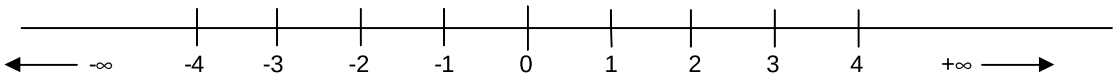
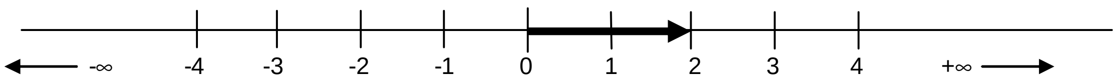
Complex Numbers & Impedance

j

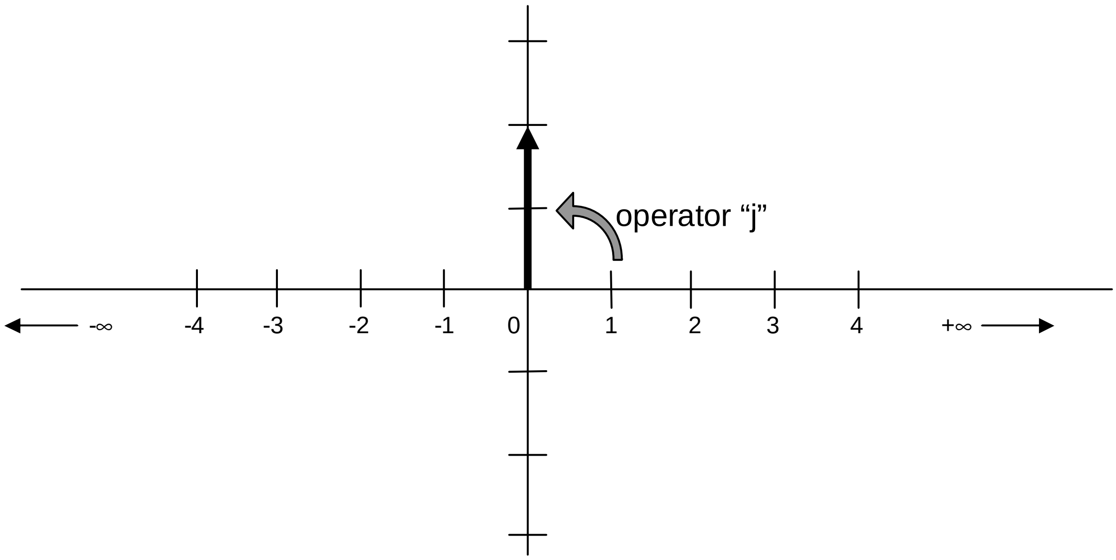
The concept of a *number line* is well-known.



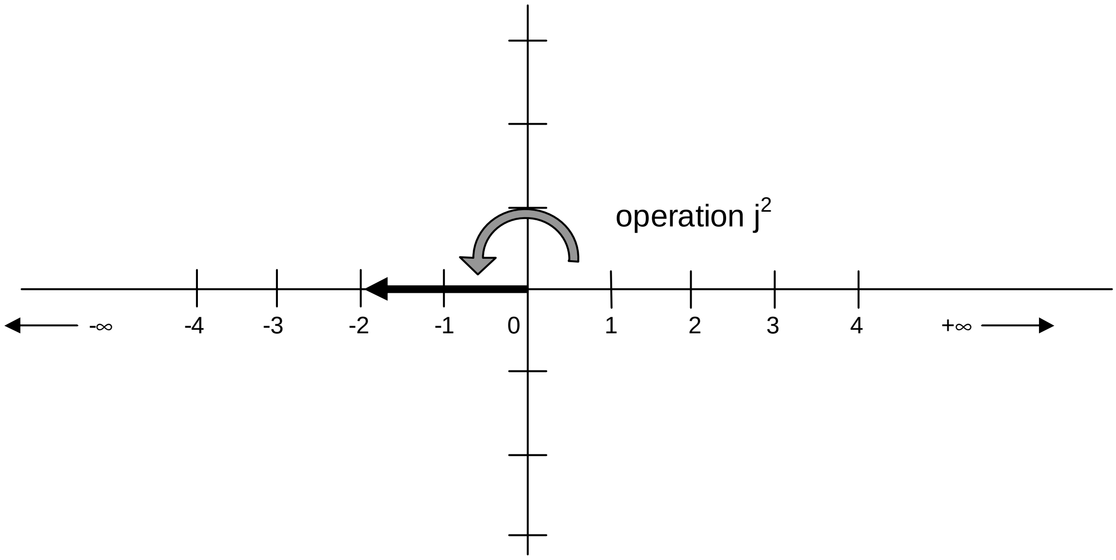
All the integers, rational and irrational numbers lie along this line. We can envisage each number as a vector on this line, thus the number 2 is shown like this.



