Intercollegiate post-graduate course in High Energy Physics

Paper 1 : The Standard Model

Friday, 29 January 2010

Time allowed for Examination : 3 hours

Answer 6 from 8 questions

Books and notes may be consulted
Question 1 (20 marks)

For a four-momentum, $p_{\mu}$, show that $p_{\mu}p^{\mu}$ is a Lorentz invariant, by considering a Lorentz transformation along a spatial axis of your choice. [5]

At a collider, two high energy particles, $A$ and $B$ with energies $E_A$ and $E_B$, which are much greater than their rest masses, collide head on. Derive the expression for the centre-of-mass energy. Using this expression, what would be the centre-of-mass energy of a proposed future facility (“LHeC”) which is supposed to collide 7 TeV protons with 70 GeV electrons? Now consider particle $B$ (the proton) to be at rest. Derive the formula for the centre-of-mass energy of such a fixed-target experiment. What electron beam energy would be required in the fixed-target experiment in order to achieve the same centre-of-mass energy as in the proposed LHeC facility? [5]

Obtain the relation for the centre-of-mass energy in electron–neutrino scattering

$$s = m_e(2E_{\nu} + m_e).$$ [3]

A particle of mass $M$ decays into two particles with masses $m_1$ and $m_2$. Determine the energies of the decay products in the rest frame of the parent particle. [5]

Hence write down, in terms of masses and the centre-of-mass energy, the energy in the rest frame of particle $A$ in a scattering, $AB \rightarrow CD$. [2]
Question 2 (20 marks)

Given the cross section for the scattering of an electron from a fixed Coulomb potential of point charge $Ze$:

$$\frac{d\sigma}{d\Omega} = \frac{2(Z\alpha)^2 m^2}{|q|^4} \text{Tr} \left[ \frac{\gamma_0}{2m} \left( \psi_i + m \right) \left( \psi_f + m \right) \right],$$

where $p_i$ and $p_f$ are the initial and final momenta and $q = p_f - p_i$, determine the Mott cross section:

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{4(\gamma\beta^2)(mc^2)^2 \sin^4 \theta/2} \left( 1 - \beta^2 \sin^2 \frac{\theta}{2} \right).$$

Trace theorems used should be explicitly stated. \[15\]

Show that in the non-relativistic limit

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{16E^2 \sin^4 \frac{\theta}{2}},$$

and in the extreme-relativistic limit

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2}}.$$\[5\]
Question 3 (20 marks)

Evaluate, in terms of the four-vectors (you do not have to convert to the Mandelstam variable),

\[
\text{Tr} \left[ \gamma^\mu \gamma^\nu (\not{p} + m) \gamma_\mu \gamma_\nu (\not{p'} + m) \right]
\]

and

\[
\text{Tr} \left[ \gamma^\mu \gamma^\nu (\not{p} + m) \gamma_\mu \gamma_\nu (\not{p'} + m) \right].
\]

that occur in the calculation of electron-photon scattering. Trace theorems and identities for \( \gamma \) matrices need not be derived, but should be quoted. \[10\]

In a massless limit, the terms in final squared transition amplitude for Compton scattering are

(a) \( 2e^4 \left( -\frac{u}{s} \right) \)  
(b) \( 2e^4 \left( -\frac{s}{u} \right) \)  
(c) \( 2e^4 \frac{t}{us} (s + u + t) \).

Identify the Feynman diagram(s) which contribute to each term. Hence write down the final squared transition amplitudes when the incoming photon is real and when it is virtual. \[5\]

Given the Compton condition

\[ \lambda' = \lambda + \frac{2\pi}{m} (1 - \cos \theta) \]

and the Klein-Nishina formula for Compton scattering

\[
\frac{d\sigma}{d\Omega}(\lambda, \lambda') = \frac{\alpha^2}{4m^2} \left( \frac{\omega'}{\omega} \right)^2 \left[ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon^* \cdot \epsilon)^2 - 2 \right]
\]

determine the cross section in the low-energy limit, i.e. \( \omega \to 0 \), in terms of the fine structure constant, \( \alpha \), the mass of the electron, \( m \), and the polarisation vectors of the photon, \( \epsilon \) and \( \epsilon^* \). \[5\]
Question 4 (20 marks)

