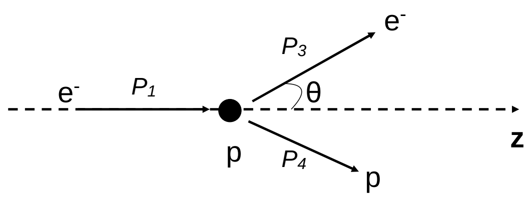
PHASM442/2015 Model Answers [Part marks]

1. The four-momenta of the *relativistic* electron and stationary proton in the lab frane shown in the Figure below can be written as,

$$p_1 = (E_1, 0, 0, E_1), \quad p_2 = (M, 0, 0, 0), \quad p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta) \quad p_4 = (E_4, \vec{p}_4)$$



(a) **[Unseen]**.
From 4-momentum conservation, $p_1 - p_3 = p_4 - p_2$. Squaring the left-hand side and writing them explicitly,

$$(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 \approx -2p_1 \cdot p_3 = -2E_1E_3(1 - \cos\theta)$$

where we have taken into account that the electron is relativistic, i.e. $E \approx |\vec{p}|$. Similarly for the right-hand side

$$(p_2 - p_4)^2 = p_2^2 + p_4^2 - 2p_2 \cdot p_4 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Therefore $E_1E_3(1 - \cos\theta) = M(E_1 - E_3)$. Hence,

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

[5]

(b) **[Unseen]**.
Using 4-momentum kinematic relations,

$$Q^2 = -q^2 = -(p_1 - p_3)^2 = 2E_1E_3(1 - \cos\theta)$$

And using the previous result for E_3,

$$Q^2 = 2E_1(1 - \cos\theta)\frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

Or numerically,

$$Q^2 = 2 \cdot 530(1 - \cos 75^\circ)\frac{530 \cdot 938}{938 + 530 - 530 \cos 75^\circ} = 293156 MeV^2$$

corresponding to the momentum transfer $Q = 541$ MeV.
[NOTE: Students can assume the proton mass to be in $930 - 1000$ MeV range.] **[3]**

(c) **[Covered in lectures. Part Unseen]** .
Taking into account the isospin symmetry, i.e. $d^n(x) = u^p(x)$ and $u^n(x) = d^p(x)$ we can write:

$$F_2^{ep}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right)$$

$$F_2^{en}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right)$$

Resolving into valence and sea quark contributions:

$$u(x) = u_V(x) + u_S(x) \quad d(x) = d_V(x) + d_S(x) \quad \overline{u}(x) = \overline{u}_S(x) \quad \overline{d}(x) = \overline{d}_S(x)$$

and assuming $u_S(x) = d_S(x) = \overline{u}_S(x) = \overline{d}_S(x) = S(x)$, we obtain:

$$F_2^{ep}(x) = x\left(\frac{4}{9}u_V(x) + \frac{1}{9}d_V(x) + \frac{10}{9}S(x)\right)$$

$$F_2^{en}(x) = x\left(\frac{4}{9}d_V(x) + \frac{1}{9}u_V(x) + \frac{10}{9}S(x)\right)$$

Giving the ratio:

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} = \frac{4d_V(x) + u_V(x) + 10S(x)}{4u_V(x) + d_V(x) + 10S(x)}$$

The sea component arises from processes such as $g \to u\overline{u}$. Due to the $1/q^2$ dependence of the gluon propagator it is much more likely to produce low energy gluons. Hence, the sea quarks will comprise of low energy $q\overline{q}$. Therefore at low x the sea quarks will dominate giving

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 1 \quad as \quad x \to 0$$

At high x the sea contribution is small, so for $x \to 1$:

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} = \frac{4d_V(x) + u_V(x)}{4u_V(x) + d_V(x)}$$

Taking into account that $u_V = 2d_V$ we obtain:

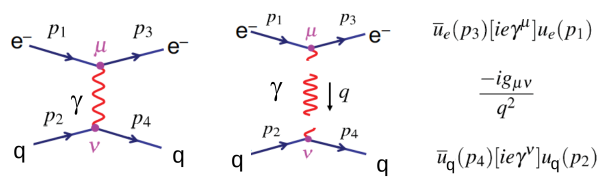
$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 2/3 \quad as \quad x \to 0$$

[6]

(d) **[Covered in lectures]** .
The prediction at low x, $\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 1$, is observed experimentally. However experimentally at high x the ratio is approximately $1/4$ rather than predicted $2/3$. It is sufficient if the answer mentions that experimentally it is $< 2/3$. **[2]**

(e) **[Covered in lectures]** .

