Before we begin

"The hardest part of a subject is the beginning. Once a certain stage is passed, we gain confidence and feel that, if need be, we could carry on by ourselves. The process of learning is very much the same whether in swimming or mechanics — an initial feeling of insecurity is followed by a feeling of power."

from "Principles of Mechanics" by John L. Synge and Byron A. Griffith.

These notes, and indeed most of my part of the circuits course, are addressed primarily to those who have very little background in electrical engineering.

If you have covered basic DC and AC theory before, then you don't need to attend my lectures. However, you should have copies of these notes and also the tutorial questions, since they define the scope of the exam.

My approach to teaching this introductory course is to deal first with all the basic laws and methods using the simplest possible circuits — namely those containing only resistances and batteries or idealized DC sources. When those fundamentals have been covered, we can progress

to time-varying voltages and to the use of "complex numbers" (a most unfortunate, misleading and initially disheartening name for what turns out to be a powerful method for the analysis of AC circuits).

If this is to produce a steady build-up of confidence in your ability to solve circuit problems, it is <u>ESSENTIAL</u> that you work through and thoroughly understand <u>all</u> the tutorial questions.

As well as giving you the tutorial questions, I shall provide outline solutions. I emphasize "outline" because the effort to fully understand can only be made by you.

Basic definitions and units

This is where we review the jargon — the technical vocabulary — of electrical engineering.

"All professions are conspiracies against the laity" wrote George Bernard Shaw, if my memory serves. Indeed they are: lawyers, doctors, scientists and engineers all use their own coded language. It is a convenient shorthand for them, but a barrier to outsiders.

It is part of your task on this course to learn the basic vocabulary. This chore does have some unexpected bonuses, though...

At a job interview, if you find yourself in a tough spot, you might try something like "Ah, but there are bound to be non-linear effects at high signal levels." That should make 'em pause for thought, especially if you happen to be discussing the salary structure at the time.

You can use electro-speak to serve many purposes: it has the power to clarify or confuse. In our own discussions I think it is best if we concentrate on the essential ideas, and use as little jargon as possible. In the exam, please only use electro-speak if you are sure you know what it means.

Fortunately, electrical circuits can be quite

usefully visualized in terms of water flowing in pipes. Many helpful analogies come from this sort of picture — and if the subject is new to you, analogies can be useful stepping-stones along the way to understanding the more abstract material.

Voltage and current

If we think of current flow in a wire as analogous to water flow in a pipe, then the voltage analogy is that the pressure at one end of the pipe is what causes the current to flow. More accurately it is the pressure difference between the ends of the pipe that tends to produce flow.

Returning to electrical current flow, it is the electrical potential difference (measured in volts) that tends to produce current flow (measured in amps, or amperes if you prefer).

What is actually happening in a wire that is connected to a battery and carrying current? The application of the voltage produces an electrical pressure gradient along the wire, and this pressure gradient is called an electric field. Electrons in the field experience a force which tends to accelerate them towards the positive terminal (since the electrons themselves are negatively charged).

Collisions with the fixed lattice atoms in the wire will repeatedly deflect the electrons from their preferred paths (along the direction of the field) but nevertheless there is a general drift of charge towards the positive terminal.

The extent to which these lattice collisions impede the flow of electrons in a solid is a characteristic property of the particular material and determines its electrical resistance.

Materials such as copper, which offer low resistance to current flow, are called conductors.

Intermediate materials such as silicon, are known as semi-conductors (or semiconductors).

Substances such as glass, which offer very high resistance to current flow, are called insulators.

Increasing the voltage applied to a conductor will increase the electric field and hence the current flow. Increasing the resistance will reduce the flow.

All this is neatly summarized by an approximate but very useful law. This law will form an excellent starting-point for our study of electrical circuits.

After all, clarity begins at Ohm. Now please turn over, if you haven't already.

The nature of electric current

The flow of electric current is the net movement of electric charge. In ordinary wires, electrons are the charge carriers and by convention they are defined to have negative charge.

The unit of electric charge is the coulomb (C) and one electron has a charge

$$q = 1.6 \times 10^{-19} \text{ C}$$

The ampere is a charge flow of 1 coulomb per second, corresponding to about

6.25×10^{18} electrons per second.

