Diffractive parton density functions

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Ringberg Workshop: New Trends in HERA Physics 2005

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Eur. Phys. J. C **44** (2005) 69 (hep-ph/0504132) Eur. Phys. J. C **37** (2004) 285 (hep-ph/0406224)

Outline

- Diffractive deep-inelastic scattering (DDIS) is characterised by a large rapidity gap due to Pomeron (vacuum quantum number) exchange.
- How do we extract diffractive parton density functions (DPDFs) from DDIS data?
- 1. Demise of the 'Regge factorisation' approach currently used by H1/ZEUS, where the exchanged Pomeron is treated as a hadron-like object.
- Rise of the 'perturbative QCD' approach, where the exchanged Pomeron is a parton ladder. Treatment of diffractive PDFs has more in common with photon PDFs than proton PDFs.

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Diffractive DIS kinematics



•
$$t \equiv (p - p')^2 \approx 0, (p - p') \approx x_{\mathbb{P}} p$$

►
$$M_X^2 \equiv (q + p - p')^2 = -Q^2 + x_{\mathbb{P}}(Q^2 + W^2)$$

 $\Rightarrow x_{\mathbb{P}} = \frac{Q^2 + M_X^2}{Q^2 + W^2}$
(fraction of proton's momentum carried by Pomeron)
► $\beta \equiv \frac{x_B}{x_D} = \frac{Q^2}{Q^2 + M^2}$ (fraction of Pomeron's momentum carr

• $\beta \equiv \frac{X_B}{X_P} = \frac{Q^2}{Q^2 + M_X^2}$ (fraction of Pomeron's momentum carried by struck quark)

Diffractive structure function $F_2^{D(3)}$

Diffractive cross section (integrated over t):

$$\frac{\mathrm{d}^3 \sigma^{\mathrm{D}}}{\mathrm{d} \mathbf{x}_{\mathbb{P}} \, \mathrm{d} \beta \, \mathrm{d} \, \mathrm{Q}^2} = \frac{2 \pi \alpha_{\mathrm{em}}^2}{\beta \, \mathrm{Q}^4} \, \left[1 + (1 - y)^2 \right] \, \sigma_r^{\mathrm{D}(3)}(\mathbf{x}_{\mathbb{P}}, \beta, \, \mathrm{Q}^2),$$

where $y = Q^2/(x_B s)$, $s = 4E_e E_p$, and

$$\sigma_r^{\mathrm{D}(3)} = F_2^{\mathrm{D}(3)} - \frac{y^2}{1 + (1 - y)^2} F_L^{\mathrm{D}(3)} \approx F_2^{\mathrm{D}(3)}(\mathbf{x}_{\mathbb{P}}, \beta, \mathbb{Q}^2),$$

for small y or assuming that $F_L^{\mathrm{D}(3)} \ll F_2^{\mathrm{D}(3)}$

Measurements of F₂^{D(3)} ⇒ diffractive parton distribution functions (DPDFs) a^D(x_P, z, Q²) = zq^D(x_P, z, Q²) or zg^D(x_P, z, Q²), where β ≤ z ≤ 1, cf. x_B ≤ x ≤ 1 in DIS.

$$F_2^{\mathrm{D}(3)} = \sum_{\boldsymbol{a}=q,g} C_{2,\boldsymbol{a}} \otimes \boldsymbol{a}^{\mathrm{D}} + \mathcal{O}(1/\mathrm{Q}), \tag{1}$$

where $C_{2,a}$ are the **same** coefficient functions as in inclusive DIS and where $a^{D} = zq^{D}$ or zg^{D} satisfy DGLAP evolution in Q^{2} :

$$\frac{\partial \boldsymbol{a}^{\mathrm{D}}}{\partial \ln \mathsf{Q}^{2}} = \sum_{\boldsymbol{a}'=q,g} \boldsymbol{P}_{\boldsymbol{a}\boldsymbol{a}'} \otimes \boldsymbol{a}'^{\mathrm{D}}$$
(2)

- Says nothing about the mechanism for diffraction: what is the colourless exchange ('Pomeron') which causes the large rapidity gap. Assuming a 'QCD Pomeron' we need to modify both (1) and (2).
- Factorisation is broken in hadron-hadron collisions, but hope that same formalism can be applied with extra suppression factor calculable from eikonal models.
- LO diffractive dijet photoproduction: resolved photon contribution should be suppressed. Complications at NLO → talk by M. Klasen.