Now we do something which looks outrageous, if you have never met complex numbers!



This multiplies the number 2 by an *operator* that rotates it anti-clockwise by a right angle. If we do this operation twice, j (j(2)) = j^2.2:



So $j^2.2 = -2$, therefore $j^2 = -1$ or $j = \sqrt{(-1)}$, a quantity that in elementary maths does not exist. Plotting on this plane is an *Argand Diagram*. The original number line is the *real axis* and the vertical axis is called the *imaginary axis*. Points plotted anywhere on the plane represent *complex numbers* of the form a+jb, where a is *real* and b is *imaginary*. The term *imaginary* is very unfortunate because is suggests that complex numbers are improper in some way. They are very strange when you first meet them and the term *imaginary* has been used since the time of Descartes, which shows that we are far from the first people to find them strange.

Where did they come from?

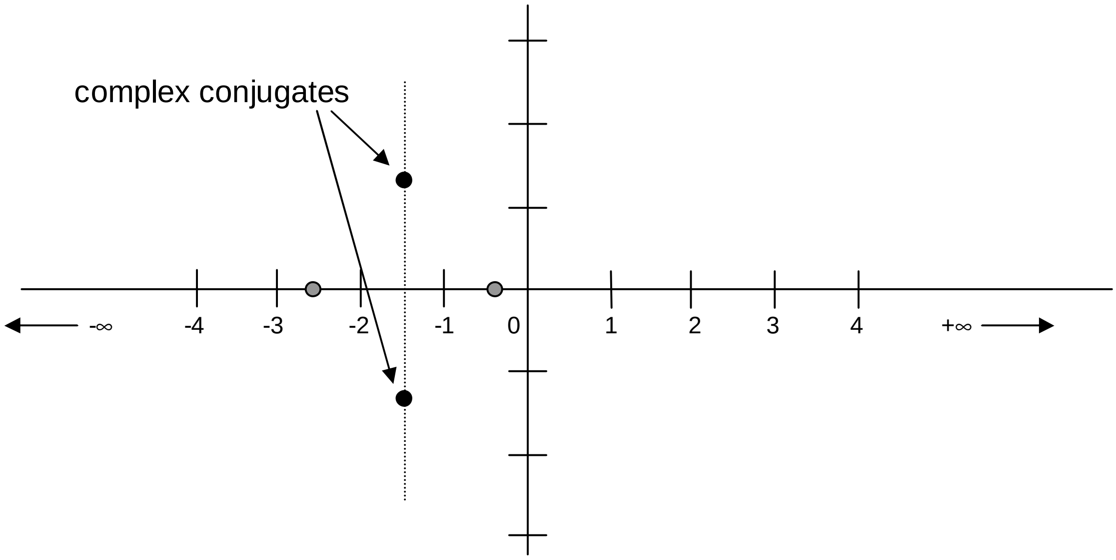
The two roots of a quadratic equation $ax^2+bx+c=0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For example, the roots of $x^2+3x+1=0$ are $x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$ which is -0.38 or -2.6. But what if the quadratic is $x^2+3x+4=0$?