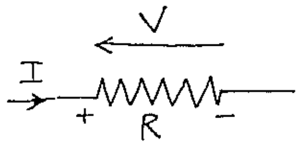
I don't expect you to remember the exact value of an electron's charge, but I do expect you to notice if you make an error in a calculation by putting 10^{+19} instead of 10^{-19}.

Current can be carried by positive charges such as protons or "holes", but for the time being we shall keep to my rule of making things as simple as possible ("— but not simpler as Albert the Great so astutely observed).

Ohm's law

In symbols:

$$I = \frac{V}{R} \quad \text{and}$$



......(1)

In words: the current flowing in the resistor (R) is directly proportional to the applied voltage and inversely proportional to the resistance R, which is measured in Ohms.

At this point you may wonder why I have included a diagram in the statement of Ohm's law. The reason, which is a very important one, applies to all future definitions and circuit diagrams in these notes:

THE DIAGRAM ACCOMPANYING THE EQUATION DEFINES THE SIGN CONVENTIONS DENOTING POSITIVE DIRECTIONS. WITHOUT THE DIAGRAM, THE DEFINITION IS INCOMPLETE.

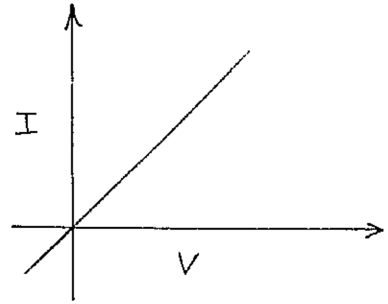
Two consequences would follow if, in the exam, you were to quote only the equation as a statement of this (or any other relevant) law:

(1) In applying it to a circuit analysis problem, you would risk getting the currents wrong in both magnitude and direction;

(2) Your mark would have neither magnitude

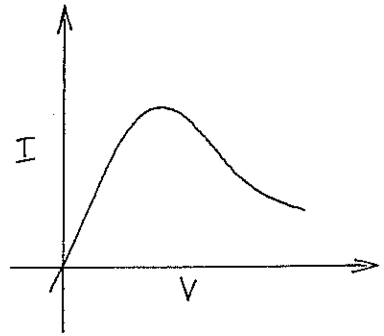
nor direction. You have been warned.

Here is another way of representing Ohm's law:



Note that the graph is linear; and that if the voltage reverses sign, so does the current.

I did say that Ohm's law is a very useful approximation, and we shall make extensive use of it for that reason. However, materials can behave in many other ways too: the sketch illustrates a rather extreme (genuine) example.



Clearly, a different approach is needed there; but again note that even this curve does begin by showing linear behaviour over a limited voltage range.

In fact, all materials depart from Ohm's law eventually. This is not something to be regretted: on the contrary, without some limit due to non-linearity, I strongly suspect that the world as we know it would not last long.

A final note on sign conventions

As we progress through the course, you will soon realize that it really does matter to take note of these seemingly arbitrary things. Here, though, is an analogy which I believe is quite appropriate.

Does it really matter whether we drive on the left or the right side of the road? In principle, no it doesn't. So, we can make an arbitrary choice. In some countries the choice is for the right, in others it is for the left.

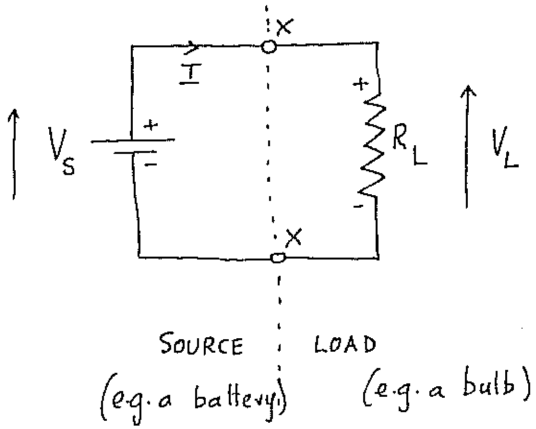
Now we move from principle to practice. Having made the arbitrary choice, it now <u>does</u> matter that everyone observes it. I rest my case.

Generalizing Ohm's law

We shall apply Ohm's law to DC circuits next. It turns out, though, that if we choose our definitions with care, it will apply equally to AC circuits — another bonus.

Sources and loads

Like so many words in the English language, these two have several meanings each. For our present purpose, a diagram will help:



It shows a battery (the power source) supplying a load (the bulb of a torch, for example).