What property of the EM interaction means that photons do not self-couple? [2]

Explain the four terms in the Lagrangian of QED:

\[ \mathcal{L} = \bar{\psi} (i \gamma_\mu \partial^\mu - m) \psi + e \bar{\psi} \gamma_\mu A^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

[4]

Briefly explain the concept of a “running” coupling constant in QED where the variation is with the scale of the process, \( Q^2 \). Draw two Feynman diagrams, one for QED and one for QCD, to illustrate the effect. And draw a diagram which leads to the QCD coupling having a different dependence. [5]

State what are meant by global and local gauge transformations. [2]

Given the phase transformations of the wave function and the electromagnetic field:

\[ \phi(x) \rightarrow \phi'(x) = \exp(iq\alpha) \phi(x) \quad A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) - \partial^\mu \alpha(x) \]

and the gauge-covariant derivative,

\[ \partial^\mu \rightarrow D^\mu = \partial^\mu + iq A^\mu \]

show that the Klein-Gordon equation is invariant under these transformations. [7]
Question 5 (20 marks)

In the decay of a $\pi^-$ at rest, $\pi^- \rightarrow e^- + \bar{\nu}_e$, show that
\[
\frac{1}{2} \left( 1 - \frac{v_e}{c} \right) = \frac{m_e^2}{m_{\pi}^2 + m_e^2}.
\]
where $v_e$ is the velocity of the electron. [5]

To lowest order, the partial decays rate for pions are:
\[
\frac{1}{\tau(\pi \rightarrow e\bar{\nu}_e)} = \frac{\alpha_\pi^2}{4\pi} \left( 1 - \frac{v_e}{c} \right) p_e^2 E_e, \quad \frac{1}{\tau(\pi \rightarrow \mu\bar{\nu}_\mu)} = \frac{\alpha_\pi^2}{4\pi} \left( 1 - \frac{v_\mu}{c} \right) p_\mu^2 E_\mu.
\]
where $\alpha_\pi$ is an effective coupling constant and $E_e, E_\mu$ and $p_e, p_\mu$ are the charged lepton’s energy and momentum. Hence show:
\[
\frac{\tau(\pi \rightarrow \mu\bar{\nu}_\mu)}{\tau(\pi \rightarrow e\bar{\nu}_e)} = \frac{m_e^2(m_{\pi}^2 - m_e^2)^2}{m_\mu^2(m_{\pi}^2 - m_\mu^2)^2}.
\]
[5]

Use the analogue of the above equation for the decay of the $K^-$ to estimate the ratio
\[
\frac{\tau(K \rightarrow \mu\bar{\nu}_\mu)}{\tau(K \rightarrow e\bar{\nu}_e)}
\]
and compare with the observed value $(2.4 \pm 0.1) \times 10^{-5}$.

Given the lifetimes $\tau(K \rightarrow \mu\bar{\nu}_\mu) = 1.948 \times 10^{-8}$ s and $\tau(\pi \rightarrow \mu\bar{\nu}_\mu) = 2.603 \times 10^{-8}$ s, estimate $\alpha_K/\alpha_\pi$.

$(m_K = 493.67$ MeV, $m_{\pi} = 139.57$ MeV, $m_\mu = 105.66$ MeV, $m_e = 0.511$ MeV.) [5]

Draw quark model diagrams for the decays $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ and $K^- \rightarrow \mu^- + \bar{\nu}_\mu$, stating which element of the CKM matrix is involved in each.