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Assume Regge factorisation [Ingelman–Schlein,'85]:

$$a^{\mathrm{D}}(x_{\mathbb{P}}, z, \mathsf{Q}^2) = f_{\mathbb{P}}(x_{\mathbb{P}}) a^{\mathbb{P}}(z, \mathsf{Q}^2)$$

Pomeron flux factor from Regge phenomenology:

$$f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}) = \int_{t_{\mathrm{cut}}}^{t_{\mathrm{min}}} \mathrm{d}t \ \mathrm{e}^{\mathcal{B}_{\mathbb{P}} t} \ \mathbf{x}_{\mathbb{P}}^{1-2\alpha_{\mathbb{P}}(t)} \qquad (\alpha_{\mathbb{P}}(t) = \alpha_{\mathbb{P}}(0) + \alpha'_{\mathbb{P}} t)$$

"Regge factorisation relates the power of x_P measured in DDIS to the power of s measured in hadron–hadron elastic scattering." [Collins,'98]

Fit to H1 F₂^{D(3)} data gives α_P(0) = 1.17 > 1.08, the value of the 'soft Pomeron' [Donnachie–Landshoff,'92]. By Collins' definition, Regge factorisation is broken. H1/ZEUS meaning of 'Regge factorisation' is that the x_P dependence factorises as a power law, with the power independent of β and Q² (also broken, see later).
 Pomeron PDFs a^P(z, Q²) = zΣ^P(z, Q²) or zg^P(z, Q²) are DGLAP-evolved from inputs at Q₀² = 3 GeV²:

$$a^{\mathbb{P}}(z, Q_0^2) = \left[A_a + B_a(2z-1) + C_a\left(2(2z-1)^2 - 1\right)\right]^2 \exp(-0.01/(1-z))$$

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 P and
 R contributions. [ZEUS: Eur. Phys. J. C 38 (2004) 43, H1prelim-01-112]
- Look for large rapidity gap (LRG). (Non-diffractive contribution is exponentially suppressed as a function of the gap size.) Proton dissociation background. Both
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- 3. Use " M_X method". Subtract non-diffractive contribution in each (W, Q^2) bin by fitting:

$$\frac{\mathrm{d}N}{\mathrm{dln}\,M_X^2} = D + \underbrace{c\,\exp(b\,\mathrm{ln}\,M_X^2)}_{\mathrm{non-diffractive}}$$

Motivated by Regge theory assuming t = 0, $\alpha_{\mathbb{P}}(0) \equiv 1$, $Q^2 \ll M_X^2$. (Validity in pQCD?) Proton dissociation background. Only \mathbb{P} contribution. [ZEUS: Nucl. Phys. B **713** (2005) 3]

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H1 vs. ZEUS M_X DPDFs



Fits and plot by F.-P. Schilling (H1)

Same procedure used to fit H1 LRG and ZEUS M_X data. (ZEUS M_X data scaled by a constant factor to account for different amount of proton dissociation.)

- Gluon from ZEUS M_X fit ~ factor two smaller than gluon from H1 LRG data, due to different Q² dependence of the data sets. H1 2002 fit gives good agreement with (LRG) DDIS dijet and D* production data.
- ▶ N.B. 2-loop α_S fixed by $\Lambda_{QCD} = 200 \text{ MeV}$ for 4 flavours. Gives α_S values much smaller than world average ⇒ H1 2002 gluon artificially enhanced. Will be corrected for H1 publication.