NOTE that the arrows for I and V_L are consistent with the sign conventions for Ohm's law defined on page 6, equation (1).

Next, note that the current and voltage arrows are in opposing directions for the load, and in the same direction for the source. There is nothing new here really – just something to note for future use.

Another thing: $V_S = V_L$. This is a consequence of Ohm's law, since the lines joining the source to the load in the diagram represent wires (if you like) having zero resistance and therefore zero voltage drop along them.

The ⊕ and ⊖ signs are just extra reminders of

the conventions we use to identify positive and negative terminals.

[Digression: in case the "XX" terminals and the dotted line mystify you, they signify the interface between the source and the load. It has very little significance for us at this stage, but becomes increasingly important in AC work as the frequency is increased – for example as we move towards the radio and microwave region and into high-speed microwave integrated circuits (MIC's) and supercomputers.

At such high frequencies – which means high data rates also – ordinary circuit theory has to be replaced by a theory which allows for the time delays in signal propagation.

This in turn means that electrical signals on cables or on integrated-circuit chips are analysed as travelling waves which can reinforce or cancel one another at different points along the line. Under these circumstances the XX interface might represent an accurately positioned coaxial connector, for example.]

Power in the load R_L

Power = volts × amps, and is measured in watts.

Power is the rate of flow of energy, so its units are joules per second.

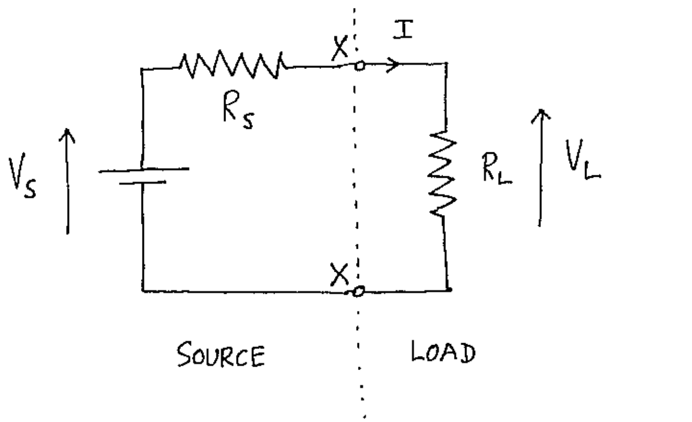
The power in the load R_L can therefore be written in a number of ways:

$$P_L = V_L I = I^2 R_L = \frac{V_L^2}{R_L} \quad \ldots\ldots\ldots\ldots (2)$$

since $V_L = I R_L$ from equation (1).

A useful result on power transfer

Suppose next that we add a resistance in series with the source battery and define everything to the left of XX to be part of the source. The new diagram becomes:



Does it look strange to just add R_S for no apparent reason?

Its relevance will be clear before long. Meanwhile, we can analyse the circuit by answering the following question: if R_S is fixed and R_L is varied in value from zero to infinity, for which value of R_L is the

maximum power delivered to the load?

$$P_L = I^2 R_L$$

$$= \frac{V_s^2 R_L}{(R_s + R_L)^2} \quad \ldots\ldots\ldots\ldots (3)$$

By equating $\dfrac{dP_L}{dR_L} = 0$ we obtain the result:

$$R_L = R_S \quad \ldots\ldots\ldots\ldots (4)$$

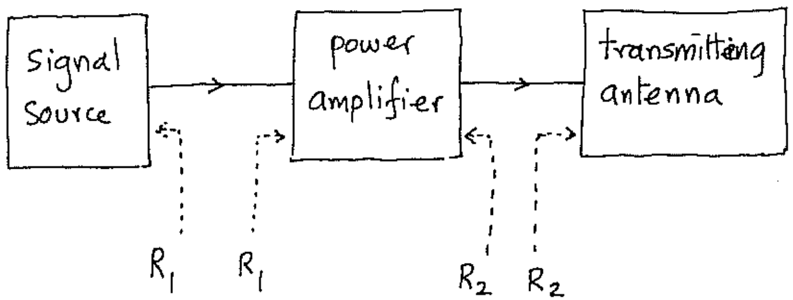
for maximum power transfer.

[A measure of the importance of this result is provided by the fact that it has been elevated to the status of a Theorem, no less: the Maximum Power Transfer Theorem. There is no need for us to be particularly impressed, but remember my helpful hints on page 3].

