FFT Normalisation for Beginners (really it's just for me)

Ryan Nichol

Parseval's Theorem

$$\sum_{i=0}^{N-1} {V_i}^2 = \frac{1}{N} \sum_{i=0}^{N-1} |H_i|^2$$

- V_i are the (real) voltage samples in the time domain
- H_i are the (complex) FFT values

- Parseval's theorem should be true for any well behaved FFT algorithm.
- It can be used to relate the normalisation of the FFT to the time domain.

Power in the Time Domain

- Many ways to define total power in wave:

(a) $$\sum_{i=0}^{N-1} {V_i}^2 \equiv \text{"sum squared amplitude"}$$

(b) $$\frac{1}{N} \sum_{i=0}^{N-1} {V_i}^2 \equiv \text{"mean squared amplitude"}$$

(c) $$\Delta t \sum_{i=0}^{N-1} {V_i}^2 \equiv \text{"time-integral squared amplitude"}$$

- "Interesting" Properties for non-periodic signals:
  - Zero-padding/increasing timespan
    - (a) doesn't change, (b) does change, (c) doesn't change
  - Changing sampling rate for same timespan
    - (a) does change, (b) doesn't change, (c) doesn't change

Power in Frequency Domain

(a) $$\sum_{i=0}^{N-1} V_i^2 = \frac{1}{N} \sum_{i=0}^{N-1} |H_i|^2$$

(b) $$\frac{1}{N} \sum_{i=0}^{N-1} V_i^2 = \frac{1}{N^2} \sum_{i=0}^{N-1} |H_i|^2$$

(c) $$\Delta t \sum_{i=0}^{N-1} V_i^2 = \frac{\Delta t}{N} \sum_{i=0}^{N-1} |H_i|^2$$

- Points to note:
  - All are valid, but must be interpreted correctly
  - (b) is what the Tektronix scope performs
    - But it is really useful for periodic (i.e CW) signals, hence the name.
  - (c) is probably best for characterising impulses as the sum stays the same when either sampling rate and timespan (or zero pad amount) are changed.

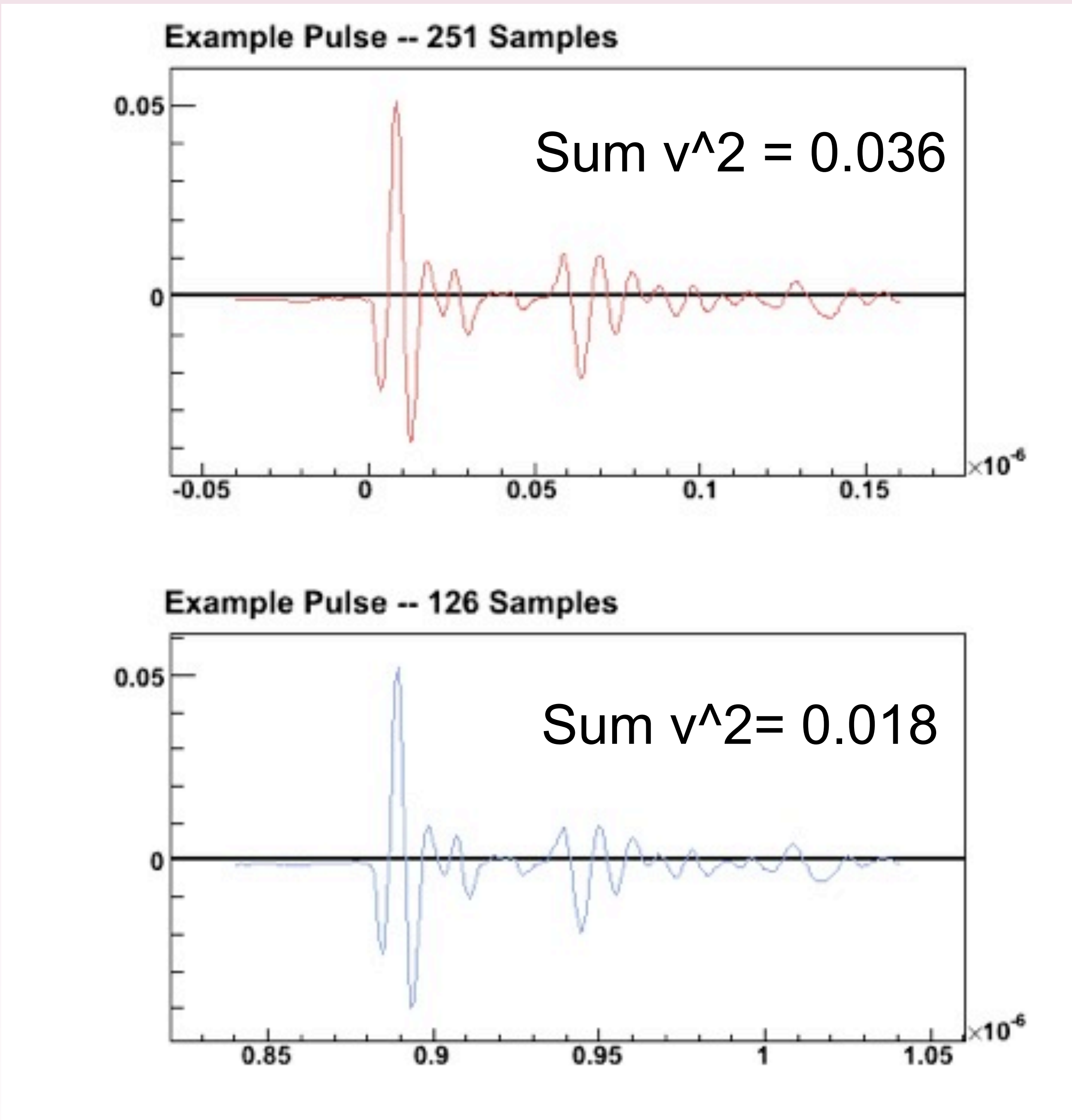
However using (c)...

