PHASM(G)442/2017 Model Answers

1. (a) [**Covered in lectures, 2 marks**].

   Massive gauge bosons responsible for weak interactions, $W^{\pm}$ and Z, break gauge invariance. **[1]**
   WW scattering calculations violate unitarity, i.e. result in a probability > 1. **[1]**

   (b) [**Unseen, 3 marks**].

   The Higgs mechanism was introduced in order to "save" the gauge invariance of the Lagrangian with massive gauge bosons. **[1]**
   Under the simplest unitary $U(1)$ transformation[1] we have $\phi \rightarrow \phi' = \phi e^{iq\chi(x)}$, and therefore the even powers of ϕ will cancel out the exponential factor:

   $$\phi'^2 = \phi'\phi'^* = \phi\phi^* e^{iq\chi(x)} e^{-iq\chi(x)} = \phi\phi^* = \phi^2.$$

   maintaining the gauge invariance. **[1]**
   An odd power of ϕ will introduce an extra $e^{iq\chi(x)}$ factor, thus breaking the gauge invariance. **[1]**

   (c) [**Unseen, 4 marks**].

   The number of Higgs events expected in the ATLAS detector is

   $$N_H = \varepsilon \times L \times \sigma_{pp},$$

   where ε is the detector efficiency, L is the integrated luminosity and σ_{pp} is the Higgs production cross-section.
   Putting the numerical data in we obtain:

   $$N_H = 1 \times 20 \cdot 10^3 \text{fb} \times 10 \text{fb}^{-1} = 2 \cdot 10^5.$$

   [2]

   And finally, taking the Standard Model branching ratios:

   $$N(H \rightarrow b\bar{b}) = 2 \cdot 10^5 \times 0.578 \approx 1.2 \cdot 10^5$$

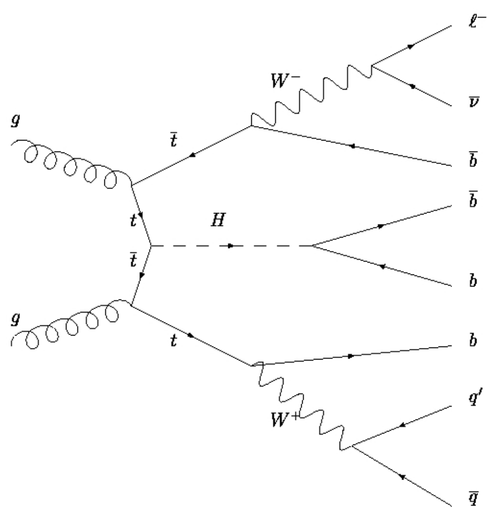
   $$N(H \rightarrow \gamma\gamma) = 2 \cdot 10^5 \times 2 \cdot 10^{-3} \times = 400.$$

   [2]

[1] U(1) symmetry group was used to explain the Higgs mechanism in the lectures due to its simplified algebra.

(d) [**Covered in lectures, 4 marks**].
Despite many more $b\bar{b}$ pairs produced in the detector from $H \to b\bar{b}$ the $H \to \gamma\gamma$ process is more likely to be observed first. This is because two γ events can be relatively easily identified without much background. They will have a clear signature in the detector with no tracks left in tracking detectors and energy depositions in electromagnetic calorimeter. **[2]**
On the contrary, $H \to b\bar{b}$ events will suffer from a large QCD background that dominates pp collisions at the LHC. Detecting a small bump on a large (and non-flat) background spectrum is very difficult and requires large statistics and good control of systematics. **[2]**

(e) [**Synthesis of ideas covered in lectures, 4 marks**].
The reference to pp collisions at LHC energies and to the mass of the Higgs boson is meant to test the knowledge of the dominant H production in gluon-gluon fusion and decay to $b\bar{b}$ as the dominant decay channel. If the Feynman diagram is not given the reference to the above will still attract 1 mark.

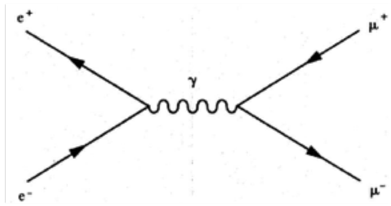


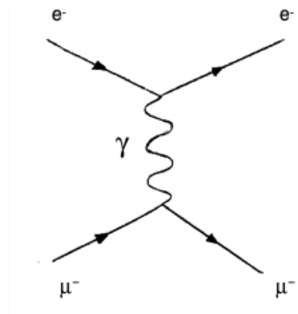
Note: Drawing the decay products of W's is not required. **[4]**

(f) [**Covered in lectures, 3 marks**].
b-quarks are identified (tagged) using the so called displaced vertex method. Given its lifetime the b-quark will travel (after hadronisation) a few mm before decaying. Due to a relatively large mass of the b-quark the decay products can be produced at a relatively large angle to the original b-quark direction.

