Maximum Likelihood and Extended Maximum Likelihoods with signal and background

With no background present, the expressions for the log likelihood and extended log likelihood are as follows:

$$-\mathcal{L}(\alpha) = -\sum_{i=1}^{N_{\rm O}} \ln P_{\rm S}(x_i,\alpha) \tag{1}$$

$$-\mathcal{L}_{\rm E}(\alpha) = -\sum_{i=1}^{N_{\rm O}} \ln P_{\rm S}(x_i,\alpha) \; + \; N_{\rm S}(\alpha) - N_{\rm O} \ln N_{\rm S}(\alpha), \tag{2}$$

where $N_{\rm O}$ is the number of events observed, $P_{\rm S}(x_i,\alpha)$ is the probability of a point arising at x_i and $N_{\rm S}(\alpha)$ is the expected number of observed events when the relvant parameter is equal to α.

When there is an expected background of $N_{\rm BG}$ events distributed according to $P_{\rm BG}$, these expressions are modified to become:

$$-\mathcal{L}'(\alpha) = -\sum_{i=1}^{N_{\rm O}} \ln \left(\frac{(N_{\rm O} - N_{\rm BG})}{N_{\rm O}} P_{\rm S}(x_i,\alpha) \; + \; \frac{N_{\rm BG}}{N_{\rm O}} P_{\rm BG}(x_i) \right) \tag{3}$$

$$\begin{aligned} -\mathcal{L}'_{\rm E}(\alpha) &= -\sum_{i=1}^{N_{\rm O}} \ln \left(\frac{N_{\rm S}(\alpha)}{N_{\rm S}(\alpha)+N_{\rm BG}} P_{\rm S}(x_i,\alpha) \; + \; \frac{N_{\rm BG}}{N_{\rm S}(\alpha)+N_{\rm BG}} P_{\rm BG}(x_i) \right) \\ &+ \quad (N_{\rm S}(\alpha)+N_{\rm BG}) \; - \; N_{\rm O} \ln (N_{\rm S}(\alpha)+N_{\rm BG}). \end{aligned} \tag{4}$$

In this way, the fit to shape only is preserved for the normal log likelihood fit, and also the expected number of events, according to α, is included in the extended log likelihood fit.

If there were an α-dependent background, of $N^{\rm D}_{\rm BG}(\alpha)$ distributed according to $P^{\rm D}_{\rm BG}(\alpha)$, the following form may be appropriate:

$$\begin{aligned} -\mathcal{L}''(\alpha) &= -\sum_{i=1}^{N_{\rm O}} \ln \left\{ \frac{(N_{\rm O} - N_{\rm BG})}{N_{\rm O}} \left(f(\alpha) P_{\rm S}(x_i,\alpha) \; + \; (1-f(\alpha)) P^{\rm D}_{\rm BG}(x_i,\alpha) \right) \right. \\ &+ \quad \left. \frac{N_{\rm BG}}{N_{\rm O}} P_{\rm BG}(x_i) \right\} \end{aligned} \tag{5}$$

$$
\begin{aligned}
-\mathcal{L}''_{\mathrm{E}}(\alpha) \;=\; & -\sum_{i=1}^{N_{\mathrm{O}}} \ln \left\{ \frac{N_{\mathrm{S}}(\alpha)}{N_{\mathrm{S}}(\alpha) + N^{\mathrm{D}}_{\mathrm{BG}}(\alpha) + N_{\mathrm{BG}}} P_{\mathrm{S}}(x_i, \alpha) \right. \\
& + \frac{N^{\mathrm{D}}_{\mathrm{BG}}(\alpha)}{N_{\mathrm{S}}(\alpha) + N^{\mathrm{D}}_{\mathrm{BG}}(\alpha) + N_{\mathrm{BG}}} P^{\mathrm{D}}_{\mathrm{BG}}(x_i, \alpha) \\
& \left. + \frac{N_{\mathrm{BG}}}{N_{\mathrm{S}}(\alpha) + N^{\mathrm{D}}_{\mathrm{BG}}(\alpha) + N_{\mathrm{BG}}} P_{\mathrm{BG}}(x_i) \right\} \\
& + \left(N_{\mathrm{S}}(\alpha) + N^{\mathrm{D}}_{\mathrm{BG}}(\alpha) + N_{\mathrm{BG}}\right) \\
& - N_{\mathrm{O}} \ln \left(N_{\mathrm{S}}(\alpha) + N^{\mathrm{D}}_{\mathrm{BG}}(\alpha) + N_{\mathrm{BG}}\right), \qquad (6)
\end{aligned}
$$

where $f(\alpha)$ is the ratio of $N_{\mathrm{S}}(\alpha)$ to $(N_{\mathrm{S}}(\alpha) + N^{\mathrm{D}}_{\mathrm{BG}}(\alpha))$.

Errors on N_{BG} and $N_{\mathrm{S}}(\alpha)$ should also be incorporated.