Where is this result used? Here are some examples, not all of which will necessarily be clear to you until later in the course:

→ When $R_L = R_S$ the source and load are said to be "matched." This is an important condition when (say) a power amplifier is used to boost a signal in a communication system: see next page...

DC circuits: the maximum power condition



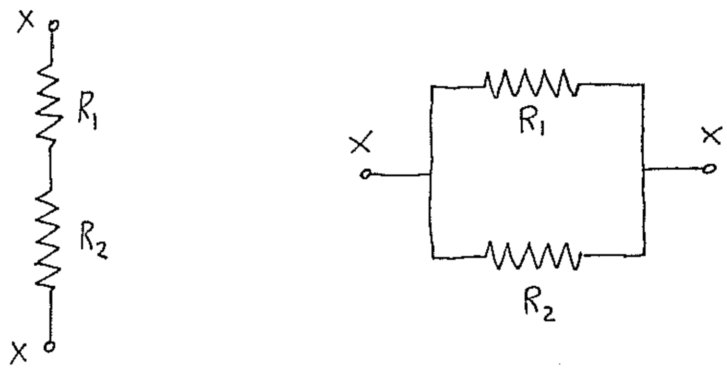
To extract the maximum possible power from the signal source (which has an output resistance R_1) the power amplifier must provide an input resistance of R_1. Similarly the output resistance of the power amplifier must be "matched" to the input resistance of the transmitting antenna, R_2.

→ In communication systems using microwaves, I have mentioned before (page 10) that electrical signals are analysed as travelling waves. The "matched" condition here has an additional significance: all the energy in the incident wave is absorbed by the load, and none is reflected back to the source. Apart from the power transfer, this can be an important requirement for the stability of the source: if power is reflected back, the signal source may become unstable or start to operate at the wrong frequency. This is particularly true of negative-resistance oscillators such as Gunn diodes or avalanche (IMPATT) diodes.

DC circuits: combinations of resistors

Resistors in series and parallel

We can see what happens when resistors are connected in different ways by remembering Ohm's Law and the analogy with water flow in pipes.



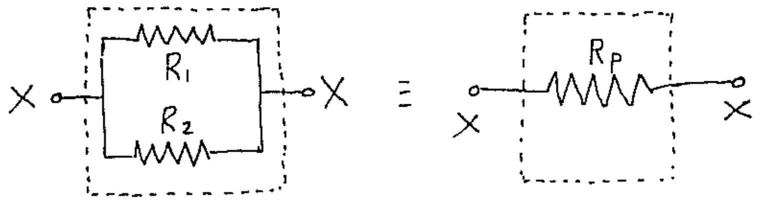
SERIES CONNECTION PARALLEL CONNECTION

For the series connection: if we imagine two water-pipes of different diameters (and hence different resistance to flow) connected together without leaks, we see that they have the same flow rate; similarly, two resistors in series carry the same current.

For the parallel connection, the same "pressure" (that is, voltage) is applied to both. They are independent, and the current that flows will depend on their individual resistances.

This exercise can be used to introduce another important idea while at the same time solving the problem of series and parallel connections.

Consider the parallel resistors:



GIVEN CIRCUIT EQUIVALENT CIRCUIT

The question is: what value of single resistor R_P is identically (electrically) equivalent to the combination of R_1 and R_2 in parallel, in the sense that if they were concealed in boxes (dotted lines) then no external measurements at XX could distinguish between them?

For the parallel resistors we find:

$$\left.\begin{aligned} &\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{which generalizes} \\ &\text{to} \\ &\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots \end{aligned}\right\} \dots \quad (5)$$

N.B. The reciprocal of resistance, $1/R$, is known by the special name <u>conductance</u>. It still looks the same, of course.

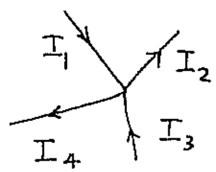
For the series resistors the result is:

$$\left.\begin{aligned} &R_S = R_1 + R_2 \quad \text{which generalizes} \\ &\text{to} \quad R_S = R_1 + R_2 + R_3 + \dots \end{aligned}\right\} \dots\dots \quad (6)$$

Kirchhoff's laws

These are pretty much common-sense rules which we could guess:

The <u>current</u> law:



At a junction (or "node") then

$$\sum I = 0 \quad \dots\dots\dots\dots \quad (7)$$

i.e., the algebraic sum of the currents entering a junction is zero. In this case:

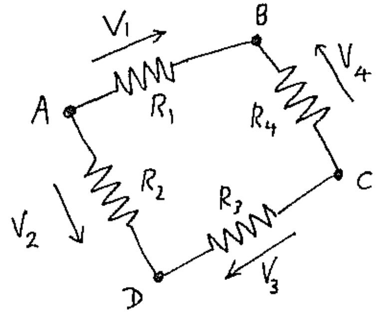
$$I_1 - I_2 + I_3 - I_4 = 0$$

Since currents leaving the junction are counted as negative.