Therefore the experimental signature in a collider detector is a jet of particles emerging from the collision point (primary vertex) and a secondary vertex from the b-quark decay displaced from the primary vertex by several mm. The identification of the displaced vertices requires a sub-mm spatial resolution and takes place in a innermost vertex (silicon) detector. **[3]**

2. (a) **[Covered in lectures, 2 marks.]**.





[2]

(b) **[Synthesis of ideas covered in lectures, 6 marks]**.

Using the definition of the Mandelstam variables and neglecting the lepton masses (i.e. $p_i^2 = m_i^2 = 0$) we can write:

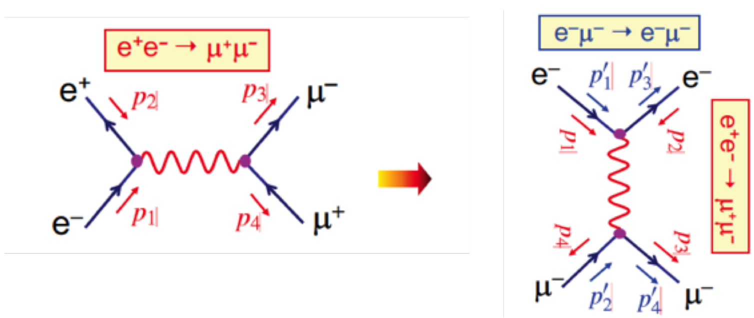
$$s = (p_1 + p_2)^2 = 2p_1 \cdot p_2 \quad t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 \quad u = (p_1 - p_4)^2 = -2p_1 \cdot p_4.$$

Therefore the spin-averaged matrix element becomes

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{(p_1 p_3)^2 + (p_1 p_4)^2}{(p_1 p_2)^2} \right).$$

[3]

Using the crossing symmetry argument illustrated in the figure below:



we obtain

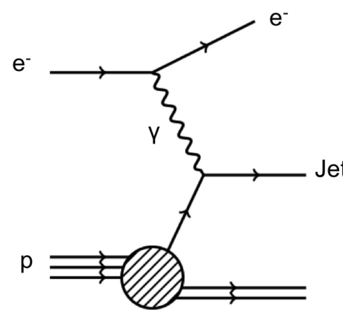
$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{(p'_1 p'_4)^2 + (p'_1 p'_2)^2}{(p'_1 p'_3)^2} \right).$$

Finally, using the Mandelstam definition above we arrive at

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{u^2 + s^2}{t^2} \right),$$

[3]

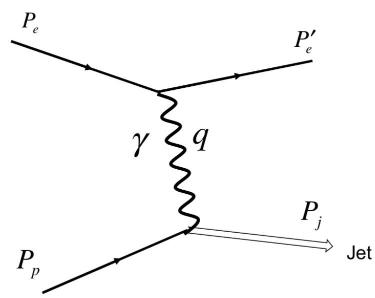
(c) [**Covered in lectures, 2 marks**].



Note: Reference to $Q^2 = 100$ GeV^2 should trigger a DIS Feynman diagram above. **[2]**

(d) [**Unseen, 6 marks**].

Using the notations in the Feynman diagram below



we have $q = P_j - P_p$ and therefore:

$$y = \frac{P_p \cdot (P_j - P_p)}{P_p \cdot P_e} = \frac{P_p \cdot P_j}{P_p \cdot P_e},$$

since $P_p^2 = M_p^2 = 0$. **[2]**

The numerator can then be written as

$$P_p \cdot P_j = E_p \cdot E_j - \vec{p}_p \cdot \vec{p}_j = E_p(E_j - p_j^z),$$

and the denominator is

$$P_p \cdot P_e = E_p \cdot E_e - \vec{p}_p \cdot \vec{p}_e = 2E_pE_e,$$

where in the above we neglected the proton and electron masses. Therefore,

$$y = \frac{E_j - p_j^z}{2E_e},$$

[4]

(e) **[Unseen, 4 marks]**.

We can write $u(x)$ and $\bar{u}(x)$ in terms of valence and sea quark contributions:

$$u(x) = u_V(x) + u_S(x) \qquad \bar{u}(x) = \bar{u}_S(x).$$

[1]

The sea component arises from contributions of virtual quark-antiquark pairs. **[1]**

It is therefore reasonable to assume that $u_S(x) = \bar{u}_S(x)$. **[1]**

Hence,

$$\int_0^1 [u(x) - \bar{u}(x)]\, dx = \int_0^1 [u_V(x) + u_S(x) - u_S(x)]\, dx = \int_0^1 u_V(x) dx = 2.$$

[2]

3. (a) [**Synthesis of ideas covered in lectures, 6 marks**].