The Feynman diagram and the Feynman rules for vertices and the propagator are shown below.



Therefore the matrix element is:

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\,\frac{-ig_{\mu\nu}}{q^2}\,[\overline{u}_q(p_4)ie\gamma^\nu u_q(p_2)]$$

[Note: $-i$ in the matrix element formula can be omitted] **[4]**

[Part marks]

2. (a) **[Unseen]**.

$(\gamma^5)^2 = 1$. Using $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$ we can write:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^3\gamma^0\gamma^1\gamma^2 = -i\gamma^3\gamma^2\gamma^0\gamma^1 = i\gamma^3\gamma^2\gamma^1\gamma^0$$

Using the identities, $(\gamma^0)^2 = 1$ and $(\gamma^\mu)^2 = -1$ for $\mu = 1, 2, 3$ we obtain

$$(\gamma^5)^2 = i^2\gamma^3\gamma^2\gamma^1\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = 1$$

$\gamma^{5\dagger} = \gamma^5$. Using the identities $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{\mu\dagger} = -\gamma^\mu$ for $\mu = 1, 2, 3$ we obtain

$$\gamma^{5\dagger} = \left(i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} = i\gamma^3\gamma^2\gamma^1\gamma^0$$

From above we have $i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^3\gamma^2\gamma^1\gamma^0$ and therefore $\gamma^{5\dagger} = \gamma^5$. **[5]**

(b) **[Covered in lectures]**.

In terms of left- and right-handed chiral components the basic QED vertex can be written as:

$$ie\overline{\psi}\gamma^\mu\psi = ie\left(\overline{\psi}_L + \overline{\psi}_R\right)\gamma^\mu\left(\psi_R + \psi_L\right)$$

$$= ie\left(\overline{\psi}_R\gamma^\mu\psi_R + \overline{\psi}_R\gamma^\mu\psi_L + \overline{\psi}_L\gamma^\mu\psi_R + \overline{\psi}_L\gamma^\mu\psi_L\right)$$

[2]

(c) **Involves synthesising few ideas covered in lectures**.

$$\begin{aligned}\overline{\psi}_R\gamma^\mu\psi_L &= \tfrac{1}{2}\psi^\dagger(1+\gamma^5)\gamma^0\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi \\ &= \tfrac{1}{4}\psi^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)\psi \\ &= \tfrac{1}{4}\overline{\psi}\gamma^\mu(1+\gamma^5)(1-\gamma^5)\psi \\ &= \tfrac{1}{4}\overline{\psi}\left(1-(\gamma^5)^2\right)\psi\end{aligned}$$

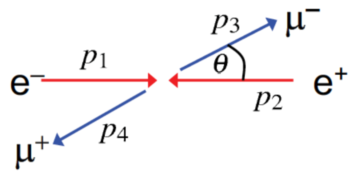
And since $(\gamma^5)^2 = 1$, $\overline{\psi}_R\gamma^\mu\psi_L = 0$. A very similar calculation leads to $\overline{\psi}_L\gamma^\mu\psi_R = 0$. **[5]**

(d) **Covered in lectures**.

Chirality is conserved in EM interactions. In the ultra-relativistic limit chirality = helicity, so only two helicity combinations at the vertices of the $e^+e^- \to \mu^+\mu^-$ process provide a non-zero contribution to the matrix element, and hence cross-section. **[2]**

(e) **Unseen**.

Using the C.o.M. notations as in Figure below:



we can write

$$p_1 = (E, 0, 0, E) \qquad p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \qquad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

This will give the following kinematic relationships:

$$p_1 \cdot p_2 = 2E^2 \qquad p_1 \cdot p_3 = E^2(1 - \cos\theta) \qquad p_1 \cdot p_4 = E^2(1 + \cos\theta)$$

leading to:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2}$$

[4]

(f) **Unseen**.
Since s-channel QED cross-sections decrease as $1/s$, as the centre-of-mass energy increases, higher instantaneous luminosities $\mathcal{L}$ are required to obtain an adequate event rate $R = \sigma\mathcal{L}$ **[2]**

[Part marks]

3. (a) **[Unseen]**.

The unitarity of the PMNS matrix means $UU^\dagger = I$, or writing it explicitly:

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore we can write:

$$U_{e1}U^*_{e1} + U_{e2}U^*_{e2} + U_{e3}U^*_{e3} = 1$$

$$U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0$$

[2]

(b) **[Covered in lectures]**.

$$|\psi(t=0)\rangle = |\nu_e\rangle = U_{e1}|\nu_1\rangle + U_{e2}|\nu_2\rangle + U_{e3}|\nu_3\rangle$$

[1]

(c) **[Covered in lectures]**.