Neglecting masses, the ratio of the CKM elements is equal to $\alpha_K/\alpha_\pi$. Hence estimate $\sin \theta_{12}$. [5]
Question 6 (20 marks)

In deep inelastic scattering at HERA, the four-momenta of the incoming and scattered electron are \((E, \mathbf{p})\) and \((E', \mathbf{p}')\), respectively. Show that the square of the four-momentum transfer is given by

\[ Q^2 = 4EE' \sin^2 \frac{\theta}{2} \]

where \(\theta\) is the angle of the scattered electron. And show that the mass of the proton is related to the energies by

\[ M(E - E') - 2EE' \sin^2 \frac{\theta}{2} = 0. \]

Give a physical description of the kinematic variables, \(x, y\) and \(Q^2\), which describe deep inelastic scattering.

The electron-quark cross section is:

\[ \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} 2e^4 q_i^2 \left[ \frac{s^2 + \hat{u}^2}{\hat{t}^2} \right]. \]

Show that this can be written in a more useful form as:

\[ \frac{d\sigma}{dy} = \frac{2\pi\alpha^2}{Q^4 q_i^2 s} \left[ 1 + (1 - y)^2 \right]. \]

The cross section for the QCD Compton and Boson-gluon fusion processes are:

\[ \frac{d\sigma}{d\Omega} \sim -\frac{\hat{t}}{s} - \frac{\hat{s}}{t} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}, \]

\[ \frac{d\sigma}{d\Omega} \sim \frac{\hat{u}}{t} + \frac{\hat{t}}{u} - \frac{2sQ^2}{\hat{t}\hat{u}}, \]

respectively. State where (e.g. with reference to Feynman diagrams or the Mandelstam channel) each term comes from.
A possible decay of the $W$ boson with associated four-momenta is

$$W(q) \rightarrow \mu(p) + \nu_\mu(k).$$

The transition amplitude squared is:

$$|T_{fi}|^2 = \frac{g_W^2}{3} \left[ p \cdot k + \frac{2}{M_W^2} (q \cdot k)(q \cdot p) \right].$$

In the $W$ rest frame, show that:

$$q \cdot k = \frac{M_W^2}{2} \left( 1 - \frac{m_\mu^2}{M_W^2} \right) \quad q \cdot p = \frac{M_W^2}{2} \left( 1 + \frac{m_\mu^2}{M_W^2} \right) \quad p \cdot k = \frac{M_W^2}{2} \left( 1 - \frac{m_\mu^2}{M_W^2} \right).$$

Then using the relationship:

$$\Gamma = \frac{1}{16\pi M_W} |T_{fi}|^2$$

derive the partial width in terms of the masses of the $W$ and $\mu$ and show that this can be simplified to

$$\Gamma = \frac{G_F M_W^3}{\sqrt{2} \ 6\pi}.$$

($m_\mu = 105.66 \text{ MeV}, \quad M_W = 80.4 \text{ GeV}$)
Question 8 (20 marks)

In the Weinberg-Salam electroweak theory, the vacuum potential, boson masses, couplings, and mixing angle are related by:

\[
M_Z = \frac{M_W}{\cos \theta_W}, \quad \frac{g'}{g_W} = \tan \theta_W, \quad M_W = \frac{v g_W}{2}
\]

Hence show that

\[
M_Z = \frac{1}{2} v \sqrt{g_W^2 + g'^2}.
\]

From the Lagrangian

\[
\frac{1}{8} \left[ g_W^2 (v + h)^2 (W_1^2 - i W_2^2) (W_1^1 + i W_2^2) - (v + h)^2 (g' B_\mu - g_W W_3^3) (g' B^\mu - g_W W_3^\mu) \right]
\]

derive the \(WWH\) and \(WWHH\) couplings and the \(ZZH\) and \(ZZHH\) couplings. (Simplify your answer to remove dependencies on both \(v\) and \(g'\).)

Given Fermi’s constant, \(G_F = 1.16637 \times 10^{-5} \text{ GeV}^{-2}\), calculate the value of \(v\).

The Higgs Boson was searched for in \(e^+e^-\) collisions at LEP. Draw the Feynman diagram of the process for Higgs production. Masses of up to about 114 GeV were ruled out; hence give the approximate maximum centre-of-mass energy of LEP.

[Total Marks = 120]