It is necessary to check whether the matrix element of the pure vector and pure axial-vector interactions is invariant under parity transformation.

Since $M \propto j_1 \cdot j_2$ we need to check the invariance of $j_1 \cdot j_2$ under parity transformation. **[1]**

For pure vector interaction:

$$j_V = \bar{\psi}\gamma^\mu\phi \xrightarrow{\hat{P}} \bar{\psi}\gamma^0\gamma^\mu\gamma^0\phi,$$

and by components:

$$j_V^0 \xrightarrow{\hat{P}} \bar{\psi}\gamma^0\gamma^0\gamma^0\phi = j_V^0$$

$$j_V^k \xrightarrow{\hat{P}} \bar{\psi}\gamma^0\gamma^k\gamma^0\phi = -\bar{\psi}\gamma^k\gamma^0\gamma^0\phi = -j_V^k,$$

where $k = 1, 2, 3$. Therefore,

$$j_{V1} \cdot j_{V2} \xrightarrow{\hat{P}} j_{V1}^0 \cdot j_{V2}^0 - \sum_{k=1,3} (-j_{V1}^k)(-j_{V2}^k) = j_{V1} \cdot j_{V2}.$$

Hence, parity is conserved in pure vector interactions (such as QED or QCD). **[3]**

Using similar arguments for axial-vector currents, $j_A^0 \xrightarrow{\hat{P}} -j_A^0$ and $j_A^k \xrightarrow{\hat{P}} j_A^k$, and therfore

$$j_{A1} \cdot j_{A2} \xrightarrow{\hat{P}} (-j_{A1}^0) \cdot (-j_{A2}^0) - \sum_{k=1,3} j_{A1}^k j_{A2}^k = j_{A1} \cdot j_{A2}.$$

Hence, a pure axial-vector interaction also conserves parity. **[2]**

(b) [**Covered in lectures, 4 marks**].

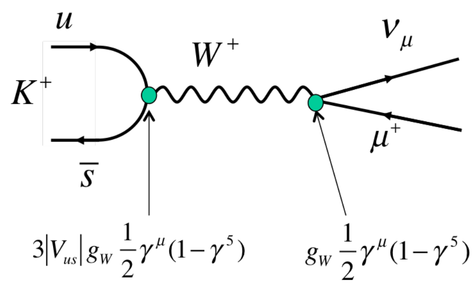
From expressions obtained in part (a):

$$j_{V1} \cdot j_{A2} \xrightarrow{\hat{P}} j_{V1}^0 \cdot (-j_{A2}^0) - \sum_{k=1,3} (-j_{V1}^k) j_{A2}^k = -j_{V1} \cdot j_{A2}.$$

Therefore $j_V \cdot j_A$ violates parity. **[3]**

Experimentally, weak interactions violate parity and therefore contain a combination of V and A currents. From experiments, such as muon decay we know that weak interaction is of $V - A$ type. **[1]**

(c) [**Similar but not this particular example is covered in lectures, 4 marks**].



[4]

(d) **[Unseen, 6 marks]**.

In the kaon rest frame from the conservation of momentum and energy we have:

$$\vec{p}_\mu = -\vec{p}_\nu \qquad |\vec{p}_\mu| = |\vec{p}_\nu| = E_\nu \qquad E_\nu = m_K - E_\mu = m_K - \gamma m_\mu.$$

From the 4-momentum conservation:

$$P_K^2 = m_K^2 = (P_\mu + P_\nu)^2 = P_\mu^2 + 2P_\mu P_\nu = m_\mu^2 + 2\left(E_\mu E_\nu - \vec{p}_\mu \cdot \vec{p}_\nu\right).$$

[2]

However from 3-momentum and energy conservation above:

$$\vec{p}_\mu \cdot \vec{p}_\nu = -E_\nu^2 = -\left(m_K - \gamma m_\mu\right)^2.$$

[1]