The wave-function evolves according to the time-evolution of the mass eigenstates:

$$|\psi(L)\rangle = U_{e1}|\nu_1\rangle e^{-i\phi_1} + U_{e2}|\nu_2\rangle e^{-i\phi_2} + U_{e3}|\nu_3\rangle e^{-i\phi_3}$$

where $\phi_i = p_i \cdot x = (E_i - |\vec{p}_i|)L$. **[2]**

(d) **[Unseen]**.

Expressing the mass eigenstates in terms of the weak eigenstates:

$$|\psi(L)\rangle = U_{e1}\left(U^*_{e1}|\nu_e\rangle + U^*_{\mu 1}|\nu_\mu\rangle + U^*_{\tau 1}|\nu_\tau\rangle\right) e^{-i\phi_1}$$

$$+U_{e2}\left(U^*_{e2}|\nu_e\rangle + U^*_{\mu 2}|\nu_\mu\rangle + U^*_{\tau 2}|\nu_\tau\rangle\right) e^{-i\phi_2}$$

$$U_{e3}\left(U^*_{e3}|\nu_e\rangle + U^*_{\mu 3}|\nu_\mu\rangle + U^*_{\tau 3}|\nu_\tau\rangle\right) e^{-i\phi_3}$$

Which can be rearranged to give:

$$|\psi(L)\rangle = \left(U_{e1}U^*_{e1}e^{-i\phi_1} + U_{e2}U^*_{e2}e^{-i\phi_2} + U_{e3}U^*_{e3}e^{-i\phi_3}\right)|\nu_e\rangle$$

$$+\left(U_{e1}U^*_{\mu 1}e^{-i\phi_1} + U_{e2}U^*_{\mu 2}e^{-i\phi_2} + U_{e3}U^*_{\mu 3}e^{-i\phi_3}\right)|\nu_\mu\rangle$$

$$+\left(U_{e1}U^*_{\tau 1}e^{-i\phi_1} + U_{e2}U^*_{\tau 2}e^{-i\phi_2} + U_{e3}U^*_{\tau 3}e^{-i\phi_3}\right)|\nu_\tau\rangle$$

From which we obtain:

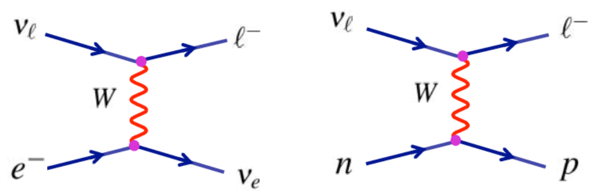
$$P(\nu_e \to \nu_\mu) = |\langle \nu_\mu|\psi(L)\rangle|^2$$

$$= \left|U_{e1}U^*_{\mu 1}e^{-i\phi_1} + U_{e2}U^*_{\mu 2}e^{-i\phi_2} + U_{e3}U^*_{\mu 3}e^{-i\phi_3}\right|^2$$

[4]

(e) [**Covered in lectures**].

The two diagrams are shown in Figure below where $l^- \equiv \mu^-$ **[2]**



(f) [**Unseen**].

Consider $\nu_\tau n \to \tau^- p$ interaction. In the laboratory frame, where the neutron is at rest, the centre-of-mass energy squared is given by:

$$s = (p_\nu + p_n)^2 = (E_\nu + m_n)^2 - E_\nu^2 = 2E_\nu m_n + m_n^2$$

The $\nu_\tau n \to \tau^- p$ interaction is only kinematically allowed if $s > (m_\tau + m_p)^2$. Therefore:

$$E_\nu > \frac{(m_p^2 - m_n^2) + m_\tau^2 + 2m_p m_\tau}{2m_n}$$

which numerically gives $E > 3.42$ GeV.

Consider $\nu_\tau e^- \to \nu_e \tau^-$ interaction. These interactions are kinematically allowed if $s > m_\tau^2$. In the laboratory frame:

$$s = (p_\nu + p_e)^2 = (E_\nu + m_e)^2 - E_\nu^2 = 2E_\nu m_e + m_e^2$$

Hence

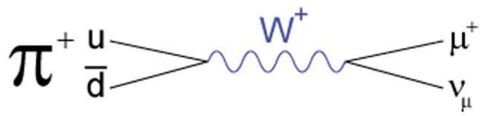
$$E_\nu > \frac{m_\tau^2 - m_e^2}{2m_e}$$

which numerically gives $E > 3.2$ TeV.

