] &= 0 \\
\text{So } Tr[\gamma^5\gamma^1\gamma^2] &= 0
\end{aligned}$$

8.3.9 $Tr[\gamma^5\gamma^\nu\gamma^\mu\gamma^\delta]=0$

The trace of an odd number of γ matrices is zero:

$$\begin{aligned}
Tr[\not{a}_1\ \not{a}_2\cdots\ \not{a}_n] &= Tr[\not{a}_a\ \not{a}_2\cdots\ \not{a}_n\gamma^5\gamma^5] \\
\text{Back-propagate } \gamma^5: & \\
Tr[\not{a}_1\ \not{a}_2\cdots\ \not{a}_n] &= (-1)^n\,Tr[\gamma^5\ \not{a}_a\ \not{a}_2\cdots\ \not{a}_n\gamma^5] \\
&= (-1)^n\,Tr[\not{a}_1\ \not{a}_2\cdots\ \not{a}_n\gamma^5\gamma^5] \\
\text{as traces are cyclic} & \\
\text{So } Tr[\not{a}_1\ \not{a}_2\cdots\ \not{a}_n &= (-1)^n\,Tr[\not{a}_1\ \not{a}_2\cdots\ \not{a}_n]
\end{aligned}$$

So as n is odd, the Trace is equal to zero.

8.3.10 $Tr[\gamma^5\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma]$

First consider $Tr[\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3]$:

$$\begin{aligned}
Tr[\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3] &= \frac{-1}{i}Tr\left[\left(\gamma^5\right)^2\right] \\
&= 4i
\end{aligned}$$

$$\begin{aligned}
Tr[\gamma^5\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma] &= 4i\epsilon_{\mu\nu\delta\sigma} \\
\text{where } \epsilon_{\mu\nu\delta\sigma} &= \begin{cases} 1 & \text{even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

8.4 Electron-muon scattering

Consider electrons and muons scattering, taking spin into account. The calculation of the cross-section can be used to predict the cross-section of other similar processes.

8.4.1 An electron in an electromagnetic field

The Dirac equation is:

$$(\alpha \cdot p + \beta m)\,\psi = E\psi$$

Now substitute $p^\mu \rightarrow p^\mu + \mathrm{e}A^\mu$

$$\begin{aligned} \text{then } E &\rightarrow E + \mathrm{e}V \\ p^k &\rightarrow p^k + \mathrm{e}A^k \end{aligned}$$

$$\alpha_k p^k + \beta m + \mathrm{e}\left(\alpha_k A^k - VI\right)\psi = E\psi$$

The amplitude for the scattering of an electron from a state ψ_i to ψ_f is:

$$\begin{aligned} T_{fi} &= -i\int \mathrm{d}^4x\psi_f^\dagger V_{DIRAC}\psi_i \\ \text{where } V_{DIRAC} &= \mathrm{e}\left(\alpha_k A^k - VI\right) \\ T_f i &= -i\mathrm{e}\int \mathrm{d}^4x\psi_f^\dagger\left(-A_0 I + \alpha^k A_k\right)\psi_i \\ &= -i\mathrm{e}\int \mathrm{d}^4x\psi_f^\dagger\gamma^0\gamma^0\left(-A_0 I + \alpha^k A_k\right)\psi_i \\ &= -i\mathrm{e}\int \mathrm{d}^4x\bar{\psi}_f\left(-\gamma^0 A_0 I + \gamma^k A_k\right)\psi_i \\ &= i\mathrm{e}\int \mathrm{d}^4x\bar{\psi}_f\gamma^\mu A_\mu\psi_i \\ &= -i\int j^\mu_{fi} A_\mu \mathrm{d}^4x \\ \text{where } j^\mu_{fi} &= -\mathrm{e}\bar{\psi}_f\gamma^\mu\psi_i \\ &= -\mathrm{e}\bar{u}_f\gamma^\mu u_i \mathrm{e}^{i(p_f-p_i)x} \end{aligned}$$

This is the electromagnetic transition current between states i and f. Recall for a spinless electron:

$$j^\mu_{fi} = -\mathrm{e}\left(p_f + p_i\right)^\mu \mathrm{e}^{i(p_f-p_i)x}$$

Consider the following diagram:

(Feynmann diagram of scattering)
The transition amplitude is then:

$$\begin{aligned}
T_{fi} &= -i\int j^1_\mu\left(\frac{-1}{q^2}\right)j_2^\mu \mathrm{d}^4x \\
&= i\mathrm{e}\int \mathrm{d}^4x\bar{u}(k')\mathrm{e}^{ik'x}\gamma_\mu u(k)\mathrm{e}^{-ikx}\left(\frac{-1}{q^2}\right)(-\mathrm{e})\,\bar{u}(p')e^{ip'x}\gamma^\mu u(p)\mathrm{e}^{-ipx} \\
&= \frac{i\mathrm{e}^2}{q^2}\int \mathrm{d}^4x\mathrm{e}^{i\left(k'+p'-k-p\right)}\Big[\bar{u}\left(k'\right)\gamma_\mu u\left(k\right)\Big]\Big[\bar{u}\left(p'\right)\gamma^\mu u\left(p\right)\Big]
\end{aligned}$$

As before for $|T_{fi}|^2$, one exponential term becomes the phase-space factor and the second becomes the product of the volume and time.

$$|T_{fi}|^2 = \frac{\mathrm{e}^4}{q^4}\Big[\bar{u}\left(k'\right)\gamma_\mu u\left(k\right)\Big]\Big[\bar{u}\left(p'\right)\gamma^\mu u\left(p\right)\Big]\Big[\bar{u}\left(k'\right)\gamma_\mu u\left(k\right)\Big]^\dagger\Big[\bar{u}\left(p'\right)\gamma^\mu u\left(p\right)\Big]^\dagger$$

The Hermitian conjugates are:

$$\begin{aligned}
\left(u^\dagger(p')\gamma^0\gamma^\mu u(p)\right)^\dagger &= u^\dagger(p)\gamma^{\mu\dagger}\gamma^{0\dagger}u(p') \\
&= u^\dagger(p)\gamma^{\mu\dagger}\gamma^0 u(p') \\
&= -u^\dagger(p)\gamma^\mu\gamma^0 u(p') \\
&= u^\dagger(p)\gamma^0\gamma^\mu u(p') \\
&= \bar{u}(p)\gamma^\mu u(p')
\end{aligned}$$

and similarly for the other term.