Putting it all together:

$$m_K^2 = m_\mu^2 + 2E_\mu E_\nu + 2\left(m_K - \gamma m_\mu\right)^2$$

$$m_K^2 = m_\mu^2 + 2\gamma m_\mu\left(m_K - \gamma m_\mu\right) + 2\left(m_K - \gamma m_\mu\right)^2$$

$$m_K^2 = m_\mu^2 + 2\gamma m_\mu m_K - 2\gamma^2 m_\mu^2 + 2m_K^2 - 4\gamma m_K m_\mu + 2\gamma^2 m_\mu^2.$$

$$m_K^2 = m_\mu^2 - 2\gamma m_\mu m_K + 2m_K^2.$$

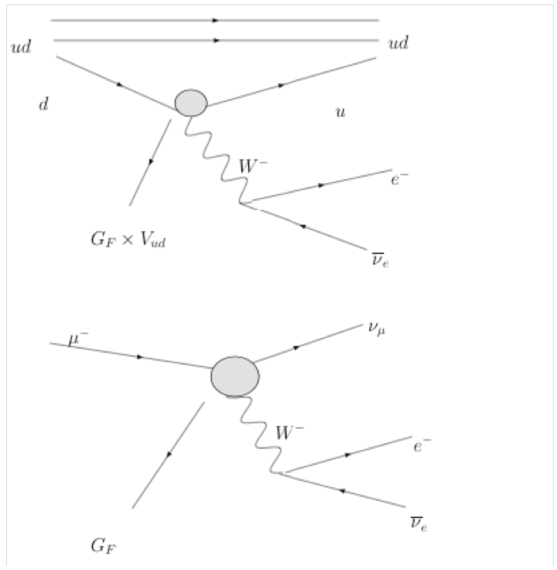
From which we have the required:

$$\gamma = \frac{m_K^2 + m_\mu^2}{2m_\mu m_K}.$$

[3]

4. (a) **[Covered in lectures, 3 marks.]**.

The two relevant Feynman diagrams are:



[2]

(b) **[Synthesis of ideas covered in lectures, 4 marks]**.

Both processes can be described as a point interaction with the Fermi coupling constant characterising the strength of the interaction. Due to the V_{ud} factor the value of G_F^β is a bit smaller than G_F^μ. **[1]**

The decay rate is $\Gamma \propto G_F^2$. Since $G_F^\mu = G_F$ and $G_F^\beta = G_F \times |V_{ud}|$ we expect

$$\frac{G_F^\beta}{G_F^\mu} = |V_{ud}| \approx 0.974, \qquad \frac{\Gamma^\beta}{\Gamma^\mu} = (|V_{ud}|)^2 \approx 0.949.$$

[3]

(c) **[Covered in lectures, 2 marks]**.

In an appearance experiment the detector placed at a distance from a source of neutrinos of a particular flavour, e.g. ν_e, searches for neutrinos of a different flavour into which the initial neutrino beam can oscillate, e.g. ν_μ.

In a disappearance experiment a deficit of the original neutrino flavour at a distant detector is searched for. **[2]**

(d) **[Unseen, 3 marks]**.

In nuclear reactors only electron anti-neutrinos are emitted. For an appearance experiment the far detector should identify either a μ^+ or τ^+ to detect $\bar{\nu}_e \to \bar{\nu}_\mu$ or $\bar{\nu}_e \to \bar{\nu}_\tau$ oscillations respectively. **[1]**

However, with the maximum energy of 9 MeV the reactor $\bar{\nu}_e$ will not be able to produce a μ^+ ($m_\mu \approx 106$ MeV) or τ^+ ($m_\tau \approx 1.8$ GeV). Thus, an appearance experiment with reactor $\bar{\nu}_e$ is not possible. **[2]**

(e) **[Unseen, 5 marks]**.

The minimum energy of the neutrino necessary for its detection via the reaction $\nu_\mu + n \to \mu^- + p$ requires the production of μ^- and p at rest. In other words this reaction is only kinematically possible if $s \geq (m_\mu + m_p)^2$, where s is the centre-of-mass energy squared.
Consequently,

$$s = (P_\nu + P_n)^2 = (E_\nu + m_n)^2 - E_\nu^2 = 2E_\nu m_n + m_n^2 \geq (m_\mu + m_p)^2.$$

[2]

For the neutrino minimum energy we have:

$$2E_\nu^{min} m_n + m_n^2 = (m_\mu + m_p)^2.$$

Assuming $m_n \approx m_p$,

$$2E_\nu^{min} m_n = m_\mu^2 + 2m_\mu m_p,$$

from which

$$E_\nu^{min} = \frac{m_\mu^2 + 2m_\mu m_p}{2m_n}.$$

[2]

Putting in the numerical values:

$$E_\nu^{min} = \frac{106^2 + 2 \cdot 106 \cdot 1000}{2 \cdot 1000} \approx 112\text{MeV}.$$

[1]

(f) **[Unseen, 4 marks]**.