This is just like saying that no water

leaks out from the junction between the pipes: it is all accounted for.

The voltage law:



For a closed loop (or "mesh")

$$\Sigma V = 0 \quad \cdots\cdots\cdots (8)$$

i.e., the algebraic sum of the voltages around a closed loop is zero. In this case:

$$V_1 - V_4 + V_3 - V_2 = 0$$

where I have chosen to call clockwise-pointing arrows positive, starting at point A and returning to it.

The "water analogy" is not so clear here, so here is another picture to help make the voltage law plausible. In the diagram above, you are looking down on a loop of mountain paths. The points A, B, C, D are at different heights. These differences correspond to gravitational potential differences on the mountain, and electrical potential differences (the voltages V_1 etc) in the circuit.

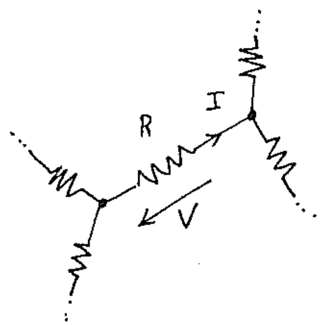
The voltage law is simply saying that if we

start at a particular height (= potential) and walk round and back to the same point, we are again at the same height (= potential). Thus the difference between the starting and finishing potential is zero, though we may have done quite a lot of work in the process.

Another note on "sign conventions"

Look again at the diagrams on pages 16 and 17. How did I know that the currents and voltages were in the directions shown?

I didn't know, and it doesn't matter; the answers would still come out right provided that Ohm's law is applied consistently. Consider a part of a complicated circuit:



I guessed the true direction of the current I, and then made the voltage arrow consistent with Ohm's law (page 6).

Suppose that, on solving all the circuit equations and finally getting a numerical result for the current I, the answer came to −3mA

A negative current? Is that possible? It is simply telling us that the true current is 3mA in the opposite direction to the one assumed. Nothing to worry about.

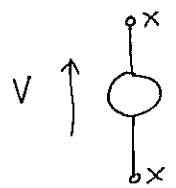
"Ideal" voltage and current sources

We use "models," that is, abstractions, to help analyse practical systems. The closer the model is to describing reality, the more accurate its predictions.

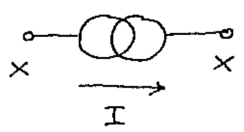
Later in the course we shall add two more components to the idealized resistance R namely inductance L and capacitance C.

For the present, we need to define two idealized sources: a "perfect" battery that always supplies the same voltage (we should be so lucky), and then something even less familiar: a current source that never varies.

The advantage of using such idealizations is that their characteristics are precisely defined; the disadvantage is that if we forget their true nature we can get apparent paradoxes.



IDEAL VOLTAGE SOURCE



IDEAL CURRENT SOURCE

These very useful analytical tools have the following properties:

Voltage source This will provide a constant voltage across any load resistance R_L connected to it, no matter how much current is drawn.

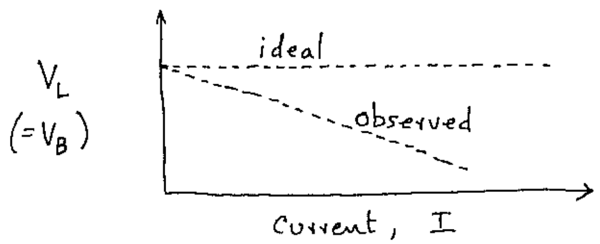
Current source This will deliver a constant current to any load resistance R_L connected to it, no matter how much voltage is required.

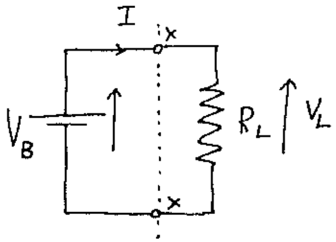
Each of these, admirable though it is, poses one possible major problem. Can you see what it is?