Then $x = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$. If the is no such thing as $\sqrt{(-7)}$, there appear to be no roots. This puzzled mathematicians for a long time.

Given j however, we can proceed like this: $x = \frac{-3 \pm \sqrt{-7}}{2} = \frac{-3 \pm j\sqrt{7}}{2} = -1.5 \pm j1.32$ which we plot on an Argand Diagram.



The grey dots show the first pair of roots (both real) and the black dots the second pair (both complex). Notice that the complex pair are mirror images of each other. This always happens when roots are found. Complex numbers like this with equal real parts and opposite imaginary parts are called *conjugate*.

Complex Arithmetic

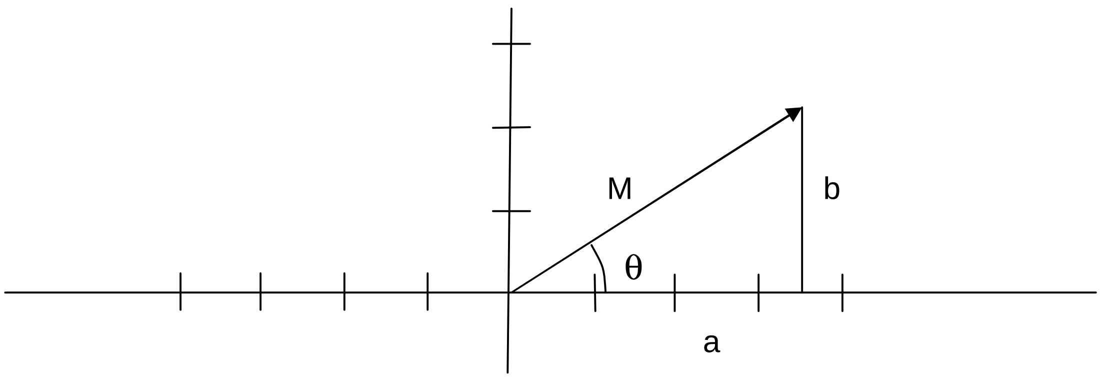
Complex numbers are often represented by the letter *z*. If $z_1=a_1+jb_1$ and $z_2=a_2+jb_2$ then

Addition: $z_1+z_2=(a_1+jb_1)+(a_2+jb_2)=(a_1+a_2)+j(b_1+b_2)$

Subtraction: $z_1-z_2=(a_1+jb_1)-(a_2+jb_2)=(a_1-a_2)+j(b_1-b_2)$

i.e. Vector addition and subtraction.

To show what happens when we multiply or divide, it is easier if we introduce the *polar* form of a complex number.



The number is represented by a magnitude M and a phase θ where $a=M.\cos(\theta)$, $b=M.\sin(\theta)$, $M=\sqrt{(a^2+b^2)}$, $\theta=\tan^{-1}(b/a)$.

If z_1 has magnitude M_1 and phase θ_1, z_2 has magnitude M_2 and phase θ_2,

Multiplication:
$$\begin{aligned} z_1 z_2 &= M_1(\cos\theta_1 + j.\sin\theta_1)\times M_2(\cos\theta_2 + j.\sin\theta_2) \\ &= M_1 M_2\{(\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2) + j(\sin\theta_1.\cos\theta_2 + \sin\theta_2.\cos\theta_1)\} \\ &= M_1 M_2\{\cos(\theta_1+\theta_2) + j.\sin(\theta_1+\theta_2)\} \end{aligned}$$

This shows that magnitude of the product of the two complex numbers is the **product of the magnitudes**, and the phase is the **sum of the phases**.