In order to carry out an appearance $\nu_e \to \nu_\mu$ experiment a *pure* ν_e beam with an energy above 112 MeV[2] needs to be created . **[1]**
ν_e beams can in principle be created from muon decays but they will be contaminated by ν_μ. **[1]**
The only pure ν_e source is available from β^+ radioactive isotopes (or K-capture). **[1]**
These isotopes can be used to create radioactive ion beams and accelerate the ions to momenta sufficient to delver ν_e beams with energies above 100 MeV. **[1]**

[2]Failing question (e) should not prevent the student from tackling question (f) based on the muon mass argument – a pure ν_e beam above 100 MeV is still needed.

5. (a) **[Covered in lectures, 4 marks]**.

Consider the derivatives of the free-antiparticle plane wave solution $\psi = v(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}$:

$$\partial_0\psi = \frac{\partial\psi}{\partial t} = iE\psi; \quad \partial_1\psi = \frac{\partial\psi}{\partial x} = -ip_x\psi; \quad ...$$

[2]

Substituting these into the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ gives:

$$\left(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m\right) v = 0$$

which can be written:

$$\left(\gamma^\mu p_\mu + m\right) v = 0$$

[2]

(b) **[Unseen, 6 marks]**.

Taking the Hermitian conjugate of the Dirac equation:

$$\left[\left(\gamma^\mu p_\mu - m\right) u\right]^\dagger = u^\dagger \left(\gamma^{\mu\dagger} p_\mu - m\right) = 0$$

This can be rewritten as

$$u^\dagger\gamma^0\gamma^0 \left(\gamma^{\mu\dagger} p_\mu - m\right) = u^\dagger\gamma^0 \left(\gamma^0\gamma^{\mu\dagger} p_\mu - \gamma^0 m\right) = 0.$$

[2]

Using the identity $\gamma^0\gamma^{\mu\dagger} = \gamma^\mu\gamma^0$ that follows from:

$$\gamma^0\gamma^{0\dagger} = \gamma^0\gamma^0; \qquad \gamma^0\gamma^{k\dagger} = -\gamma^0\gamma^k = \gamma^k\gamma^0 \quad \text{where} \quad k = 1,2,3,$$

[2]

we have:

$$\bar{u}\left(\gamma^\mu\gamma^0 p_\mu - m\gamma^0\right) = \bar{u}\left(\gamma^\mu p_\mu - m\right)\gamma^0 = 0.$$

Finally, multiplying both parts by γ^0 we obtain the required:

$$\bar{u}\left(\gamma^\mu p_\mu - m\right) = 0.$$

[2]

(c) **[Covered in lectures, 4 marks]**.

For the particle solution $\psi = ue^{+i(\vec{p}\cdot\vec{r}-Et)}$:

$$\hat{H}\psi = i\frac{\partial}{\partial t}\left[ue^{+i(\vec{p}\cdot\vec{r}-Et)}\right] = i^2(-E)ue^{+i(\vec{p}\cdot\vec{r}-Et)} = E\psi,$$

Therefore E is the real eigenvalue of $\hat{H}$ and represent the physical energy of the particle. **[2]**

For the *anti*-particle solution $\psi = ve^{-i(\vec{p}\cdot\vec{r}-Et)}$ the same argument as above will lead to a negative energy solution.

By swapping the sign of the $\hat{H}$ operator we obtain

$$\hat{H}^{(v)}\psi = -i\frac{\partial}{\partial t}ve^{-i(\vec{p}\cdot\vec{r}-Et)} = (-i)(-i)(-E)ve^{-i(\vec{p}\cdot\vec{r}-Et)} = E\psi,$$

i.e. positive energy solutions for *anti*-particles. **[2]**

(d) **[Unseen, 6 marks.]**.

Using the identity $(\gamma^5)^2 = 1$,

$$P_R P_R = \frac{1}{4}\left(1+\gamma^5\right)\left(1+\gamma^5\right) = \frac{1}{4}\left(1+\gamma^5+\gamma^5+\left(\gamma^5\right)^2\right) =$$

$$\frac{1}{4}\left(2+2\gamma^5\right) = \frac{1}{2}\left(1+\gamma^5\right) = P_R.$$

By analogy $P_L P_L = P_L$ and hence $P_i^2 = P_i$. **[3]**

$$P_R P_L = \frac{1}{4}\left(1+\gamma^5\right)\left(1-\gamma^5\right) = \frac{1}{4}\left(1-\left(\gamma^5\right)^2\right) = 0.$$

[2]

$$P_L + P_R = \frac{1}{2}\left(1+\gamma^5+1-\gamma^5\right) = 1.$$

[1]