Consequently for detecting ν_τ via CC reactions only interactions on nucleons are relevant (typical energies in these experiments are GeV). The energy threshold for CC interactions of ν_τ with atomic electrons are impractically high. **[5]**

(g) [**Covered in lectures**].

The Feynman diagram of pion decay is shown below.



Despite a smaller phase space the decay to $\mu^+\nu_\mu$ is much more likely than to $e^+\nu_e$ due to the spin structure of the weak interaction. Weak interaction only couples to left-handed particle states. Since neutrinos are (almost) massless, it must be in left-handed state. Therefore to conserve angular momentum μ^+ is emitted in the left-handed state. (*Students might include a diagram to demonstrate that*). As μ^+ is not relativistic (due to a relatively small difference between the masses of pion and muon) it can have a significant "wrong" helicity component, proportional to m_μ/m_π. The decay to $e^+\nu_e$ is strongly suppressed for that reason. Energetically there are no other final states π^+ could decay to without violating lepton number. **[4]**

[Part marks]

4. (a) [**Unseen**].
Consider the derivatives of the free-particle plane wave solution $\psi(\mathbf{x}, t) = u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$:

$$\partial_0\psi = \frac{\partial\psi}{\partial t} = -iE\psi; \quad \partial_1\psi = \frac{\partial\psi}{\partial x} = ip_x\psi; \quad ...$$

Substituting these into the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ gives:

$$\left(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m\right) u = 0$$

which can be written:

$$(\gamma^\mu p_\mu - m)\, u = 0$$

[3]

(b) [**Involves synthesising few ideas covered in lectures**].
Under the transformation $\psi(x) \to \psi'(x) = e^{iq\chi(x)}\psi(x)$ the Lagrangian becomes

$$\begin{aligned} \mathcal{L}' &= i\overline{\psi'}\gamma^\mu\partial_\mu\psi' - m\overline{\psi'}\psi' \\ &= ie^{-iq\chi}\overline{\psi}\gamma^\mu\left[e^{iq\chi(x)}\partial_\mu\psi + iq(\partial_\mu\chi)e^{iq\chi(x)}\psi\right]\psi - me^{-iq\chi(x)}\overline{\psi}e^{iq\chi(x)}\psi \\ &= \mathcal{L} - q\overline{\psi}\gamma^\mu(\partial_\mu\chi)\psi \end{aligned}$$

If χ is constant the term $(\partial_\mu\chi)$ disappears and the Lagrangian is restored (invariant). However if χ is a function of x the Lagrangian is *not* invariant under the local phase transformation. **[4]**

(c) [**Covered in lectures**].
To restore the Lagrangian invariance we need to cancel the term $q\overline{\psi}\gamma^\mu(\partial_\mu\chi)\psi$. The cancellation is achieved by introducing a new field which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi$$

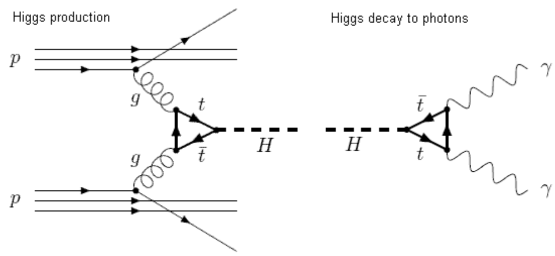
Therefore the gauge-invariant Lagrangian for a spin-half fermion becomes:

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - q\overline{\psi}\gamma^\mu A_\mu\psi$$

The term $q\overline{\psi}\gamma^\mu A_\mu\psi$ describes the interaction of the fermion with the new field A_μ, which can be identified as the photon. Therefore the requirement of gauge invariance introduces the interaction between fermions in QED via exchange of gauge bosons (photons). **[4]**

(d) [**Involves synthesising few ideas covered in lectures**].
The terms $\frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \lambda v^2 h^2$ describe a massive h scalar.
The terms $\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g^2v^2B_\mu B^\mu$ describe a massive gauge boson.
The terms $g^2vB_\mu B^\mu h + \frac{1}{2}g^2B_\mu B^\mu h^2$ describe interactions between the scalar h and the gauge boson B. Finally the terms $\lambda vh^3 - \frac{1}{4}\lambda h^4$ describe self-interactions of the scalar h. **[4]**

(e) [**Unseen**].
Example diagrams are below. Other suitable diagrams will be accepted (e.g. W-fusion for Higgs production). **[2]**