H1 vs. ZEUS M_X vs. ZEUS LPS DPDFs



- No correction made for different amounts of proton dissociation.
- GLP = Groys–Levy–Proskuryakov (ZEUS) fit to ZEUS M_X data, gives much too low prediction for ZEUS (LRG) DDIS dijets.
- ZEUS LPS fit describes dijets well, but:

"The shape of the fitted PDFs changes significantly depending on the functional form of the initial parameterisation, a consequence of the relatively large statistical uncertainties of the present sample. Therefore, these data cannot constrain the shapes of the PDFs." [ZEUS: Eur. Phys. J. C 38 (2004) 43]

Q² dependence of effective Pomeron intercept



- Recall that 'Regge factorisation' fits assume that $\alpha_{\mathbb{P}}(0)$ is independent of β and Q^2 .
- α_P(0) clearly rises with Q², but is smaller than in inclusive DIS, indicating that the x_P dependence is controlled by some scale μ² < Q².
- α_ℙ(0) > 1.08 [Donnachie–Landshoff,'92] indicating that the Pomeron in DDIS is not the 'soft' Pomeron exchanged in hadron–hadron collisions ⇒ should use pQCD instead of Regge phenomenology. In pQCD, Pomeron exchange can be described by two-gluon exchange.

How to reconcile two-gluon exchange with DPDFs?



Two-gluon exchange calculations are the basis for the colour dipole model description of DDIS (\rightarrow talk by G. Shaw).



Right: x_PF^{D(3)}₂ for x_P = 0.0042 as a function of β

[Golec-Biernat-Wüsthoff,'99].

- dotted lines: γ^{*}_T → qq̄g,
- dashed lines: $\gamma_T^* \to q\bar{q}$,
- dot-dashed lines: $\gamma_L^* \rightarrow q\bar{q}$,

important at low, medium, and high β respectively.

γ^{*}_L → qq̄ is higher-twist, but DPDFs only include leading-twist contributions, therefore H1/ZEUS DPDFs are artificially large at high z.



p.11/27

The QCD Pomeron is a parton ladder

- Generalise γ^{*} → qq̄ and γ^{*} → qq̄g to arbitrary number of parton emissions [Ryskin,'90; Levin–Wüsthoff,'94].
- ► Work in Leading Logarithmic Approximation (LLA) ⇒ transverse momenta are strongly ordered.



 New feature: integral over scale μ² (starting scale for DGLAP evolution of Pomeron PDFs).

$$F_2^{\mathrm{D}(3)} = \int_{\mu_0^2}^{Q^2} \frac{\mathrm{d}\mu^2}{\mu^2} f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}};\mu^2) F_2^{\mathbb{P}}(\beta, Q^2;\mu^2)$$
$$f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}};\mu^2) = \frac{1}{\mathbf{x}_{\mathbb{P}}B_D} \left[R_g \frac{\alpha_{\mathcal{S}}(\mu^2)}{\mu} \mathbf{x}_{\mathbb{P}}g(\mathbf{x}_{\mathbb{P}},\mu^2) \right]^2$$
$$\frac{1}{2}(\beta, Q^2;\mu^2) = \sum_{a=q,g} C_{2,a} \otimes a^{\mathbb{P}}$$

²₀ ~ 1 GeV², B_D from t-integration, R_g from skewedness [Shuvaev et al.,'99]
 Pomeron PDFs a^P(z, Q²; μ²) DGLAP-evolved from an input scale μ² up to Q².

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Gluonic and sea-quark Pomeron



- Pomeron structure function F^P₂(β, Q²; μ²) calculated from quark singlet Σ^P(z, Q²; μ²) and gluon g^P(z, Q²; μ²) DGLAP-evolved from an input scale μ² up to Q².
- Input Pomeron PDFs $\Sigma^{\mathbb{P}}(z, \mu^2; \mu^2)$ and $g^{\mathbb{P}}(z, \mu^2; \mu^2)$ are Pomeron-to-parton splitting functions.

LO Pomeron-to-parton splitting functions



 LO Pomeron-to-parton splitting functions calculated in Eur. Phys. J. C 44 (2005) 69.