$$\Rightarrow |T_{fi}|^2 = \frac{\mathrm{e}^4}{q^4}\Big[\bar{u}(k')\gamma_\mu u(k)\Big]\Big[\bar{u}(k)\gamma_\nu u(k')\Big]\Big[\bar{u}(p')\gamma^\mu u(p)\Big]\Big[\bar{u}(p)\gamma^\nu u(p')\Big]$$

$$\begin{array}{ll}
\Big(\bar{u}(k')\gamma_\mu u(k)\Big]\Big(\bar{u}(k)\gamma_\nu u(k')\Big] & \text{is the electron tensor} \\
\Big[\bar{u}(p')\gamma_\mu u(p)\Big]\Big[\bar{u}(p)\gamma_\nu u(p')\Big] & \text{is the muon tensor}
\end{array}$$

$$|T_{fi}|^2 = \frac{\mathrm{e}^4}{q^4}{}^{\mathrm{e}}L_{\mu\nu}{}^{\mu}L^{\mu\nu}$$

In order to calculate the transition amplitude correctly the amplitude must be summed over all initial states, summed over all final spin states and averaged over all initial spin states:

$${}^{\mathrm{e}}L_{\mu\nu} = \frac{1}{2}\sum_S\sum_{S'}\bar{u}(k')\gamma_\mu u(k)\bar{u}(k)\gamma_\nu u(k')$$

Writing this explicitly in terms of individual matrix elements, α, β, γ, δ:

$$^{\mathrm{e}}L_{\mu\nu} = \frac{1}{2}\sum_S\sum_{S'}\bar{u}(k')_\alpha\gamma_\mu^{\alpha\beta}u(k)_\beta\bar{u}(k)_\gamma\gamma_\nu^{\gamma\delta}u(k')_\delta$$

where the factor of 1/2 is due to the averaging over initial spin states. These values are all elements of tensors, so they can be reordered:

$$\begin{aligned}
^{\mathrm{e}}L_{\mu\nu} &= \frac{1}{2}\sum_S\sum_{S'}u(k')_\delta\bar{u}(k')_\alpha\left(\gamma_\mu^{\alpha\beta}\right)u(k)_\beta\bar{u}(k)_\gamma\gamma_\nu^{\gamma\delta} \\
&= \sum_S\sum_{S'}\left(\not{k}'+m\right)_{\delta\alpha}\left(\gamma_\mu^{\alpha\beta}\right)\left(\not{k}+m\right)_{\beta\gamma}
\end{aligned}$$

So $^{\mathrm{e}}L_{\mu\nu}$ is reduced to the trace of the product of four 4×4 matrices:

$$\begin{aligned}
^{\mathrm{e}}L_{\mu\nu} &= \frac{1}{2}Tr\Big[\left(\not{k}'+m\right)(\gamma_\mu)\left(\not{k}+m\right)(\gamma_\nu)\Big] \\
_{\mu}L_{\mu\nu} &= \frac{1}{2}Tr\Big[\left(\not{p}'+m\right)(\gamma_\mu)\left(\not{p}+m\right)(\gamma_\nu)\Big]
\end{aligned}$$

Denoting the electron mass by m and the muon mass by M gives:

$$\begin{aligned}
|T_{fi}|^2 &= \frac{\mathrm{e}^4}{q^4}\frac{1}{2}Tr\Big[\left(\not{k}'+m\right)\gamma_\mu\left(\not{k}+m\right)\gamma_\nu\Big]\frac{1}{2}Tr\Big[\left(\not{p}'+M\right)\gamma^\mu\left(\not{p}+M\right)\gamma^\nu\Big] \\
&= \frac{\mathrm{e}^4}{4q^4}Tr\Big[\left(\not{k}'+m\right)\gamma_\mu\left(\not{k}+m\right)\gamma_\nu\Big]Tr\Big[\left(\not{p}'+M\right)\gamma^\mu\left(\not{p}+M\right)\gamma^\nu\Big]
\end{aligned}$$

The only non-zero terms are terms involving two or four γ matrices. eg $\left(\not{k}'\gamma_\mu m\gamma_\nu\right)=0$.

$$\begin{aligned}
\text{So } |T_{fi}|^2 &= \frac{\mathrm{e}^4}{4q^4}Tr[\gamma_\delta\gamma_\mu\gamma_\sigma\gamma_\nu k'^\delta k^\sigma+\gamma_\mu\gamma_\nu m^2]Tr[\gamma^\delta\gamma^\mu\gamma^\sigma\gamma^\nu p'_\delta p_\sigma+\gamma^\mu\gamma^\nu M^2] \\
&= \frac{\mathrm{e}^4}{q^4}\left((g_{\delta\nu}g_{\nu\sigma}-g_{\delta\sigma}g_{\nu\mu}+g_{\delta\mu}g_{\sigma\nu})\,k'^\delta k^\sigma+g^{\mu\nu}m^2\right) \\
&\quad \times4\left(\left(g^{\delta\nu}g^{\nu\sigma}-g^{\delta\sigma}g^{\nu\mu}+g^{\delta\mu}g^{\sigma\nu}\right)p'_\delta p_\sigma+g_{\mu\nu}M^2\right) \\
&\quad \text{Multiplying out:} \\
|T_{fi}|^2 &= \frac{8\mathrm{e}^4}{q^4}\left((k'\cdot p')(k\cdot p)+(k'\cdot p)(k\cdot p')-m^2(p'\cdot p)-M^2(k'\cdot k)+m^2M^2\right)
\end{aligned}$$

In the relativistic limit the m^2 and M^2 terms can be neglected.

$$\Rightarrow |T_{fi}|^2 = \frac{8\mathrm{e}^4}{q^4}\left((k'\cdot p')(k\cdot p)+(k'\cdot p)(k\cdot p)\right)$$

Expressing this in terms of the Mandelstran variables gives:

$$
\begin{aligned}
s &= (k+p)^2 \\
&\sim 2k\cdot p \\
&\sim 2k'\cdot p' \\
q^2 &= t \\
&= (k-k')^2 \\
&= (p-p')^2 \\
&\sim -2k\cdot k' \\
&\sim 2p\cdot p' \\
u &= (k-p')^2 \\
&\sim -2p'\cdot k \\
&\sim -2k'\cdot p \\
\text{So } |T_{fi}|^2 &= \frac{8\mathrm{e}^4}{t^2}\left(\frac{s}{2}\frac{s}{2} + \left(\frac{-u}{2}\right)\left(\frac{-u}{2}\right)\right) \\
&= \frac{2\mathrm{e}^4}{t^2}\left(s^2+u^2\right) \\
&= 2\mathrm{e}^4\left(\frac{s^2+u^2}{t^2}\right) \\
\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}|T_{fi}|^2 \\
&= \frac{\mathrm{e}^4}{32\pi^2 s}\left(\frac{s^2+u^2}{t^2}\right)
\end{aligned}
$$

8.5 Cross section for $\mathrm{e^+e^-} \to \mu^+\mu^-$

This cross section can be easily derived from $e\mu \to e\mu$ scattering.