The equivalent circuit of a real battery

This is an opportunity to show, by means of a simple example, how the concept of an equivalent circuit can greatly simplify analysis.

If we test a torch battery (for example) to see whether it is an ideal voltage source, we find something like this:





[The current I is varied by varying R_L]

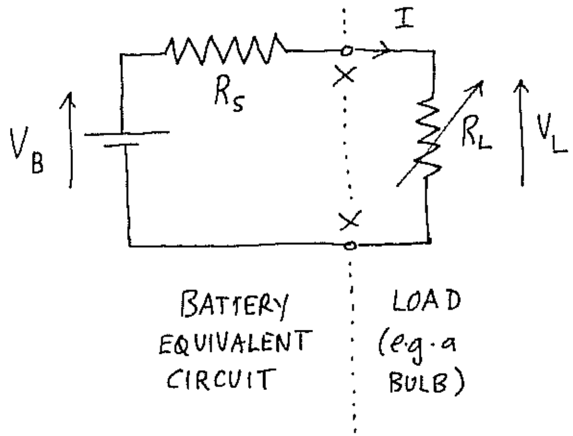
Instead of the voltage supplied to the load remaining constant, it falls as the current is increased. Clearly, then, the circuit on the right is not equivalent to a real battery.

Notice that the voltage drop is not a sign that the battery is wearing out, because voltage returns towards its maximum value as the current is reduced to zero.

A battery is a complex package of chemicals. We could try to understand its chemistry in detail (don't panic: we won't). Or,

we could try to invent a simple electrical model which, at the terminals XX, represents the behaviour of the battery reasonably well.

This is the "black box" approach. The battery behaves *as if* it were an ideal voltage source in series with an internal resistance:



The arrow through R_L signifies that it is variable (to vary I).

R_S = internal resistance of the source.

Now we find, using Ohm's law, that

$$V_L = V_B - I R_S$$

and this describes the "observed" line in the graph (page 21) quite well.

NOTE

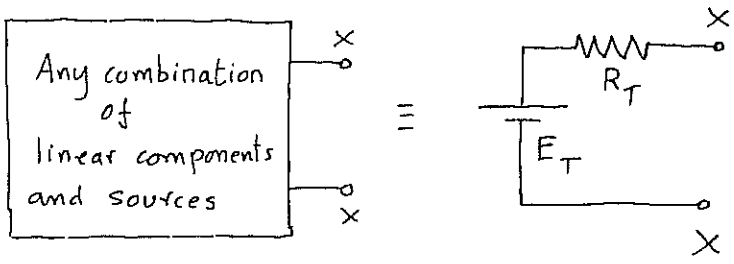
An ideal battery would have $R_S = 0$ so we conclude that:

an ideal voltage source has ZERO internal resistance.

Thevenin's theorem

Here is another result of such wide use that it has been elevated...

It turns out that the simple picture of a voltage source and an internal resistance has very wide application:



A GENERAL LINEAR CIRCUIT

THE "THÉVENIN EQUIVALENT CIRCUIT"

The box can contain any number of linear components and voltage or current sources, yet the whole lot behaves as though it is a single voltage source E_T in series with a single resistance R_T.

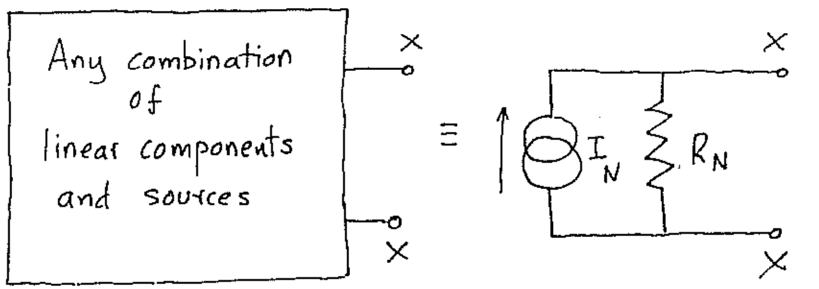
R_T, the Thévenin resistance, is defined as the resistance that would be measured at XX if all the sources were replaced by their internal resistances.

E_T, the Thévenin voltage, is defined as the

voltage that would appear at XX with nothing connected, i.e., with terminals XX open-circuited.

