1) Rotation Matrices [5 marks]

In the lectures, we looked at how rotation matrices could be derived for rotations about the \( y \)-axis. Here we will consider what happens when we used the \( x \)-axis.

Consider a 3D representation of SU(2) with a generator:

\[
J_x = \frac{i}{2} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

Evaluate the rotation matrix \( \langle jm' | \exp(i\theta J_x) | jm \rangle \) where \( j = 1 \) and \( m \) & \( m' = -1,0,1 \).

How do the rotation matrix elements differ from those we saw in the lectures (and you derived in your homework) when we considered the rotation about the \( y \)-axis? Does it matter?

(The standard rotation matrices can be seen on the attached page from the PDG Book.)

2) Lie Algebra in SU(2) [8 marks]

Using the properties of the raising and lowering operators in SU(2), derive the Lie Algebra (commutators) for the generators \( J_1, J_2, J_3 \).

The properties you should use are:

I. \( J_x = J_1 \pm iJ_2 \)

II. \( J_3 | j, m \rangle = m | j, m \rangle \)

III. \( J_\pm | j, m \rangle = \frac{\sqrt{(j \pm m)(j \pm m + 1)}}{j} | j, (m \pm 1) \rangle \)

Hints:

Firstly, compare \( J_3 J_x | j, m \rangle \) and \( J_x J_3 | j, m \rangle \)

Secondly, compare \( J_- J_+ | j, m \rangle \) and \( J_+ J_- | j, m \rangle \)
3) Combining Spins – Clebsch-Gordon Coefficients [2 marks]

Spin-1 states can be made from combining

a) Spin-1/2 and Spin-1/2, or
b) Spin-1 and Spin-1, or
c) Spin-3/2 and Spin-1/2

Use the Clebsch-Gordon coefficients on the attached sheet from the PDG Book to show how the Spin-1 state

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

ie. \( f=1, m=0 \)

can be constructed from the pairs of states in each of the 3 lists (a), (b) and (c) above.

4) Combining Spins – Young Tableaux [5 marks]

In part (a) of the previous question, it was stated that a Spin-1 state can be constructed from the combination of two Spin-1/2 states for SU(2). In terms of Young Tableaux, this looks like:

\[
\begin{array}{c}
\begin{array}{c}
\text{Spin} \\
\text{Multiplicity}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}
\end{array}
\oplus \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}
\end{array}
\]

The combination of the 2 Spin-1/2 states results in Spin-1 and Spin-0 multiplets.

Show how this would look for the combinations in parts (b) and (c) of the previous question.

Hints:
You don’t need to think about the 3rd component of spin – just the multiplets.
For the representations for Spin-1 and Spin-3/2, you will need to represent the Tableaux by 2 and 3 boxes in a row, respectively.
[Don’t panic if you don’t appear to identify all the states you might expect for the combination of Spin-1 with Spin-1 – we didn’t discuss how to combine complex YT.]
35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND $d$ FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-\sqrt{3/5}$ read $-\sqrt{3}/\sqrt{5}$.

\[
\begin{array}{c|c|c|c|c}
1/2 \times 1/2 & 1/2 \times 1/2 \\
\hline
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
\end{array}
\]

\[
Y^m_l \pm = \sqrt{\frac{2}{N_l}} \cos \theta \\
Y^m_l \mp = \sqrt{\frac{2}{N_l}} \sin \theta e^{\hat{x}} \\
Y^m_l = \sqrt{\frac{2}{N_l}} \sin \theta e^{\hat{x}} \\
Y^m_l = \sqrt{\frac{2}{N_l}} \cos \theta e^{\hat{x}} \\
3/2 \times 1/2 \\
\hline
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
\end{array}
\]

\[
d^m_{l,l} = \frac{(-1)^{m-m'}d_{m-m'}^{m'}}{\sqrt{1+\frac{1}{2}}} \\
3/2 \times 3/2 \\
\hline
3/2 & 3/2 & 3/2 & 3/2 \\
\end{array}
\]