Comparing the vertices:

$$
\begin{array}{ccc}
\text{For } I & k \to k' & \text{in } e\mu \to e\mu \\
 & k \to -p & \text{in } e^+e^- \to \mu^+\mu^- \\
\text{For } II & p \to p' & \text{in } e\mu \to e\mu \\
 & -k' \to p' & \text{in } e^+e^- \to \mu^+\mu^-
\end{array}
$$

So between the two processes there is an interchange of k' with $-p$ and p with $-k'$.

Recall for $e^-\mu^-$ scattering:

$$
|T_{fi}|^2 = \frac{8\mathrm{e}^4}{q^4}\left((k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p')\right)
$$

$$\text{and } q^4 = 4\left(k \cdot k\right)^2$$

So $|T_{fi}|^2$ for e^+e^- is:

$$\begin{aligned}
|T_{fi}|^2 &= \frac{8\mathrm{e}^4}{4\left(k \cdot p\right)^2}\left(\left(p \cdot p'\right)\left(k \cdot k'\right) + \left(p \cdot k'\right)\left(k \cdot p\right)\right) \\
&= \frac{2\mathrm{e}^4}{\left(k \cdot p\right)^2}\left(\left(p \cdot p'\right)\left(k \cdot k'\right) + \left(p \cdot k'\right)\left(k \cdot p'\right)\right) \\
&= \frac{2\mathrm{e}^4}{\left(\frac{s}{2}\right)^2}\left(\left(\frac{-t}{2}\right)\left(\frac{-t}{2}\right) + \left(\frac{-u}{2}\right)\left(\frac{-u}{2}\right)\right) \\
\Rightarrow |T_{fi}|^2_{e^+e^-} &= 2\mathrm{e}^4\left(\frac{t^2+u^2}{s^2}\right) \\
\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{e^+e^-} &= \frac{1}{64\pi^2} \times 2\mathrm{e}^4\left(\frac{t^2+u^2}{s^2}\right) \\
&= \frac{\mathrm{e}^4}{32\pi^2}\left(\frac{t^2+u^2}{s^2}\right)
\end{aligned}$$

In comparison with $e\mu \to e\mu$, the values of s and t are simply interchanged.

8.5.1 Total cross-section

To find the total cross-section, integrate with respect to Ω and evaluate s, t and u in terms of the energy and angle of scattering in the centre of mass frame:

$$\begin{aligned}
s &\sim 2k \cdot p \\
&= 4E_{e^-}E_{e^+} \\
t^2 &= 4\left(k \cdot k'\right)^2 \\
&= 4\left(E_{e^-}E_{\mu^+} - p_{e^-} \cdot p_{\mu^+}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= 4E_{e^-}E_{\mu^+}\left(1-\cos\theta\right)^2 \\
u^2 &= 4\left(k\cdot p'\right)^2 \\
&= 4E_{e^-}E_{\mu^-}\left(1+\cos\theta\right)^2 \\
\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}2\mathrm{e}^4\frac{4\left(E_{e^-}E_{\mu^+}\left(1-\cos\theta\right)^2+E_{e^-}E_{\mu^-}\left(1+\cos\theta\right)^2\right)}{16E_{e^+}E_{e^-}}
\end{aligned}
$$

But in the centre of mass system:

$$E_{e^-}=E_{e^+}=E_{\mu^-}=E_{\mu^+}$$

$$
\begin{aligned}
\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\mathrm{e}^4\left(1+\cos^2\theta\right) \\
\sigma &= \int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega \\
\mathrm{d}\Omega &= 2\pi\mathrm{d}\left(\cos\theta\right) \\
\alpha &= \frac{\mathrm{e}^2}{4\pi} \\
\Rightarrow \sigma &= \int_0^{\pi}\frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)2\pi\mathrm{d}\left(\cos\theta\right) \\
&= \frac{2\pi\alpha^2}{4s}\int_1^{-1}\left(1+x^2\right)\mathrm{d}x \\
(\text{as } x &= \cos\theta) \\
&= \frac{2\pi\alpha^2}{4s}\Big[-\cos\theta-\frac{1}{3}\cos^3\theta\Big]_0^{\pi} \\
&= \frac{4\pi\alpha^2}{3s}
\end{aligned}
$$

The annihilation process falls off as a function of $1/s$. Had it not been for $q\bar{q}$ resonances, eg Z^0, particle physics would have become rather uninteresting.

Note that using similar methods it is possible to calculate Bhabha scattering.

8.6 The ratio R at $\mathrm{e}^+\mathrm{e}^-$ colliders

$$R=\frac{\sigma(e^+e^-\to hadrons)}{\sigma(e^+e^-\to\mu^+\mu^-)}$$

At low energies e^+e^- can annihilate into systems containing u or d quarks with quarks subsequently hadronising. They can annihilate through the virtual photon to make $q\bar{q}$ resonances such as the $\rho(770)$. As the energy increases $s\bar{s}$, $c\bar{c}$ and $b\bar{b}$ states can be formed. At very high energies $t\bar{t}$ states can be formed, although such e^+e^- colliders have yet to be built.

Consider the contribution to R from on generation of $q\bar{q}$ pairs:

$$\begin{aligned} R &= \frac{\left(\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2\right) \times \mathrm{e}^2 \times 3}{\mathrm{e}^2} \\ &= \frac{15}{9} \text{ per generation} \end{aligned}$$

By the time the energy is $\sqrt{s} > 2m_b$, R should be be $\sim \frac{11}{3}$.

Superimposed are the resonances, such as the J/ψ and Υ family. There are resonances at $\rho(770)$, $J/\psi(3088)$ and $\Upsilon(10588)$. At Babar, the e^+e^- collider runs at the $\Upsilon(4S)$ resonances, which decays almost exclusively to $b\bar{b}$ pairs.

Although the simple model predicting $\frac{11}{3}$ for R for $\sqrt{s} > 2m_b$ gives a reasonable description of the data, this is not the complete picture. The actual value is somewhat higher due to gluon radiation in the final state of the hadronic system:

So there is a higher order correction to $R \sim \frac{11}{3}$.

At the Z^0 mass resonance, an analogous quantity is the ratio of the partial decay widths:

$$R_Z = \frac{\Gamma(Z^0 \rightarrow hadrons)}{\Gamma(Z^0 \rightarrow \mu^+\mu^-)}$$

To lowest order, $R_Z = 20.09$, however the measured value is 20.79 ± 0.04. This 3.5% discrepency is due entirely to higher order QCD corrections and gives a good way to measure α_s.