Division:
$$\frac{z_1}{z_2} = \frac{M_1(\cos\theta_1 + j.\sin\theta_1)}{M_2(\cos\theta_2 + j.\sin\theta_2)}$$

Now multiply top and bottom by the complex conjugate of the denominator.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{M_1(\cos\theta_1 + j.\sin\theta_1)(\cos\theta_2 - j.\sin\theta_2)}{M_2(\cos\theta_2 + j.\sin\theta_2)(\cos\theta_2 - j.\sin\theta_2)} \\ &= \frac{M_1\{(\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2) + j(\sin\theta_1.\cos\theta_2 - \sin\theta_2.\cos\theta_1)\}}{M_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{M_1}{M_2}(\cos(\theta_1-\theta_2) + j.\sin(\theta_1-\theta_2)) \end{aligned}$$

So the magnitude of the quotient is the **quotient of the magnitudes**, and the phase is the **difference between the phases**.

So this section has shown what the effects of the 4 arithmetic operations are on pairs of complex numbers, seen as pairs of vectors in the Argand Diagram or *complex plane*.

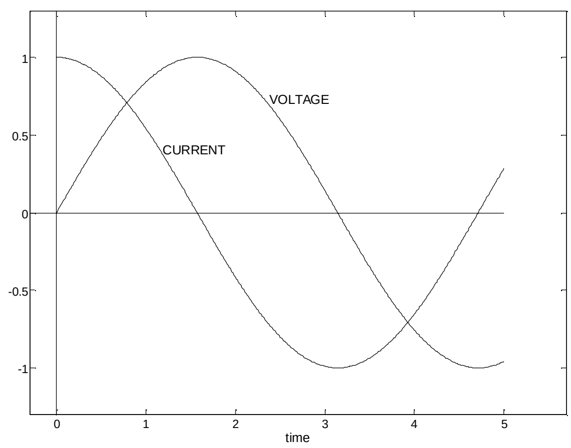
AC Theory

Alternating Current Theory allows us to analyse circuits when the currents and voltages are all sinusoidal. More complicated waveforms can be broken down into components at many frequencies by Fourier Analysis, so we can understand much about circuit behaviour from AC Theory, treating the complicated waveform as many superimposed frequencies.

A capacitor has a capacitance (C) measured in farads (usually micro-, nano- or picofarads). It has the basic property that the charge stored is proportional to the applied voltage: q = C.v.

Suppose that we apply a voltage $v=v_p.\sin(\omega t)$: what will the current be? ω is the angular frequency (rads/s) where ω=2πf and f is the frequency in Hz.

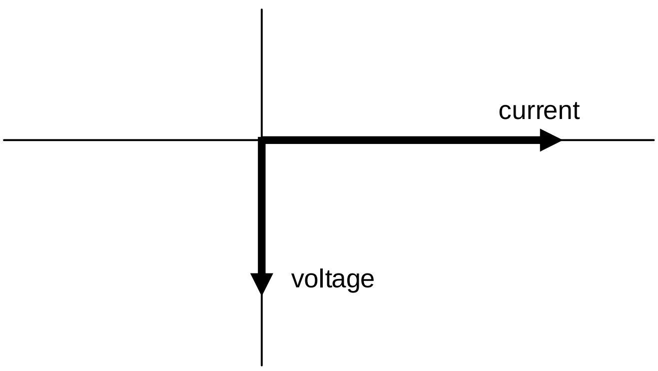
Differentiating $\frac{dq}{dt} = C\frac{dv}{dt}$ but $\frac{dv}{dt} = \omega v_p \cos(\omega t)$ and the current is $\frac{dq}{dt}$, therefore $i = \omega C v_p \cos(\omega t)$. This shows that the current has an amplitude that is proportional to the voltage (Ohm's Law) but that the current *leads* the voltage by 90°.



We can represent the ratio of the magnitude of the current to the magnitude of the voltage by the *reactance* X.

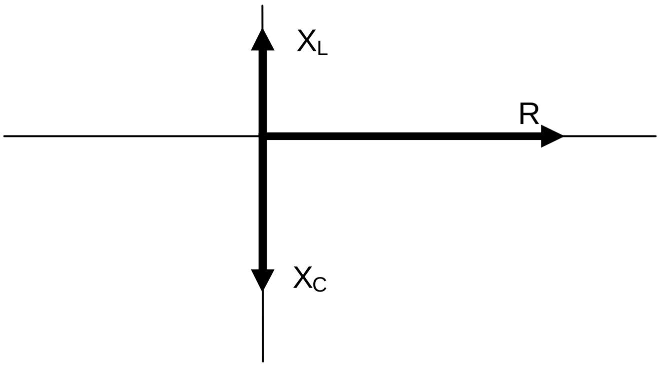
$X = \frac{|v|}{|i|} = \frac{1}{\omega C}$ but how can we represent the phase relationship?