▶ Notation: ' $\mathbb{P} = G$ ' means gluonic Pomeron, ' $\mathbb{P} = S$ ' means sea-quark Pomeron, ' $\mathbb{P} = GS$ ' means interference between these.

$$\begin{split} z\Sigma^{\mathbb{P}=G}(z,\mu^2;\mu^2) &= P_{q,\mathbb{P}=G}(z) = z^3 (1-z), \\ zg^{\mathbb{P}=G}(z,\mu^2;\mu^2) &= P_{g,\mathbb{P}=G}(z) = \frac{9}{16} (1+z)^2 (1-z)^2 \\ z\Sigma^{\mathbb{P}=S}(z,\mu^2;\mu^2) &= P_{q,\mathbb{P}=S}(z) = \frac{4}{81} z (1-z), \\ zg^{\mathbb{P}=S}(z,\mu^2;\mu^2) &= P_{g,\mathbb{P}=S}(z) = \frac{1}{9} (1-z)^2, \\ z\Sigma^{\mathbb{P}=GS}(z,\mu^2;\mu^2) &= P_{q,\mathbb{P}=GS}(z) = \frac{2}{9} z^2 (1-z), \\ zg^{\mathbb{P}=GS}(z,\mu^2;\mu^2) &= P_{g,\mathbb{P}=GS}(z) = \frac{1}{4} (1+2z) (1-z)^2 \end{split}$$

Evolve these input Pomeron PDFs from μ^2 up to Q^2 using NLO DGLAP evolution.

Contribution to $F_2^{D(3)}$ as a function of μ^2

$$\begin{aligned} F_2^{\mathrm{D}(3)} &= \int_{\mu_0^2}^{Q^2} \frac{\mathrm{d}\mu^2}{\mu^2} f_{\mathbb{P}}(\boldsymbol{x}_{\mathbb{P}}; \mu^2) F_2^{\mathbb{P}}(\beta, Q^2; \mu^2) \\ f_{\mathbb{P}}(\boldsymbol{x}_{\mathbb{P}}; \mu^2) &= \frac{1}{\boldsymbol{x}_{\mathbb{P}} B_D} \left[R_g \frac{\alpha_{\mathcal{S}}(\mu^2)}{\mu} \, \boldsymbol{x}_{\mathbb{P}} g(\boldsymbol{x}_{\mathbb{P}}, \mu^2) \right]^2 \end{aligned}$$

- Naïvely, f_ℙ(x_ℙ; µ²) ~ 1/µ², so contributions from large µ² are strongly suppressed.
- But x_Pg(x_P, μ²) ~ (μ²)^γ, where γ is the anomalous dimension. In BFKL limit γ ≃ 0.5, so f_P(x_P; μ²) ~ constant.
- HERA domain is in an intermediate region: γ is not small, but is less than 0.5.
- ▶ Plot integrand as a function of μ^2 (using MRST2001 NLO PDFs) ⇒ large contribution from large μ^2 .
- ► H1 (ZEUS) fits assume that $\mu^2 < Q_0^2$, where $Q_0^2 = 3$ (2) GeV² for H1 (ZEUS) fits.