8.7 Helicity conservation at high energies

It is possible to gain further insight into cross-section calculations and their angular distributions by looking at the helicity of particles. The states are:

$$\begin{aligned}
U_L &= \frac{1}{2}\left(1-\gamma^5\right)u \\
U_R &= \frac{1}{2}\left(1+\gamma^5\right)u \\
\bar{U}_L &= U_L^\dagger \gamma^0 \\
&= \frac{1}{2}\left(U^\dagger\right)\left(1-\gamma^5\right)^\dagger \gamma^0 \\
&= \frac{1}{2}U^\dagger \left(1-\gamma^5\right)\gamma^0 \\
&= \frac{1}{2}U^\dagger \gamma^0 \left(1+\gamma^5\right) \\
&= \frac{1}{2}\bar{U}\left(1+\gamma^5\right)
\end{aligned}$$

At high energies the electromagnetic interaction conserves helicity.

Consider the electromagnetic current:

$$\begin{aligned}
\bar{u}\gamma^\mu u &= \left(\bar{U}_L+\bar{U}_R\right)\gamma^\mu\left(U_L+U_R\right) \\
\bar{U}_L\gamma^\mu U_R &= \frac{1}{2}\bar{u}\left(1+\gamma^5\right)\gamma^\mu \frac{1}{2}\left(1+\gamma^5\right)u \\
&= \frac{1}{4}\bar{u}\gamma^\mu\left(1-\gamma^5\right)\left(1+\gamma^5\right)u \\
&= \frac{1}{4}\bar{u}\gamma^\mu\left(1-\left(\gamma^5\right)^2\right)u \\
&= 0
\end{aligned}$$

Helicity conservation requires that the incoming electron and positron have opposite helicities, as do the outgoing muons.

In the centre of mass system:

The reaction proceeds via a photon of spin-1 so the amplitudes are proportional to the rotation matrices:

$$d^j_{\lambda\lambda'}(\theta) = \langle j\lambda'|e^{-i\theta J_y}|j\lambda\rangle$$

where y is perpendicular to the reaction plane. The rotation matrices can be calculated using angular momentum theory.

$$\begin{array}{ccccccc} d^1_{1,1}(\theta) & = & d^1_{-1,-1}(\theta) & = & \frac{1}{2}(1+\cos\theta) & \sim & -\frac{u}{s} \\ d^1_{1,-1}(\theta) & = & d^1_{-1,1}(\theta) & = & \frac{1}{2}(1-\cos\theta) & \sim & -\frac{t}{s} \end{array}$$

Squaring and adding the above:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto \frac{t^2+u^2}{s^2}$$

Chapter 9

Massless spin-1 particles (photons)

9.1 Maxwell's equations and the definition of classical potentials

$$
\begin{array}{cccc}
I & \underline{\nabla} \cdot \underline{E} & = & \rho \\
II & \underline{\nabla} \cdot \underline{B} & = & 0 \\
III & \underline{\nabla} \times \underline{E} & = & -\dot{\underline{B}} \\
IV & \underline{\nabla} \times \underline{B} & = & \underline{J} + \dot{\underline{E}}
\end{array}
$$

The potentials are defined as:

$$
\begin{aligned}
\underline{B} &= \underline{\nabla} \times \underline{A} \\
\underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{\nabla} \times \underline{A}}{\partial t} \\
&= -\underline{\nabla} \times \left(\frac{\partial \underline{A}}{\partial t} \right) \\
\text{So } \underline{\nabla} \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t} \right) &= \underline{0}
\end{aligned}
$$

The solution is:

$$\underline{E} + \frac{\partial \underline{A}}{\partial t} = -\underline{\nabla} \phi$$

as the gradient of a scalar function has zero curl everywhere.
I and IV give:

$$\begin{aligned}
\underline{\nabla}\cdot\underline{E} &= \rho \\
\underline{E} &= -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \\
\Rightarrow -\nabla^2\phi - \frac{\partial}{\partial t}\left(\underline{\nabla}\cdot\underline{A}\right) &= \rho \\
\text{So } \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{A} &= \rho \qquad (9.1)\\
\underline{\nabla}\times\underline{B} &= \underline{J} + \frac{\partial \underline{E}}{\partial t} \\
\underline{\nabla}\times\underline{\nabla}\times\underline{A} &= \underline{J} + \frac{\partial \underline{E}}{\partial t} \\
&= \underline{J} + \frac{\partial}{\partial t}\left(-\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}\right) \\
&= \underline{J} - \underline{\nabla}\frac{\partial\phi}{\partial t} - \frac{\partial^2\underline{A}}{\partial t^2} \\
\text{or } \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A}\right) - \nabla^2\underline{A} &= \underline{J} - \underline{\nabla}\frac{\partial\phi}{\partial t} - \frac{\partial^2\underline{A}}{\partial t^2} \\
\Rightarrow \nabla^2\underline{A} - \frac{\partial^2\underline{A}}{\partial t^2} - \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A}\right) - \frac{\partial}{\partial t}\underline{\nabla}\phi &= \underline{J} \qquad (9.2)\\
& \qquad (9.3)
\end{aligned}$$

Equations (21) and (22) can be written as:

$$\begin{aligned}
\Box^2 A^\mu - \partial^\mu\partial_\nu A^\nu &= J^\mu \\
\text{where } \Box^2 &= \left(\frac{\partial^2}{\partial t^2}, -\nabla^2\right) \\
A^\mu &= (\phi, \underline{A}) \\
J^\mu &= (\rho, \underline{J})
\end{aligned}$$

Therefore the electromagnetic field is given by the four-potential A^μ which satisfies the above equation. For a free electromagnetic field $(\rho, \underline{J}) = 0$.

Consider the polarisation states for a free photon. Since there is a four-potential there appears to be four polarisation states. These states reduce to the well known two polarisations states of the free photon. The four states for a photon travelling in the $z-$direction are:

$\|1,0,0,0\rangle$	time-like polarisation
$\|0,1,0,0\rangle$	polarisation in the x direction
$\|0,0,1,0\rangle$	polarisation in the y direction
$\|0,0,0,1\rangle$	polarisation in the z direction

For virtual photons all four polarisation states exist, whereas for real photons only tranverse polarisation states exist.

Applying the Lorentz condition:

$$\begin{aligned}\partial_\mu A^\mu &= 0\\ \Box^2 A^\mu &= J^\mu\end{aligned}$$

This makes the time-like component depend on the spatial components so that the time-like component is no longer independent. For free photons $J^\mu = 0^\mu$, so $\Box^2 A^\mu = 0^\mu$ and the solutions are plane waves:

$$A^\mu = \epsilon_i^\mu \mathrm{e}^{-iqx}$$

where ϵ_i are the four polarisation states.