This is where complex numbers come in. In the complex plane, we can represent these as *phasors*



$$V = -j\frac{1}{\omega C}I = \frac{1}{j\omega C}I$$

Here the voltage and current are in capitals because these are not functions of time (as v and i were), but they represent the amplitudes of the phasors. Conventionally they are the r.m.s. values (root mean square) so that the power is the same as a direct current with the same numerical value. $X = \frac{1}{\omega C}$ is the reactance of a capacitor and is a capacitive reactance (i.e. multiplying by -j). I will not show the derivation, but inductors have a complementary reactance $X = \omega L$ and because the current lags the voltage by 90°, inductive reactances point up the imaginary axis.



We can now represent the effect of more than one component, just as we can for several resistors, but when there are capacitors and/or inductors present too. The relationship between the voltage and current is now, generally, neither *in-phase* (purely resistive) or *in quadrature* (purely reactive), and is called the **impedance.**

For example if a capacitor C and a resistor R are connected in series, the impedance $Z = R + \frac{1}{j\omega C} = R - \frac{j}{\omega C}$.

For components in parallel, we can use the same rule as for two resistors in parallel: divide the product by the sum. For example, an inductor L in parallel with a resistor R will have an impedance $Z = \frac{(R)(j\omega L)}{(R)+(j\omega L)}$.

Examples of Complex Number and Impedance Calculations that we shall need later on

1. Algebraic Product.

$$(a + jb)(c + jd) = ac + jbc + jda + j^2bd = (ac - bd) + j(bc + da)$$

2. Algebraic Division

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{(ac-bd)+j(bc-da)}{c^2+d^2}$$

3. Numerical Division

$$\frac{4+j5}{2+j3} = \frac{(4+j5)(2-j3)}{(2+j3)(2-j3)} = \frac{(10+j10-j12-j^2 15)}{(10+j6-j6+j^2 9)} = 25 - j22$$

4. An Ohm’s Law example. If a current is flowing which is I=(5+j4) mA, relative to the reference phase, and it passes through a resistor of 2 kΩ and a capacitive reactance of 3 kΩ, what are the magnitude and phase of the voltage across both components together?

$$V = IZ = (5+j4)(2-j3) = 22-j7$$

This has a magnitude of $\sqrt{(22^2+7^2)}$ = 23.1 volts, and a phase of $\tan^{-1}(-7/22)$ = -17.6° (i.e. lagging relative to the reference).

5. A choke has an inductance of 3mH and a series loss resistance of 10 Ω at 5000 Hz. How much will the current lag the voltage at this frequency?

The inductive reactance is $X_L = \omega L = 2\pi f L = 2\pi \times 5000 \times 3 \times 10^{-3} = 94.2\ \Omega$. Its total impedance is therefore Z=10 + j 94.2.

If the driving voltage were 1 V amplitude, the current could be calculated as

$$I = \frac{1}{10 + j94.2}$$

You could divide this out (multiple by complex conjugate), and then calculate the phase. However, remembering that in complex division, the phase is subtracted, in this case the phase merely changes sign, so all we need do is find the phase of Z and reverse the sign.

i.e. the phase of Z is $\tan^{-1}(94.2/10) = 84°$, so the current lags the voltage 84°.

Suggested Books

Alternating current, inductors and capacitors, all treated without complex numbers with many examples and questions for you to tackle (with answers given).

Schaum's Outlines: "Basic Electricity", McGraw-Hill, £13.99

Circuit Theorems, Complex impedance.

Schaum's Outlines: "Electric Circuits", McGraw-Hill, £13.99