Inhomogeneous evolution of DPDFs

$$F_{2}^{D(3)} = \sum_{a=q,g} C_{2,a} \otimes a^{D},$$

where $a^{D}(\mathbf{x}_{\mathbb{P}}, \mathbf{z}, \mathbf{Q}^{2}) = \int_{\mu_{0}^{2}}^{\mathbf{Q}^{2}} \frac{d\mu^{2}}{\mu^{2}} f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}; \mu^{2}) a^{\mathbb{P}}(\mathbf{z}, \mathbf{Q}^{2}; \mu^{2})$
$$\implies \frac{\partial a^{D}}{\partial \ln \mathbf{Q}^{2}} = \int_{\mu_{0}^{2}}^{\mathbf{Q}^{2}} \frac{d\mu^{2}}{\mu^{2}} f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}; \mu^{2}) \frac{\partial a^{\mathbb{P}}}{\partial \ln \mathbf{Q}^{2}} + f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}; \mu^{2}) a^{\mathbb{P}}(\mathbf{z}, \mathbf{Q}^{2}; \mu^{2}) \Big|_{\mu^{2} = \mathbf{Q}^{2}}$$
$$= \int_{\mu_{0}^{2}}^{\mathbf{Q}^{2}} \frac{d\mu^{2}}{\mu^{2}} f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}; \mu^{2}) \sum_{a'=q,g} P_{aa'} \otimes a'^{\mathbb{P}} + f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}; \mathbf{Q}^{2}) a^{\mathbb{P}}(\mathbf{z}, \mathbf{Q}^{2}; \mathbf{Q}^{2})$$
$$= \underbrace{\sum_{a'=q,g} P_{aa'} \otimes a'^{D}}_{\mathsf{DGLAP term}} + \underbrace{f_{\mathbb{P}}(\mathbf{x}_{\mathbb{P}}; \mathbf{Q}^{2}) P_{a\mathbb{P}}(\mathbf{z})}_{\mathsf{Extra inhomogeneous term}}$$

Inhomogeneous evolution of DPDFs is not a new idea:

"We introduce a diffractive dissociation structure function and show that it obeys the DGLAP evolution equation, **but**, with an additional inhomogeneous term." [Levin–Wüsthoff,'94]

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Pomeron structure is analogous to photon structure Diffractive structure function



Photon structure function



Dijets in diffractive photoproduction Resolved photon Dire



Effect of direct Pomeron coupling was considered by Kniehl–Kohrs–Kramer [Z. Phys. C 65 (1995) 657],

but with Pomeron coupling to quarks like a photon: $\mathcal{L}_{int} = c \, \bar{q}(x) \gamma_{\mu} q(x) \phi^{\mu}(x)$.

Need for NLO calculations

- NLO analysis of DDIS data is not yet possible.
- Need C_{2,P} and P_{aP} at NLO (help wanted!). Calculable with usual methods, e.g. LO diagrams are:



Dimensional regularisation: work in $4 - 2\varepsilon$ dimensions, collinear singularity appears as $1/\varepsilon$ pole multiplied by $P_{q\mathbb{P}}$, subtract in e.g. \overline{MS} factorisation scheme to leave finite remainder $C_{2,\mathbb{P}}$.

- Simplified analysis: take NLO C_{2,a} and P_{aa'} (a, a' = q, g), but LO C_{2,ℙ} and P_{aℙ}.
 - ▶ Work in Fixed Flavour Number Scheme (no charm DPDF), with charm production via NLO $\gamma^* g^{\mathbb{P}} \rightarrow c\bar{c}$ [Riemersma *et al.*,'95] and LO $\gamma^* \mathbb{P} \rightarrow c\bar{c}$ [Levin–Martin–Ryskin–Teubner.'97].
 - ▶ For light quarks, include LO $\gamma_L^* \mathbb{P} \to q\bar{q}$ (higher-twist), but not $\gamma_T^* \mathbb{P} \to q\bar{q}$ (leading-twist).

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Description of DDIS data

Take input quark singlet and gluon densities at Q₀² = 3 GeV² in the form:

$$egin{aligned} & z \Sigma^{\mathrm{D}}(x_{\mathbb{P}},z,Q_{0}^{2}) = \mathit{f}_{\mathbb{P}}(x_{\mathbb{P}}) \; \mathit{C}_{q} \, z^{A_{q}} (1-z)^{B_{q}}, \ & z g^{\mathrm{D}}(x_{\mathbb{P}},z,Q_{0}^{2}) = \mathit{f}_{\mathbb{P}}(x_{\mathbb{P}}) \; \mathit{C}_{g} \, z^{A_{g}} (1-z)^{B_{g}}, \end{aligned}$$

- Take f_P(x_P) as in the H1 2002 fit with α_P(0), C_a, A_a, and B_a (a = q, g) as free parameters.
- Treatment of secondary Reggeon as in H1 2002 fit.
- Fit H1 LRG and ZEUS M_X data separately with cuts M_X > 2 GeV and Q² > 3 GeV². Allow overall normalisation factors of 1.10 and 1.43 respectively to account for proton dissociation.
- Statistical and systematic experimental errors added in quadrature.
- Two types of fits:
 - "Regge" = 'Regge factorisation' approach (i.e. no C_{2,ℙ} or P_{aℙ}) as H1/ZEUS do.
 - ▶ "pQCD" = 'perturbative QCD' approach with LO C_{2,P} and P_{aP}.