The Lorentz condition gives:

$$\begin{aligned}\partial_\mu A^\mu &= \partial_\mu \epsilon_i^\mu \mathrm{e}^{-iq_\mu x^\mu}\\ &= 0\\ \Rightarrow -iq_\mu \epsilon_i^\mu \mathrm{e}^{-iq_\mu x^\mu} &= 0\end{aligned}$$

So the Lorentz condition reduces to:

$$\begin{aligned}q_\mu \epsilon_i^\mu &= 0\\ \Rightarrow q_0 \epsilon_i^0 &= q_k \epsilon_i^k\end{aligned}$$

So the time-like component is dependent on the space-like components. To reduce to two polarisation vector a gauge transformation is applied. Recall that $\underline{E}$ and $\underline{B}$ in classic electromagnetism come from the field tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$F^{\mu\nu}$ is unchanged and this $\underline{E}$ and $\underline{B}$ are unchanged under the gauge transformation:

$$A'^\mu \rightarrow A^\mu + \partial^\mu \Lambda$$

where Λ is a scalar field. Λ satisfies the Lorentz condition.

$$\begin{aligned}\text{Let } \Lambda &= ia\mathrm{e}^{-iq_\mu x^\mu}\\ \partial_\mu \partial^\mu \Lambda &= (-iq_\mu)(-iq^\mu)\, ia\mathrm{e}^{-iq_\mu x^\mu}\\ &= -iaq^2 \mathrm{e}^{-iqx}\end{aligned}$$

But $q^2 = E^2 - p^2$ and is equal to zero for a real photon, so $\partial^\mu_\mu = 0$ and the Lorentz condition is satisfied. Substituting Λ and $A^\mu = \epsilon^\mu \mathrm{e}^{-iq_\mu x^\mu}$ into the gauge tranformation gives:

$$\begin{aligned} A'^\mu &\rightarrow \epsilon^\mu \mathrm{e}^{-iqx} + (-iq^\mu)\, ia\mathrm{e}^{-iqx} \\ &= \epsilon^\mu \mathrm{e}^{-iqx} + aq^\mu \mathrm{e}^{-iqx} \end{aligned}$$

So the gauge transformation simplifies to:

$$\epsilon'^\mu \rightarrow \epsilon^\mu + aq^\mu$$

So two polarisation vectors ϵ and ϵ'^μ which differ by a multiple of q^μ, describe the same photon. This means the time component must be zero. $\epsilon_0 = 0$.

So the Lorentz condition reduces to $\underline{\epsilon} \cdot \underline{q} = 0$. From this only two independent polarisation vectors can exist and they must be perpendicular to $\underline{q}$. So the states are:

$$\begin{aligned} \underline{\epsilon}_1 &= (1, 0, 0) \\ \underline{\epsilon}_2 &= (0, 1, 0) \end{aligned}$$

They can also be expressed as circular polarisation:

$$\begin{aligned} \underline{\epsilon}_R &= \frac{1}{\sqrt{2}} (\underline{\epsilon}_1 + i\underline{\epsilon}_2) \\ \underline{\epsilon}_L &= \frac{1}{\sqrt{2}} (\underline{\epsilon}_1 - i\underline{\epsilon}_2) \end{aligned}$$

9.1.1 Virtual photons and the photon propagator

For virtual photons, by imposing the Lorentz condition:

$$\begin{aligned} \Box^2 A^\mu &= J^\mu \\ &= g^{\mu\nu} J_\nu \\ \text{By inspection: } A^\mu &= -\frac{g^{\mu\nu}}{q^2} J_\nu \end{aligned}$$

The solution can be derived by the propagator approach:

$$A^\mu(x') = \int G(x'; x) j^\mu(x) \mathrm{d}^4 x \tag{9.4}$$

From the Lorentz condition:

$$\begin{aligned}
\Box^2 A^\mu(x') &= j^\mu(x') \\
&= \int \mathrm{d}^4x\delta^4(x'-x)j^\mu(x) \\
\text{Also from (24):}\Box^2 A^\mu(x') &= \int \Box^2 G(x',x)j^\mu(x)\mathrm{d}^4x
\end{aligned}$$

Comparing the expressions for $\Box^2 A^\mu(x')$:

$$\Box^2 G(x',x) = \delta^4(x'-x)$$

Translating into four-momentum space via a Fourier transform:

$$\begin{aligned}
\frac{1}{(2\pi)^4}\int \mathrm{d}^4q\,(-iq)^2\,G(q)\mathrm{e}^{-iq(x-x')} &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4q\mathrm{e}^{-iq(x'-x)} \\
\Rightarrow G(q) &= -\frac{1}{q^2} \\
\text{So } A^\mu(x) &= -\frac{j^\mu(x)}{q^2} \\
&= -\frac{-g^{\mu\nu}j_\nu(x)}{q^2}
\end{aligned}$$

When considering the propagator approach theory it is found that the propagator in four-momentum space is the inverse of the equation describing the free propagation of virtual particles. In four-momentum space the propagator for a Klein-Gordon particle is obtained by inverting the Klein-Gordon equation, multiplied by i.

$$i\left(\Box^2+m^2\right)\phi = -iV\phi$$

So the Klein-Gordon propagator is:

$$\begin{aligned}
\frac{1}{i\left(\Box^2+m^2\right)} &= \frac{-i}{\Box^2+m^2} \\
\Box^2 &= \partial_\mu\partial^\mu \\
&= \frac{i\partial_\mu i\partial^\mu}{i^2} \\
&= -p^\mu p_\mu \\
&= -p^2
\end{aligned}$$

So the propagator is $\dfrac{i}{p^2-m^2}$.

Consider the Dirac equation:

$$\Big[(\alpha p + \beta m) + \mathrm{e}\left(\alpha A - A^0 I\right) \Big]\psi = E\psi$$

where the Dirac potential is $\alpha A - A^0 I$.
Converting this equation to convariant form by premultiplying by β gives:

$$\Big[\beta\alpha p + \beta^2 m + \mathrm{e}\left(\beta\alpha A - \beta A^0 I\right) \Big]\psi = \beta E\psi$$

Rearranging:

$$\begin{aligned}
\left(\beta E - \beta\alpha p - \beta^2 m\right)\psi &= \mathrm{e}\left(\beta\alpha A - \beta A^0\right)\psi \\
\left(\gamma^0 E - \gamma^k p_k - Im\right)\psi &= -\mathrm{e}\left(\beta A^0 - \beta\alpha A\right)\psi \\
\left(\not{p} - m\right)\psi &= -\mathrm{e}\ \not{A}\psi \\
\Rightarrow -i\left(\not{p} - m\right)\psi &= i\mathrm{e}\ \not{A}\psi \\
&= -iV\psi
\end{aligned}$$

So the propagator is:

$$\begin{aligned}
\frac{1}{-i\left(\not{p} - m\right)} &= \frac{i\left(\not{p} + m\right)}{\left(\not{p} - m\right)\left(\not{p} + m\right)} \\
&= \frac{i\left(\not{p} + m\right)}{\not{p}^2 - m^2} \\
&= \frac{i\left(\not{p} + m\right)}{p^2 - m^2} \\
&= \frac{i\sum_{spins} u\bar{u}}{p^2 - m^2} \text{via the completeness relation}
\end{aligned}$$

This is the general form of the propagator of a virtual particle where the sum is over eg all spin states of the electron or polarisation states of the photon.

