University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M4 (2003–2004)

1. Rewrite the equations in matrix form,

$$\frac{d^2\underline{x}}{dt^2} = \underline{A}\,\underline{x}\,,$$

where

$$\underline{A} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix} .$$ **[1]**

This has eigenvalues λ given by

$$\begin{vmatrix} (-\frac{5}{2}-\lambda) & \frac{3}{2} \\ \frac{3}{2} & (-\frac{5}{2}-\lambda) \end{vmatrix} = 0\,,$$

which has solutions $\lambda_1 = -1$ and $\lambda_2 = -4$. The normal modes therefore satisfy the uncoupled equations **[2]**

$$\frac{d^2y_1}{dt^2} + \;\; y_1 = 0\,,$$
$$\frac{d^2y_2}{dt^2} + 4y_2 = 0\,.$$ **[1]**

To relate the normal modes to the original coordinates, we must find the rotation matrix $\underline{R}$, i.e. the eigenvectors of $\underline{A}$. For $\lambda_1 = -1$, we require

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

By inspection, the (normalised) solution is

$$\underline{r}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} .$$ **[2]**

For the other eigenvalue of $\lambda_2 = -4$, we require

$$\begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} r_{12} \\ r_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

By inspection, the (normalised) solution is

$$\underline{r}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} .$$

[2]

The rotation matrix

$$\underline{R} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$ **[1]**

and the two sets of coordinates are related by

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(y_1 + y_2) \quad &:& \quad y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)\,, \\ x_2 &= \frac{1}{\sqrt{2}}(y_1 - y_2) \quad &:& \quad y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)\,. \end{aligned}$$ **[1]**

The general solutions of the uncoupled differential equations are

$$\begin{aligned} y_1 &= \alpha_1 \cos t + \beta_1 \sin t\,, \\ y_2 &= \alpha_2 \cos 2t + \beta_2 \sin 2t\,. \end{aligned}$$ **[1]**

At time $t = 0$, $\dot{y}_1 = \dot{y}_2 = 0$, $y_1 = a\sqrt{2}$, and $y_2 = -a\sqrt{2}$. At later times, therefore,

$$\begin{aligned} y_1 &= a\sqrt{2}\cos t\,, \\ y_2 &= -a\sqrt{2}\cos 2t\,. \end{aligned}$$ **[2]**

Rotating back to the original coordinates,

$$\begin{aligned} x_1 &= a(\cos t - \cos 2t)\,, \\ x_2 &= a(\cos t + \cos 2t)\,. \end{aligned}$$ **[1]**

Students can actually solve this simple two-degree-of-freedom problem by much easier methods. Adding and subtracting the two original equations gives

$$\begin{aligned} 2\frac{d^2x_1}{dt^2} + 2\frac{d^2x_2}{dt^2} &= -2x_1 - 2x_2\,, \\ 2\frac{d^2x_1}{dt^2} - 2\frac{d^2x_2}{dt^2} &= -8x_1 - 8x_2\,. \end{aligned}$$

We can see directly here the uncoupled equations in $x_1 \pm x_2$ and all the subsequent manipulations should come out. However it does not use the matrix diagonalisation technique asked for. The maximum mark is therefore only 10/14.

2. For the matrix

$$\underline{A} = \begin{pmatrix} 1 & i & 3i \\ -i & 1 & -3 \\ -3i & -3 & -3 \end{pmatrix}$$

the eigenvalue equation is

$$| \underline{A} - \lambda \mathbf{I} | = \begin{vmatrix} 1-\lambda & i & 3i \\ -i & 1-\lambda & -3 \\ -3i & -3 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(1-\lambda)(-3-\lambda) - 9] - i[-i(-3-\lambda) - 9i] + 3i[3i + 3i(1-\lambda)]$$
$$= (1-\lambda)(\lambda^2 + 2\lambda - 12) + (\lambda - 6) + 3(3\lambda - 6) = 0\,.$$

The characteristic equation is therefore

$$\lambda^3 + \lambda^2 - 24\lambda + 36 = 0\,.$$ **[3]**

By inspection, $\lambda = 2$ is one solution and, factorising this out,

$$(\lambda - 2)(\lambda^2 + 3\lambda - 18) = (\lambda - 2)(\lambda - 3)(\lambda + 6) = 0\,,$$

and hence the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -6$. **[2]**

(i) The sum of the eigenvalues is $2+3-6 = -1$, whereas the trace is $1+1-3 = -1$, as expected. **[1]**

(ii) The product of the eigenvalues is $2 \times 3 \times -6 = -36$. The determinant

$$| \underline{A} | = \begin{vmatrix} 1 & i & 3i \\ -i & 1 & -3 \\ -3i & -3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & i & 3i \\ 0 & 0 & -6 \\ -3i & -3 & -3 \end{vmatrix} = -36\,,$$

as predicted. **[2]**

The eigenvector equation in the case of $\lambda = 2$ is

$$\begin{pmatrix} 1-\lambda & i & 3i \\ -i & 1-\lambda & -3 \\ -3i & -3 & -3-\lambda \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \end{pmatrix} = \begin{pmatrix} -1 & i & 3i \\ -i & -1 & -3 \\ -3i & -3 & -5 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

leading to the two independent equations

$$\begin{aligned} -u_{11} + iu_{21} + 3iu_{31} &= 0\,, \\ -3iu_{11} - 3u_{21} - 5u_{31} &= 0\,. \end{aligned}$$

This has solution

$$\underline{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix},$$ **[3]**

where the factor in front has been inserted to ensure that the eigenvector is normalised, $\underline{u}_1^\dagger \, \underline{u}_1 = 1$. Note that this involves complex conjugation and students might forget this point.

For $\lambda = 3$,

$$\begin{pmatrix} -2 & i & 3i \\ -i & -2 & -3 \\ -3i & -3 & -6 \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \\ u_{32} \end{pmatrix} = 0 \, .$$

This requires

$$\begin{aligned} -2u_{12} + iu_{22} + 3iu_{32} &= 0 \, , \\ -iu_{12} - u_{22} - 2u_{32} &= 0 \, , \end{aligned}$$

from which

$$\underline{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix} \, .$$ **[3]**

Finally, for $\lambda = -6$,

$$\begin{pmatrix} 7 & i & 3i \\ -i & 7 & -3 \\ -3i & -3 & 3 \end{pmatrix} \begin{pmatrix} u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = 0 \, .$$

Hence

$$\begin{aligned} -iu_{13} + 7u_{23} - 3u_{33} &= 0 \, , \\ -3iu_{13} - 3u_{23} + 3u_{33} &= 0 \, . \end{aligned}$$

This has solution

$$\underline{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -i \\ 1 \\ 2 \end{pmatrix} \, .$$ **[3]**

Taking the scalar products,

$$\begin{aligned} \underline{u}_2^\dagger \, \underline{u}_1 &\propto -i \times i - 1 \times 1 + 1 \times 0 = 0 \, , \\ \underline{u}_3^\dagger \, \underline{u}_2 &\propto i \times i + 1 \times (-1) + 2 \times 1 = 0 \, , \\ \underline{u}_1^\dagger \, \underline{u}_3 &\propto -i \times (-i) + 1 \times 1 + 0 \times 2 = 0 \, . \end{aligned}$$ **[3]**

Hence the eigenvectors are orthogonal to each other.