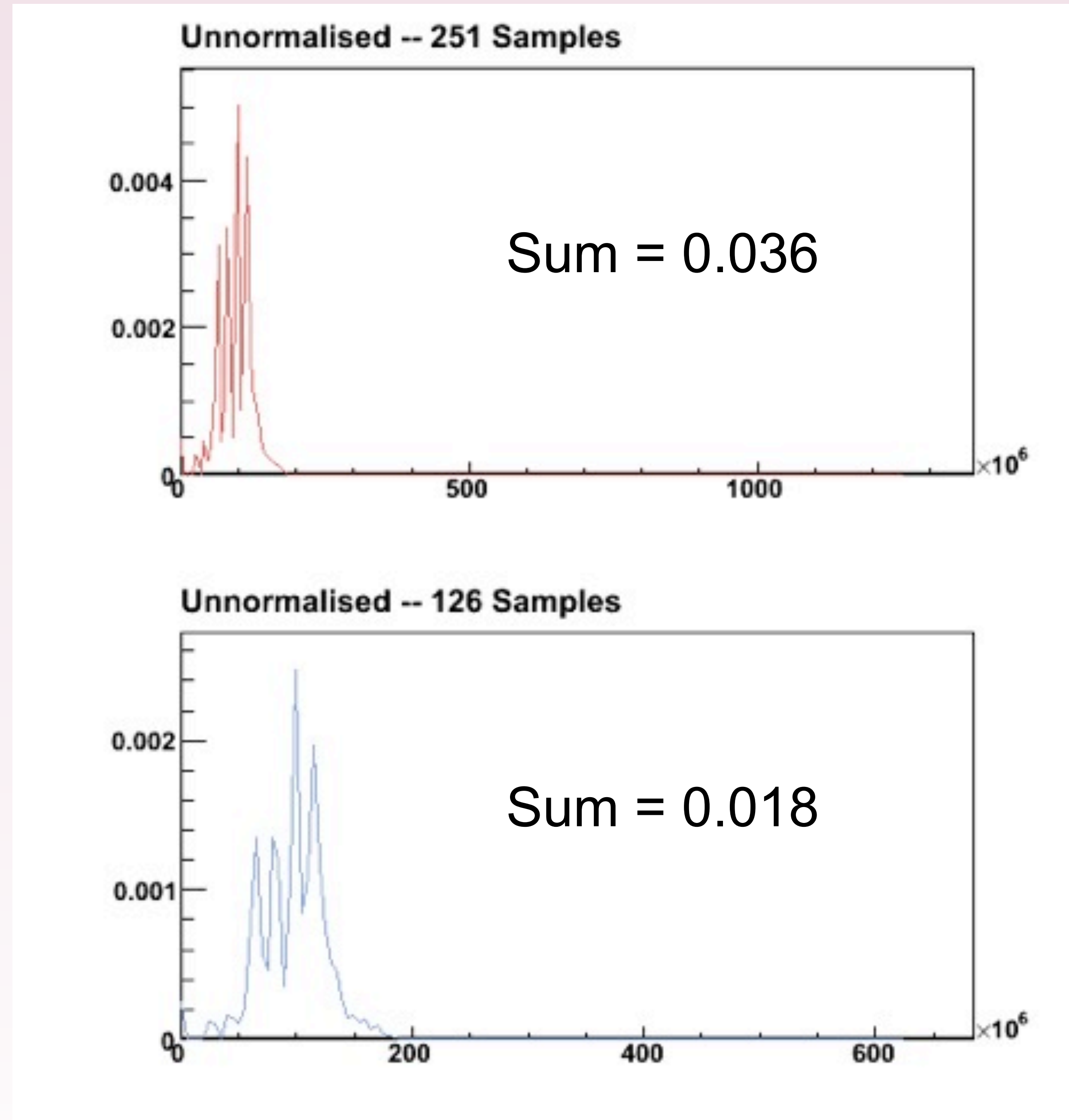
- With this normalisation the sum remains constant
  - But changing N (zero padding, etc.) changes the number of frequency bins --> so the values in the bins must change.
  - In order to compare the bin values between two FFT with different N, need to divide by Δf.
    - Then need to change the summation to an integral to retain physical meaning for the power.
  - Therefore (in my opinion) the correct normalisation is:

$$y_i = \frac{\Delta t}{N \Delta f} |H_i|^2 = \Delta t^2 |H_i|^2$$

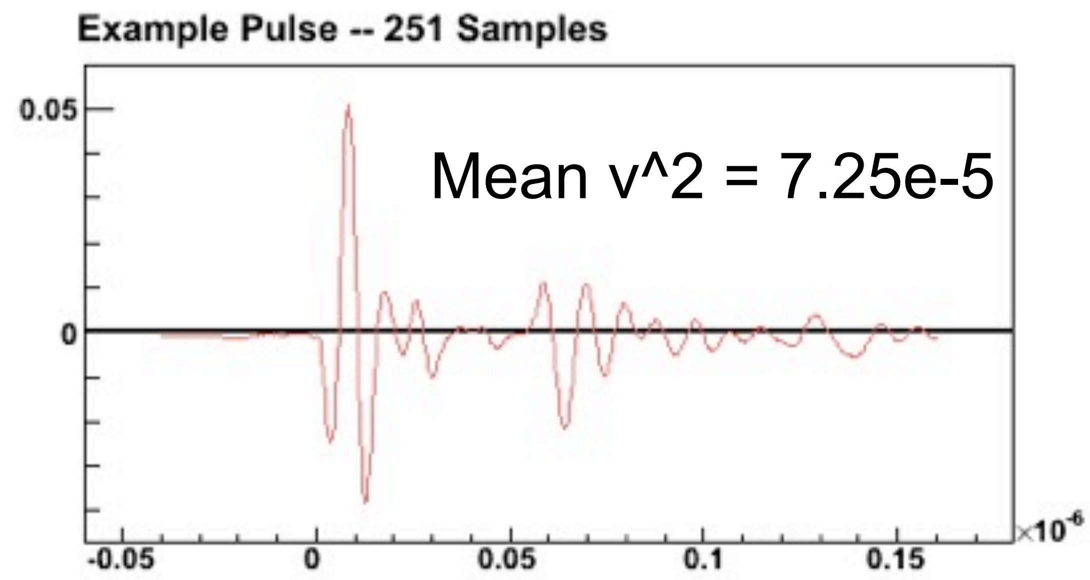
    - But one must integrate (i.e times each bin by Δf) when computing the total power (or power in a frequency range).
    - And of course this is just normalised to the area under the voltage squared curve.