9.1.2 Real and virtual photons and the significance of longitudinal and time-like polarisations

Consdier a typical process involving photon exchange ie the photon is sandwiched between two currents:

$$\begin{aligned}
j^A_\mu(x)\left(\frac{-g^{\mu\nu}}{q^2}\right) j^B_\nu(x) &= -j^A_\mu(x)\frac{1}{q^2} j^{B\mu}(x) \\
&= \frac{1}{q^2}\Big[j^A_1(x) j^{1B}(x) + j^A_2(x) j^{2B}(x) + j^A_3(x) j^{3B}(x) - j^A_0(x) j^{0B}(x) \Big]
\end{aligned}$$

However, electromagnetic current is conserved:

$$\partial_\mu j^\mu = 0 \Rightarrow q_\mu j^\mu = 0$$

Proof Consider $j^\mu = \bar{u}_f e^{ip_f x}\gamma^\mu u_i e^{-ip_i x}$. Then if q is the momentum of the exchanged photon and $q = p_f - p_i$ then:

$$\partial_\mu j^\mu \propto i\left(p_f - p_i\right)$$

Therefore if $\partial_\mu j^\mu = 0$ then $q_\mu j^\mu = 0$.
Since q can be taken to be parallel to the x^3 axis without loss of generality:

$$q_3 j^1 = q_3 j^2 = 0$$

Applying the condition $q_\mu j^\mu = 0$ to the longitudinal and time-like components:

$$\begin{aligned} q_\mu j^\mu &= q_0 j^0 - q_3 j^3 \\ \Rightarrow j^3 &= \frac{q_0 j^0}{q_3} \end{aligned}$$

Substituting this back into the amplitude:

$$\begin{aligned} \frac{1}{q_3^2 q^2} q_0^2 j_0^A j^{0B} - \frac{1}{q^2} j_0^A j^{0B} &= \frac{1}{q^2} j_0^A(x) j^{0B}(x) \left(\frac{q_0^2 - q_3^2}{q_3^2}\right) \\ \text{but } q^2 &= q_0^2 - q_3^2 \\ \text{so } \textit{amplitude} &= \frac{j_0^A(x) j^{0B}(x)}{q_3^2} \end{aligned}$$

which is Coulomb's law in three-momentum space.
The completeness relation for real photons is:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} (10) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (01) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

However the same completeness relation can be used as for virtual photons.

$$\text{ie} - g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the $\{\cdots\}$ denotes the real parts of the polarisation.
The generalised form of virtual photons $(-g^{\mu\nu})$ is used for the completeness relation for virtual photons.

Chapter 10

Massive spin-1 particles

For a massless photon the four-potential satisfies:

$$\Box^2 A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu$$

For a free particle $j^\mu = 0$

$$\begin{aligned} \text{so } \Box^2 A^\mu - \partial^\mu \partial_\nu &= 0 \\ \left(\frac{\partial^2}{\partial t^2}\right) A^\mu - \partial^\mu \partial_\nu A^\nu &= 0 \\ \left(E^2 + p^2\right) A^\mu - \partial^\mu \partial_\nu A^\nu &= 0 \end{aligned}$$

For massive particles $E^2 = p^2 + m^2$, so a massive particle satisfies:

$$\Box^2 A^\mu + m^2 A^\mu - \partial^\mu \partial_\nu A^\nu = \begin{cases} 0 & \text{real} \\ j^\mu & \text{virtual} \end{cases}$$

This the Proca equation. Differentiating with respect to ∂_μ:

$$\begin{aligned} \partial_\mu \Box^2 A^\mu + m^2 \partial_\mu A^\mu - \partial_\mu \partial^\mu \partial_\nu A^\nu &= \partial_\mu j^\mu \\ &= 0 \\ \partial_\mu \partial_\mu \partial^\mu A^\mu + m^2 \partial_\mu A^\mu - \partial_\mu \partial^\mu \partial_\nu A^\nu & \\ \text{so } m^2 \partial_\mu A^\mu &= 0 \quad (m^2 \neq 0) \end{aligned}$$

Therefore the (free) Proca equation field satisfies the Lorentz condition. The polarisation vectors for free massive vector bosons is:

$$A^\mu = \epsilon^\mu \mathrm{e}^{-ip_\mu x^\mu}$$

As $\partial_\mu A^\mu = 0$, $ip_\mu \epsilon^\mu = 0$.
In the rest frame:

$$\begin{aligned}
p^\mu &= (m,0,0,0) \\
\epsilon_1 &= (0,1,0,0) \\
\epsilon_2 &= (0,0,1,0) \\
\epsilon_3 &= (0,0,0,1)
\end{aligned}$$

then $p^\mu \epsilon_\mu = 0$.

Consider the polarisation states when m is boosted along z. ϵ_1 and ϵ_2 remain unchanged as they are perpendicular to the boost. The particle is now described by the four-vector:

$$(E,0,0,-p_z)$$

It is possible to determine ϵ_3 by requiring the Lorentz condition.

$$\begin{aligned}
\Rightarrow (p_z,0,0,E) \times \frac{1}{m}(E,0,0,-p) &= 0 \\
\text{So } \epsilon_3 &= \frac{1}{m}(p_z,0,0,E)
\end{aligned}$$

Consider the completeness relation for massive vector bosons:

$$\begin{aligned}
\sum_i \epsilon_i \epsilon_i^\star &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}(0100) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}(0010) + \frac{1}{m^2}\begin{pmatrix} p_z \\ 0 \\ 0 \\ E \end{pmatrix}(p_z 0 0 E) \\
&= \begin{pmatrix} \frac{p_z^2}{m^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{E^2}{m^2} \end{pmatrix} \\
&= -g^{\mu\nu} + \frac{p^\mu p^\nu}{m}
\end{aligned}$$

The 00 term is:

$$\begin{aligned}
-g^{00} + \frac{p^0 p^0}{m^2} &= -1 + \frac{E^2}{m^2} \\
&= \frac{E^2 - m^2}{m^2} \\
&= \frac{p_z^2}{m^2}
\end{aligned}$$

And the 33 term is:

$$
\begin{aligned}
-g^{33} + \frac{p^3 p^3}{m^2} &= 1 + \frac{p_z^2}{m^2} \\
&= \frac{p_z^2 + m^2}{m^2} \\
&= \frac{E^2}{m^2}
\end{aligned}
$$