\[
d_{1,1} = \frac{1 + \cos \theta}{2} \\
d_{2,1} = -\frac{1 + \cos \theta}{2} \\
d_{2,2} = \frac{1 + \cos \theta}{2} \\
d_{2,3} = -\frac{1 + \cos \theta}{2} \\
d_{2,4} = \frac{1 + \cos \theta}{2} \\
d_{2,5} = -\frac{1 + \cos \theta}{2} \\
d_{2,6} = \frac{1 + \cos \theta}{2} \\
d_{2,7} = -\frac{1 + \cos \theta}{2} \\
d_{2,8} = \frac{1 + \cos \theta}{2} \\
d_{2,9} = -\frac{1 + \cos \theta}{2} \\
d_{2,10} = \frac{1 + \cos \theta}{2} \\
d_{2,11} = -\frac{1 + \cos \theta}{2} \\
d_{2,12} = \frac{1 + \cos \theta}{2} \\
d_{2,13} = -\frac{1 + \cos \theta}{2} \\
d_{2,14} = \frac{1 + \cos \theta}{2} \\
d_{2,15} = -\frac{1 + \cos \theta}{2} \\
d_{2,16} = \frac{1 + \cos \theta}{2} \\
d_{2,17} = -\frac{1 + \cos \theta}{2} \\
d_{2,18} = \frac{1 + \cos \theta}{2} \\
d_{2,19} = -\frac{1 + \cos \theta}{2} \\
d_{2,20} = \frac{1 + \cos \theta}{2} \\
d_{2,21} = -\frac{1 + \cos \theta}{2} \\
d_{2,22} = \frac{1 + \cos \theta}{2} \\
d_{2,23} = -\frac{1 + \cos \theta}{2} \\
d_{2,24} = \frac{1 + \cos \theta}{2} \\
d_{2,25} = -\frac{1 + \cos \theta}{2} \\
d_{2,26} = \frac{1 + \cos \theta}{2} \\
d_{2,27} = -\frac{1 + \cos \theta}{2} \\
d_{2,28} = \frac{1 + \cos \theta}{2} \\
d_{2,29} = -\frac{1 + \cos \theta}{2} \\
d_{2,30} = \frac{1 + \cos \theta}{2} \\
d_{2,31} = -\frac{1 + \cos \theta}{2} \\
d_{2,32} = \frac{1 + \cos \theta}{2} \\
d_{2,33} = -\frac{1 + \cos \theta}{2} \\
d_{2,34} = \frac{1 + \cos \theta}{2} \\
d_{2,35} = -\frac{1 + \cos \theta}{2} \\
d_{2,36} = \frac{1 + \cos \theta}{2} \\
d_{2,37} = -\frac{1 + \cos \theta}{2} \\
d_{2,38} = \frac{1 + \cos \theta}{2} \\
d_{2,39} = -\frac{1 + \cos \theta}{2} \\
d_{2,40} = \frac{1 + \cos \theta}{2} \\
d_{2,41} = -\frac{1 + \cos \theta}{2} \\
d_{2,42} = \frac{1 + \cos \theta}{2} \\
d_{2,43} = -\frac{1 + \cos \theta}{2} \\
d_{2,44} = \frac{1 + \cos \theta}{2} \\
d_{2,45} = -\frac{1 + \cos \theta}{2} \\
d_{2,46} = \frac{1 + \cos \theta}{2} \\
d_{2,47} = -\frac{1 + \cos \theta}{2} \\
d_{2,48} = \frac{1 + \cos \theta}{2} \\
d_{2,49} = -\frac{1 + \cos \theta}{2} \\
d_{2,50} = \frac{1 + \cos \theta}{2} \\
d_{2,51} = -\frac{1 + \cos \theta}{2} \\
d_{2,52} = \frac{1 + \cos \theta}{2} \\
d_{2,53} = -\frac{1 + \cos \theta}{2} \\
d_{2,54} = \frac{1 + \cos \theta}{2} \\
d_{2,55} = -\frac{1 + \cos \theta}{2} \\