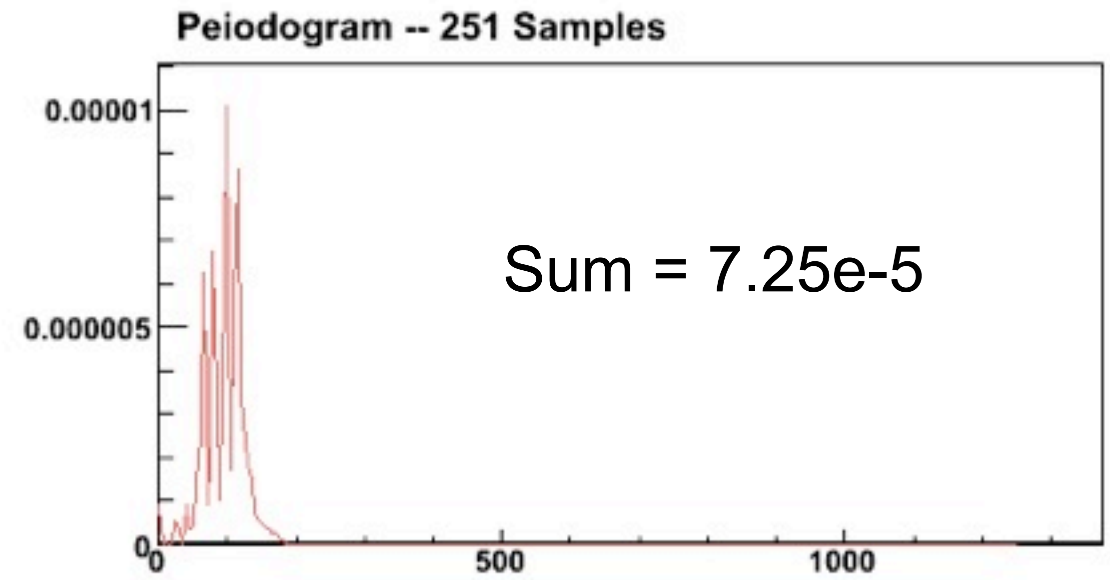
Un-normalised FFT -- Summed Power

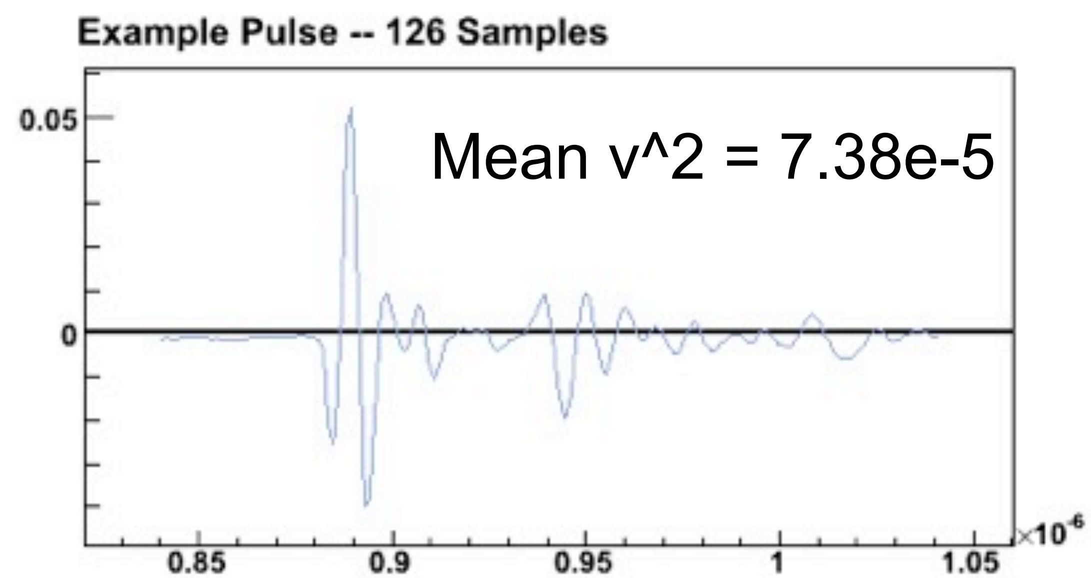