(f) [**Covered in lectures**].
$H \to b\bar{b}$ suffers from a much higher QCD background ($b\bar{b}$ are produced copiously in strong interactions).
Photons from $H \to \gamma\gamma$ will produce a much cleaner (less background) signal in the EM calorimeter, easier to identify EM showers than to tag b-jets.
The energy resolution of the EM calorimeter is better than that of the hadronic calorimeter, the Higgs invariant mass can be reconstructed with a better resolution. **[3]**

[Part marks]

5. (a) **[Unseen]**.
The total decay rate is determined by the spin-averaged matrix element squared, i.e.:

$$\langle |M_{fi}|^2\rangle = \tfrac{1}{3}\left(|M_-|^2 + |M_L|^2 + |M_+|^2\right)$$
$$= \tfrac{1}{3} g_W^2 m_W^2 \left[\tfrac{1}{4}(1+\cos\theta)^2 + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}(1-\cos\theta)^2\right]$$
$$= \tfrac{1}{3} g_W^2 m_W^2$$

Therefore the $W^- \to e^-\overline{\nu}_e$ decay rate is

$$\Gamma = \frac{p^*}{32\pi^2 m_W^2}\int \langle |M_{fi}|^2\rangle \, d\Omega^*$$

$$= \frac{p^*}{32\pi^2 m_W^2} 4\pi \tfrac{1}{3} g_W^2 m_W^2 = \frac{g_W^2 m_W}{48\pi}$$

[6]

(b) **Unseen**.
From the lepton universality and the small (compared to the W mass) differences between lepton masses we have:

$$\Gamma(W^- \to e^-\overline{\nu}_e) = \Gamma(W^- \to \mu^-\overline{\nu}_\mu) = \Gamma(W^- \to \tau^-\overline{\nu}_\tau)$$

The W boson can also decay to all flavours of quarks with the exception of the top quark, which is too massive ($m_t > m_W$). The decay rate of the W boson to a particular quark flavour needs to account for the elements of the CKM matrix and the three possible colours of the final state quarks.

$$\Gamma(W^- \to d\overline{u}) = 3|V_{ud}|^2\Gamma_{e\nu}, \quad \Gamma(W^- \to d\overline{c}) = 3|V_{cd}|^2\Gamma_{e\nu}$$

$$\Gamma(W^- \to s\overline{u}) = 3|V_{us}|^2\Gamma_{e\nu}, \quad \Gamma(W^- \to s\overline{c}) = 3|V_{cs}|^2\Gamma_{e\nu}$$

$$\Gamma(W^- \to b\overline{u}) = 3|V_{ub}|^2\Gamma_{e\nu}, \quad \Gamma(W^- \to b\overline{c}) = 3|V_{cb}|^2\Gamma_{e\nu}$$

From the unitarity of the CKM matrix:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \quad \text{and} \quad |V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1$$

Therefore the lowest order prediction for the W boson decay to quarks is:

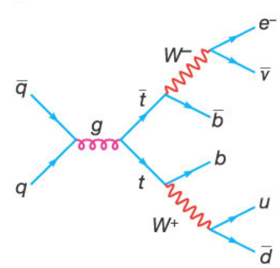
$$\Gamma(W^- \to q\overline{u}') = 6\Gamma(W^- \to e^-\overline{\nu}_e)$$

Thus the total decay rate of the W boson is

$$\Gamma_W = 9\Gamma_{W\to e\nu} = \frac{3g_W^2 m_W}{16\pi} \approx 2.07 GeV$$

[6]

(c) **[Unseen]**.
Any allowed combination of quarks/leptons in the diagram (below) is ok. **[2]**



(d) **[Unseen]**.
Because $|V_{tb}| >> |V_{ts}| > |V_{td}|$. **[1]**

(e) **[Covered in lectures]**.
b-quarks will be tagged in the vertex (silicon) detector using displaced vertices from b-quark decay. The jets are detected in the hadronic calorimeter.
Electrons will be detected in the EM calorimeter with a characteristic shape and depth of the EM shower. The sign of the electrons (or positrons) can be detected by the sign of the curvature in the magnetic field using the tracking detector. Neutrino will be associated with missing energy.
The hadronic final states (pions) of the second W decay will be detected in the hadronic calorimeter. **[3]**

(f) **[Covered in lectures]**.
By reconstructing the invariant mass of the final state particles, $W^2 = (\sum E)^2 - (\sum \mathbf{p})^2$, and comparing them with the known mass of W boson and top quark (taking into account detector resolution effects) one can statistically separate W and t from background events that are not peaked at the invariant mass values. **[2]**