Similarly it is possible to determine the polarisation vectors for virtual photons. Imposing the Lorentz condition removes the time-like polarisation state. For the virtual photon:

$$
\begin{aligned}
q^\mu &= (\nu, 0, 0, q_z) \\
q^2 &= q_\mu q^\mu \\
&= \nu^2 - q_z^2 \\
\Rightarrow q_z^2 &= \nu^2 - q^2 \\
&= \nu^2 + Q^2 \\
\Rightarrow q_z &= (\nu, 0, 0, \sqrt{\nu^2 + Q^2})
\end{aligned}
$$

In a deep elastic scattering experiment Q^2 and ν are known from the kinematics of the lepton vertex:

(Feynmann diagram of deep inelastic scattering)

Repearing the argument for the massive vector boson, the polarisation states for a virtual photon are:

$$
\begin{aligned}
\epsilon_1 &= (0, 1, 0, 0) \\
\epsilon_2 &= (0, 0, 1, 0) \\
\epsilon_3 &= \frac{1}{\sqrt{Q^4}} (\sqrt{\nu^2 + Q^2}, 0, 0, \nu)
\end{aligned}
$$

10.0.3 Massive virtual vector boson propagator

$$
\begin{aligned}
\left(\Box^2 + m^2\right) A^\mu - \partial^\mu \partial_\nu A^\nu &= j^\mu \\
\text{But } m^2 \partial_\nu A^\nu &= \partial_\mu j^\mu \\
\Rightarrow \partial_\nu A^\nu &= \frac{1}{m^2} \partial_\mu j^\mu \\
\Rightarrow \left(\Box^2 + m^2\right) A^\mu - \frac{\partial^\mu}{m^2} \partial_\mu j^\mu &= j^\mu \\
\Rightarrow \left(\Box^2 + m^2\right) A^\mu &= \frac{\partial^\mu}{m^2} \partial_\nu j^\nu + g^{\mu\nu} j_\nu \\
&= \frac{\partial^\mu}{m^2} \partial^\nu j_\nu + g^{\mu\nu} j_\nu
\end{aligned}
$$

$$
\begin{aligned}
&= \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m^2}\right) j_\nu \\
\text{using } \partial^\nu j_\nu &= (-iq) j^\nu
\end{aligned}
$$

So the propagator is:

$$\frac{g^{\mu\nu} - \frac{q^\mu q^\nu}{m^2}}{-q^2 + m^2}$$

The above expression is the propagator for the exchange of the massive soin−1 particle. However if the particle is formed in an annihilation process then it is a real particle which can decay. The propagator of the spin−1 particle in the s-channel is modified in the following manner:

$$\Gamma = \frac{1}{m^2} |T_{fi}|^2 p_f \frac{1}{32\pi^2} 4\pi$$

The quantum state of a decaying particle in the rest frame must be of the form:

$$\psi = \mathrm{e}^{-iMt} \mathrm{e}^{-\frac{\Gamma t}{2}}$$

such that:

$$\psi^\star \psi = \mathrm{e}^{-\Gamma t}$$

This suggests that for a decaying particle $-iM$ should be replaced with $-iM - \Gamma/2$ in the propagator.

$$
\begin{aligned}
\text{So } \textit{propagator} &= \frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2}}{q^2 - \left(m - \frac{i\Gamma}{2}\right)^2} \\
&\simeq \frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m}}{q^2 - m^2 + im\Gamma}
\end{aligned}
$$

Chapter 11

Compton scattering

Feynmann diagrams for the process $e\gamma \to e\gamma$ are:

These processes are closely related to e^+e^- annihilation:

Calculating the cross-section for Compton scattering is also useful for deriving the cross-section for the QCD Compton process.

Boson-gluon fusion (BGF):

The above diagrams cause structure functions to evole with Q^2 of the probe (in this case the photon).

$$
\begin{aligned}
s &= (k+p)^2 \\
t &= (k-k')^2 \\
u &= (k-p')^2
\end{aligned}
$$

This is a double scattering process. Recall:

$$
\begin{aligned}
T_{fi} &= -i\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \phi^\star(2)V(2)G_0(2,1)V(1)\phi(1) \\
&= -i\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2\, (\mathrm{e})\, \bar{u}\mathrm{e}^{ip'x_2}\mathrm{e}^{ik'x_2}\epsilon^\star_\nu\gamma^\nu \frac{1}{(2\pi)^4}\mathrm{e}^{-i(p+k)(x_2-x_1)} \\
&\quad \times i\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\,(-\mathrm{e})\,\epsilon_\mu \mathrm{e}^{-ikx_2}\gamma^\mu u\mathrm{e}^{-ipx_1}
\end{aligned}
$$

In the above expression the integration $\int \mathrm{d}^4x\mathrm{e}^{-i(p+k)(x_2-x_1)}$ gives $\delta^4(x_2-x_1)$, so x_2 and x_1 can be replaced with a dummy variable x. The exponential in terms of the four-momentum, combined with other terms gives the volume and time and the number of state vectors as before. Therefore T_{fi} can be more simply expressed as:

$$T_{fi} = -i\bar{u}(p')(-\mathrm{e})\epsilon^\star_\nu\gamma^\nu\left(\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\right)(-\mathrm{e})\epsilon_\mu\gamma^\mu u(p)$$

Assume $m \to 0$, then:

$$
\begin{aligned}
T_{fi} &= -\frac{i\mathrm{e}^2}{s}\bar{u}(p')\epsilon^\star_\nu\gamma^\nu\left(\not{p}+\not{k}\right)\epsilon_\mu\gamma^\mu u(p) \\
|T_{fi}|^2 &= \frac{\mathrm{e}^4}{s^2}\left(\epsilon^\star_{\mu'}\epsilon_{\nu'}\epsilon^\star_\nu\epsilon_\mu\bar{u}(p)\gamma^{\mu'}\left(\not{p}+\not{k}\right)\gamma^{\nu'}u(p')\bar{u}(p')\gamma^\nu\left(\not{p}+\not{k}\right)\gamma^\mu u(p)\right)
\end{aligned}
$$