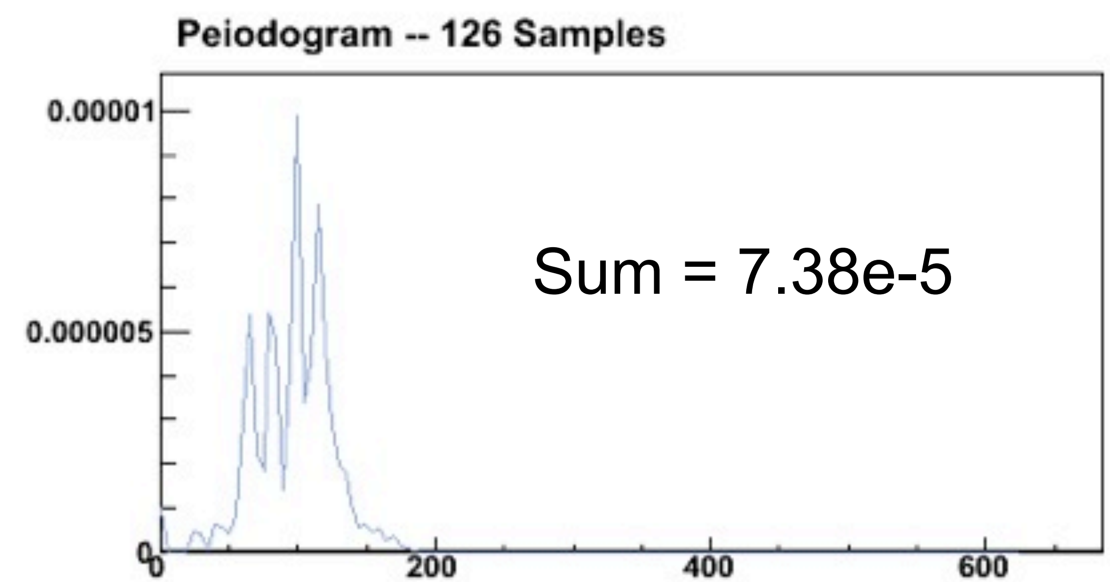


Periodogram -- Average Power