d_{2,56} = \frac{1 + \cos \theta}{2} \\
d_{2,57} = -\frac{1 + \cos \theta}{2} \\
d_{2,58} = \frac{1 + \cos \theta}{2} \\
d_{2,59} = -\frac{1 + \cos \theta}{2} \\
d_{2,60} = \frac{1 + \cos \theta}{2} \\
d_{2,61} = -\frac{1 + \cos \theta}{2} \\
d_{2,62} = \frac{1 + \cos \theta}{2} \\
d_{2,63} = -\frac{1 + \cos \theta}{2} \\
d_{2,64} = \frac{1 + \cos \theta}{2} \\
d_{2,65} = -\frac{1 + \cos \theta}{2} \\
d_{2,66} = \frac{1 + \cos \theta}{2} \\
d_{2,67} = -\frac{1 + \cos \theta}{2} \\
d_{2,68} = \frac{1 + \cos \theta}{2} \\
d_{2,69} = -\frac{1 + \cos \theta}{2} \\
d_{2,70} = \frac{1 + \cos \theta}{2} \\
d_{2,71} = -\frac{1 + \cos \theta}{2} \\
d_{2,72} = \frac{1 + \cos \theta}{2} \\
d_{2,73} = -\frac{1 + \cos \theta}{2} \\
d_{2,74} = \frac{1 + \cos \theta}{2} \\
d_{2,75} = -\frac{1 + \cos \theta}{2} \\
d_{2,76} = \frac{1 + \cos \theta}{2} \\
d_{2,77} = -\frac{1 + \cos \theta}{2} \\
d_{2,78} = \frac{1 + \cos \theta}{2} \\
d_{2,79} = -\frac{1 + \cos \theta}{2} \\
d_{2,80} = \frac{1 + \cos \theta}{2} \\
d_{2,81} = -\frac{1 + \cos \theta}{2} \\
d_{2,82} = \frac{1 + \cos \theta}{2} \\
d_{2,83} = -\frac{1 + \cos \theta}{2} \\
d_{2,84} = \frac{1 + \cos \theta}{2} \\
d_{2,85} = -\frac{1 + \cos \theta}{2} \\
d_{2,86} = \frac{1 + \cos \theta}{2} \\
d_{2,87} = -\frac{1 + \cos \theta}{2} \\
d_{2,88} = \frac{1 + \cos \theta}{2} \\
d_{2,89} = -\frac{1 + \cos \theta}{2} \\
d_{2,90} = \frac{1 + \cos \theta}{2} \\
d_{2,91} = -\frac{1 + \cos \theta}{2} \\
d_{2,92} = \frac{1 + \cos \theta}{2} \\
d_{2,93} = -\frac{1 + \cos \theta}{2} \\
d_{2,94} = \frac{1 + \cos \theta}{2} \\
d_{2,95} = -\frac{1 + \cos \theta}{2} \\
d_{2,96} = \frac{1 + \cos \theta}{2} \\
d_{2,97} = -\frac{1 + \cos \theta}{2} \\
d_{2,98} = \frac{1 + \cos \theta}{2} \\
d_{2,99} = -\frac{1 + \cos \theta}{2} \\
d_{2,100} = \frac{1 + \cos \theta}{2} \\
\end{array}
\]

Figure 35.1: The sign convention is that of Wigner (Group Theory, Academic Press, New York, 1969), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge Univ. Press, New York, 1955), Rose (Elementary Theory of Angular Momentum, Wiley, New York, 1957), and Cohen (Tables of the Clebsch-Gordan Coefficients, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients have been calculated using computer programs written independently by Cohen and at LBL.