This must be summed over initial and final spin states and averaged over the initial spin states, 1/4. For the sum over the initial photon polarisation states:

$$\sum \epsilon^\star_\mu\epsilon_{\mu'} = -g_{\mu\mu'}$$

The sum over the initial and final electron states is again performed by using the completeness relation for $u\bar{u}$ as in $e\mu$ scattering:

$$
\begin{aligned}
|T_{fi}|^2 &= \frac{\mathrm{e}^4}{4s^2}g_{\mu\mu'}g_{\nu\nu'}Tr[(\not{p}'+m)\gamma^{\mu'}\left(\not{p}+\not{k}\right)\gamma^{\nu'}\left(\not{p}+m\right)\gamma^\nu\left(\not{p}+\not{k}\right)\gamma^\mu] \\
&= \frac{\mathrm{e}^4}{4s^2}Tr[\gamma^\mu\ \not{p}'\gamma_\mu\left(\not{p}+\not{k}\right)\gamma_\nu\ \not{p}\gamma^\nu\left(\not{p}+\not{k}\right)] \\
&= \frac{\mathrm{e}^4}{s^2}Tr[\not{p}'\left(\not{p}+\not{k}\right)\ \not{p}\left(\not{p}+\not{k}\right)] \\
&= \frac{\mathrm{e}^4}{s^2}Tr[\not{p}'\ \not{k}\ \not{p}\ \not{k}] \\
\text{where}\quad \not{p}'\ \not{p} &= m_e^2 \text{ terms have been neglected} \\
|T_{fi}|^2 &= \frac{4\mathrm{e}^4}{s^2}\left((p'\cdot k)(p\cdot k)-(p'\cdot p)(k\cdot k)+(p'\cdot k)(k\cdot P)\right) \\
k\cdot k &= 0 \text{ as the photon is massless}
\end{aligned}
$$

$$
\begin{aligned}
\text{So } |T_{fi}|^2 &= \frac{2e^4}{s^2}\left(2(p'\cdot k)2(p\cdot k)\right) \\
&= \frac{2e^4}{s^2}(-us) \\
&= -2e^4\left(\frac{u}{s}\right)
\end{aligned}
$$

For the second diagram:

$$
|T_{fi}|^2 = -2e^4\left(\frac{s}{u}\right)
$$

For the above expressions, the matrix element is needed for the calculation of the inteference between the two diagrams.

$$
\begin{aligned}
T_{fi} &= -i\frac{e^2}{u}\bar{u}(p')\epsilon_\mu\gamma^\mu\left(\not p - \not k\right)\epsilon^\star_\nu\gamma^\nu u(p) \\
|T^I_{fi}+T^{II}_{fi}|^2 &= |T^I_{fi}|^2+|T^{II}_{fi}|^2+T^{\star I}_{fi}T^{II}_{fi}+T^{\star II}_{fi}T^I_{fi} \\
T^{\star II}_{fi}T^I_{fi} &= T^{\dagger II}_{fi}T^I_{fi} \\
&= \frac{e^2}{u}\epsilon_\nu\epsilon^\star_{\mu'}\bar{u}(p)\gamma^{\nu'}\left(\not p - \not k'\right)\gamma^{\mu'}u(p')\times\frac{e^2}{s}\epsilon^\star_\nu\epsilon_\mu\bar{u}(p')\gamma^\nu\left(\not p + \not k\right)\gamma^\mu u(p) \\
&= \frac{e^4}{us}\frac{1}{4}g_{\mu\mu'}g_{\nu\nu'}\sum_{\text{spins}}\sum\bar{u}(p)\gamma^{\nu'}\left(\not p - \not k'\right)\gamma^{\mu'}u(p')\bar{u}(p')\gamma^\nu\left(\not p + \not k\right)\gamma^\mu u(p) \\
&= \frac{e^4}{4us}Tr[\not p\gamma_\nu\left(\not p - \not k'\right)\gamma_\mu\ \not p'\gamma^\nu\left(\not p + \not k\right)\gamma^\mu] \\
&= \frac{e^4}{4us}Tr[-2\left(\not p - \not k'\right)\gamma_\nu\ \not p\ \not p'\gamma^\nu\left(\not p + \not k\right)] \\
(\text{using } \gamma_\mu\ \not a\ \not b\ \not c\gamma^\mu &= -2\ \not c\ \not b\ \not a) \\
&= -\frac{e^4}{2us}Tr[\not p\ \not p' 4(p+k)(p-k')] \\
&= -\frac{2e^4}{us}(p\cdot p')(p+k)(p-k') \\
&= 4e^4\frac{t}{su}(p\cdot p+k\cdot p-p\cdot k'-k\cdot k')
\end{aligned}
$$

$$\begin{aligned}
&= 4\mathrm{e}^4\frac{t}{su}\left(0+\frac{1}{2}s+\frac{1}{2}u+\frac{1}{2}t\right)\\
&= 2\mathrm{e}^4\frac{t}{su}(s+t+u)\\
\text{But } s+t+u &= 2(m_e^2+m_\gamma^2)\simeq 0
\end{aligned}$$

Thus the inteference terms for real photons scattering off nearly massless electrons does not contribute to the cross-section. However, if the incoming photon is virtual then $s+t+u=Q^2$ and for a photon of mass $k^2=Q^2$.

So for a real photon:

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}2\mathrm{e}^4\left(\frac{-u}{s}+\frac{-s}{u}\right)\\
&= \frac{\alpha^2}{2s}\left(\frac{-u}{s}+\frac{-s}{u}\right)\\
&\text{and for a virtual photon:}\\
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}2\mathrm{e}^4\left(\frac{-u}{s}+\frac{-s}{u}+\frac{2tQ^2}{su}\right)
\end{aligned}$$

The cross-section and $|T_{fi}|^2$ for e^+e^- annihilation are the same as for the above except that s, t and u are permutated. The Compton scattering cross-section in the limits $m_e\to 0$ and $s\to\infty$ is:

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}2\mathrm{e}^4\left(\frac{-u}{s}+\frac{-s}{u}\right)\\
\lim_{s\to\infty}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2}2\mathrm{e}^4\left(\frac{-1}{u}\right)\\
u &\simeq -2p\cdot k'\\
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2}2\mathrm{e}^4\frac{-1}{-2p\cdot k'}\\
&= \frac{1}{64\pi^2}\frac{\mathrm{e}^4}{p\cdot k'}
\end{aligned}$$

$p\cdot k'$ can be evaluated in the centre of mass system and using $E_e=\sqrt{p_e^2+m_e^2}$:

$$\begin{aligned}
p\cdot k' &= p_e E_\gamma\left(1+\cos\theta+\frac{1}{2}\frac{m_e^2}{p_e^2}+\cdots\right)\\
\Rightarrow\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2}\frac{\mathrm{e}^4}{\frac{s}{4}\left(1+\cos\theta+\frac{2m_e^2}{s}\right)}\\
\mathrm{d}\sigma &= \frac{\alpha^2}{s}2\pi\frac{\mathrm{d}\cos\theta}{1+\cos\theta+\frac{2m_e^2}{s}}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \sigma &= \frac{2\pi\alpha^2}{s} \int \frac{\mathrm{d}l}{l} \\
(\text{using } l &= 1 + \cos\theta + \frac{2m_e^2}{s}) \\
\sigma &= \frac{2\pi\alpha^2}{s} \ln \left[1 + \cos\theta + \frac{2m_e^2}{s} \right]_{-1}^{1} \\
&= \frac{2\pi\alpha^2}{s} \ln \left(\frac{2\left(1 + \frac{m_e^2}{s}\right)}{2\frac{m_e^2}{s}} \right) \\
&\simeq \frac{2\pi\alpha^2}{s} \ln \left(\frac{s}{m_e^2} \right)
\end{aligned}
$$