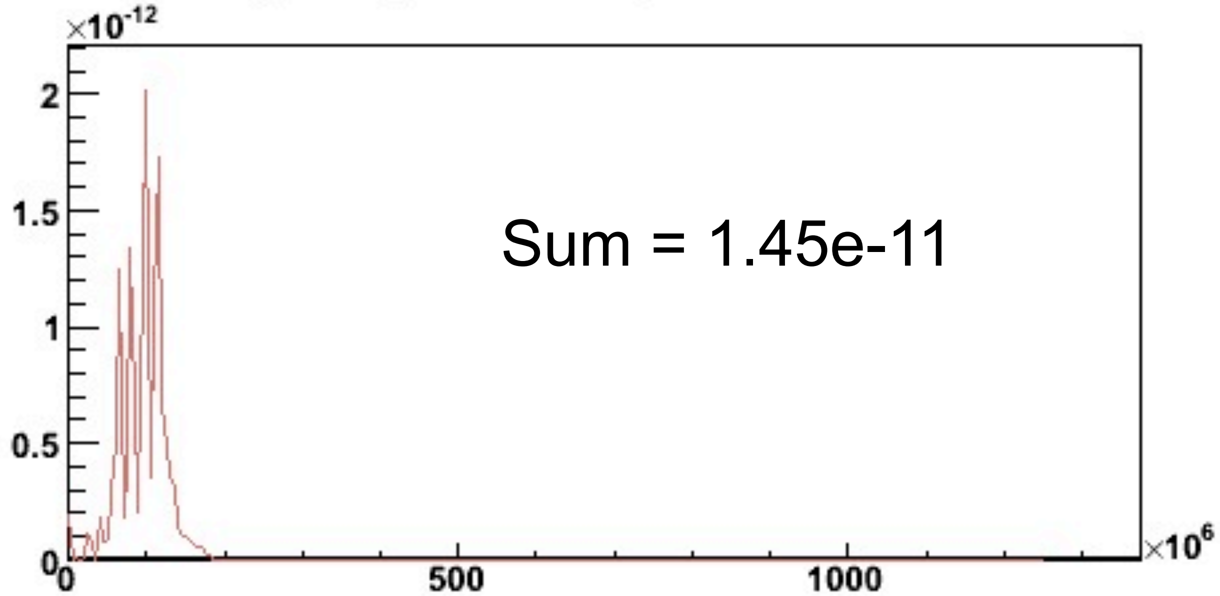




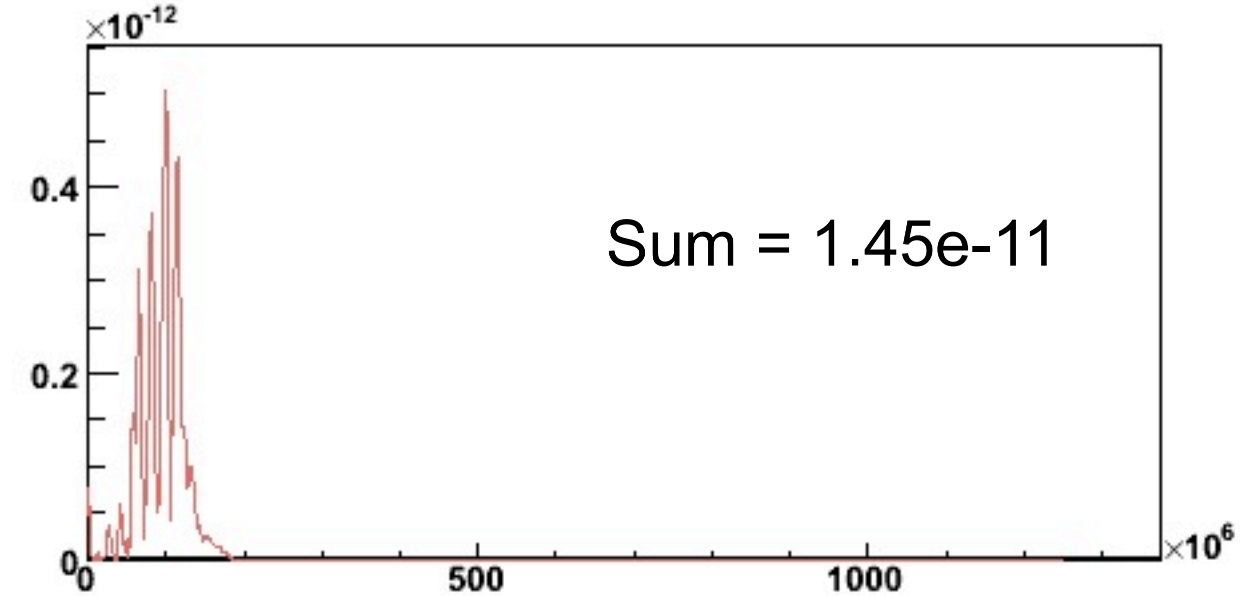


Normalised (not scaled by bin width)