"pQCD" fits to H1 and ZEUS M_X data





▶ χ^2 /d.o.f. = 0.71 (0.75 for "Regge" fit) ▶ χ^2 /d.o.f. = 0.84 (0.76 for "Regge" fit)

DPDFs from fits to H1 and ZEUS M_X data



• "pQCD" DPDFs are smaller at large z due to inclusion of the higher-twist $\gamma_L^*\mathbb{P} o qar q$.

"pQCD" DPDFs have slightly more rapid evolution due to the inhomogeneous term.

Difference between H1 and ZEUS M_X fits remains.

Q^2 slope of H1 vs. ZEUS M_X data



Peak due to threshold for $\gamma^* \mathbb{P} \to c\bar{c}$ at $\beta = Q^2/(Q^2 + 4m_c^2)$.

- Additional contributions to scaling violations apart from DGLAP contribution.
- All free parameters in 'DGLAP' part: ZEUS M_X data have smaller scaling violations.

$x_{\mathbb{P}}$ dependence of H1 vs. ZEUS M_X data

Fit σ_r^{D(3)} ∝ f_P(x_P) in each (β, Q²) bin with ≥ 4 data points and y < 0.45 (additional cut x_P < 0.01 for H1 data):</p>



- ▶ Inhomogeneous evolution can account for rise of $\alpha_{\mathbb{P}}(0)$ with Q^2 .
- Inhomogeneous evolution breaks Regge factorisation.

Predictions for diffractive charm production



Data measured using LRG method.

- H1 DPDFs give good description, ZEUS M_X DPDFs too small at low β.
- ▶ Direct Pomeron contribution significant at moderate β . These charm data points are included in the ZEUS LPS fit, but only $\gamma^* g^{\mathbb{P}} \rightarrow c\bar{c}$ contribution is included and not the $\gamma^* \mathbb{P} \rightarrow c\bar{c}$ contribution. Therefore, diffractive gluon from ZEUS LPS fit needs to be artificially large to fit the charm data.

Non-linear evolution of inclusive PDFs





- Interesting application of DDIS formalism to calculate shadowing corrections to inclusive DIS via AGK cutting rules.
- Inhomogeneous evolution of DPDFs ⇒ non-linear evolution of inclusive PDFs.
- More precise version of GLRMQ equation derived.
- Fit HERA F₂ data similar to MRST2001 NLO fit. Small-x gluon enhanced at low scales.

For more details see Phys. Lett. B 627 (2005) 97.

- Diffractive DIS is more complicated than inclusive DIS.
- Collinear factorisation holds, but need to account for direct Pomeron coupling:

$$\begin{split} \mathcal{F}_{2}^{\mathrm{D}(3)} &= \sum_{a=q,g} \, \mathcal{C}_{2,a} \otimes a^{\mathrm{D}} \, + \, \mathcal{C}_{2,\mathbb{P}} \\ \frac{\partial a^{\mathrm{D}}}{\partial \ln \, \mathsf{Q}^{2}} &= \sum_{a'=q,g} \, \mathcal{P}_{aa'} \otimes a'^{\mathrm{D}} \, + \, \mathcal{P}_{a\mathbb{P}}(z) \, f_{\mathbb{P}}(x_{\mathbb{P}}; \, \mathsf{Q}^{2}) \end{split}$$

Analogous to the photon case.

- Outlook
 - Are the LRG and M_X methods compatible? (If so, is the amount of proton dissociation Q² dependent?)
 - ▶ Need $C_{2,\mathbb{P}}$ and $P_{a\mathbb{P}}$ (a = q, g) at NLO.
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