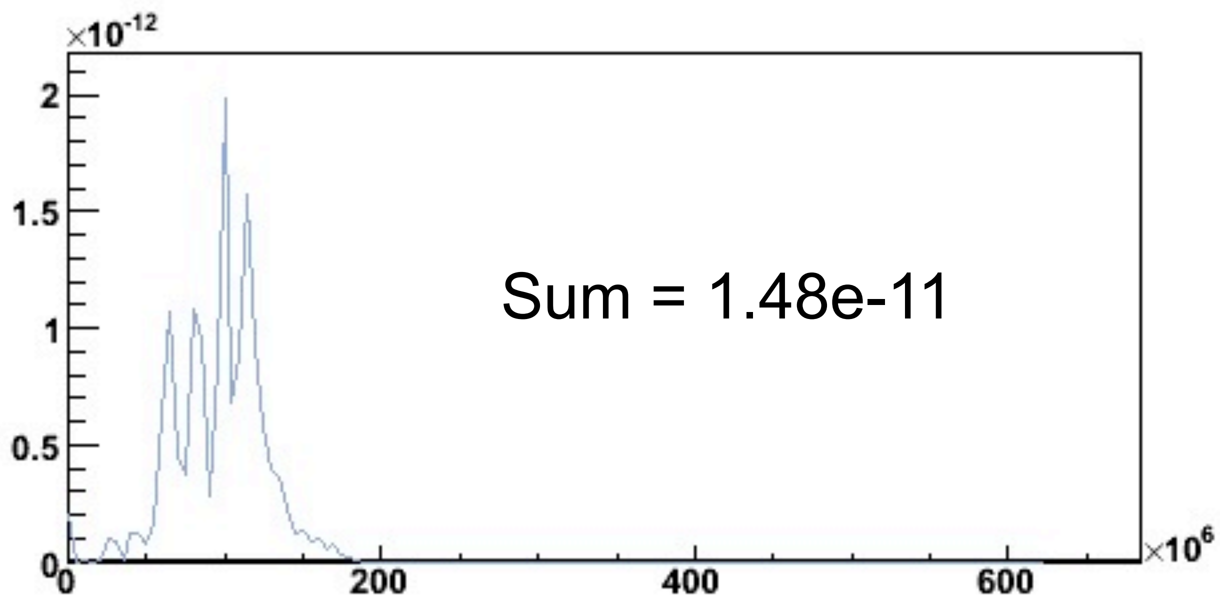
Normalised (not Δf) -- 251 Samples



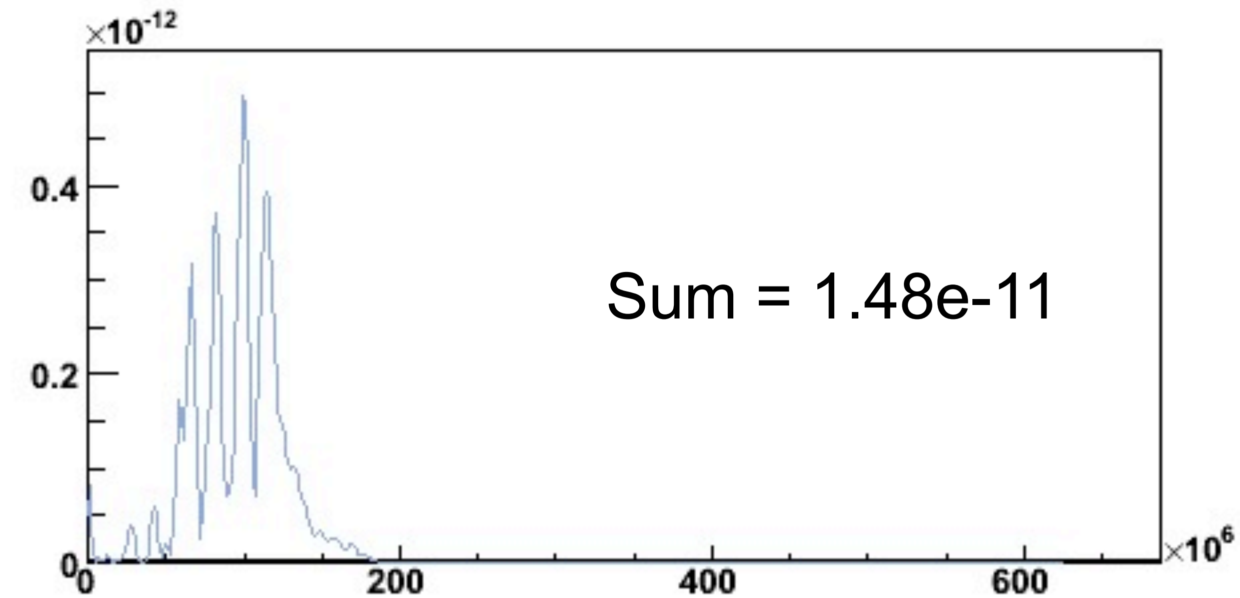
Normalised (not Δf)-- 251 Samples (x4 Zero Padding)



Normalised (not Δf) -- 126 Samples



Normalised (not Δf) -- 126 Samples (x4 Zero Padding)



Normalised (scaled by bin width)