The Norton circuit

This is a kind of dual or mirror-image of the Thévenin circuit just described, because it represents the same general circuit using a current generator and a parallel resistance instead of a voltage generator and a series resistance:



A GENERAL LINEAR CIRCUIT

THE "NORTON EQUIVALENT CIRCUIT

R_N, the Norton resistance, is defined in the same way as R_T, the Thévenin resistance.

I_N, the Norton current, is defined as the current that would flow between terminals XX if they were short-circuited, i.e. connected by a wire having zero resistance. (That sounds rather drastic).

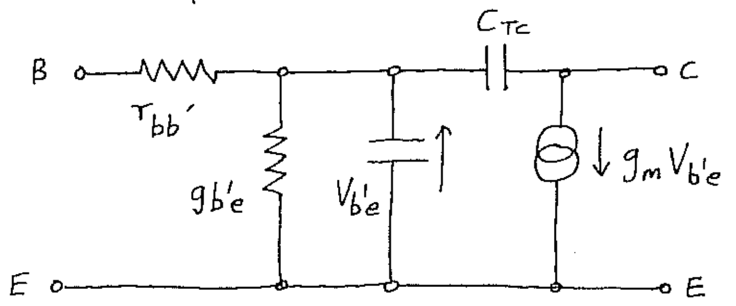
Because the Thévenin and Norton circuits are each equivalent to the general network, it follows that they are equivalent to one another.

Thus:

$$\left.\begin{aligned} R_N &= R_T \\ \text{and}\quad I_N R_N &= E_T \end{aligned}\right\} \quad \ldots\ \ldots\ \ldots \quad (9)$$

Uses of equivalent circuits

The most common application is in the representation of electron devices such as transistors. Here is an example from transistor amplifiers:



(Hybrid π circuit)

This is taken from "Fundamentals of Semiconductor Devices" by Edward S. Yang. The point for us to note is that the equivalent circuit contains a current generator whose value depends on a voltage elsewhere in the circuit. When that happens, the current generator is called a <u>controlled</u> current source.

Postscript on Thévenin & Norton

Both these theorems contain references to "all sources being replaced by their internal resistances." We have already noted this for an ideal voltage source – here is a summary for both types of ideal source:

Internal resistance of voltage source is zero

Internal resistance of current source is infinite.

Linearity and superposition

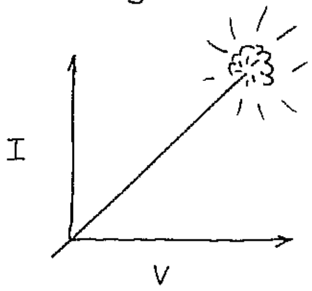
More jargon, but very straightforward and common-sense ideas.

What, after all, could be more natural and obvious than that two torch batteries in series will produce twice the current through a resistor that one battery alone produces?

"Linearity" is the fundamental concept, and "superposition" is the consequence. In the example in the last paragraph, it is the linearity of the resistor that allows us to get the total current as the sum of the currents that each (<u>ideal</u>) battery would produce separately.

To contrast linear and non-linear resistances, consider the following two V-I characteristics:

Linearity

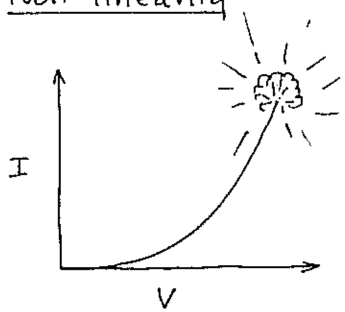


Resistor: linear, obeys Ohm's law (initially):

$$I = \frac{V}{R}$$

The test of linearity: if the voltage doubles, the current doubles.

Non-linearity



A semiconductor diode has a very non-linear characteristic, even close to the origin:

$$I = a\left[\exp(bV) - 1\right] \quad \cdots\cdots (10)$$

where a and b are constants.

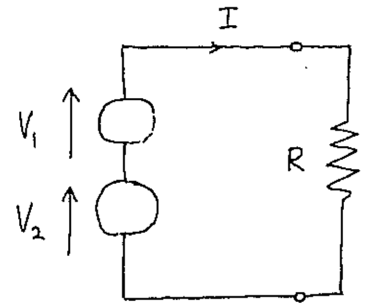
Doubling V causes I to more than double.

[Digression: non-linear devices have many applications in communications and signal processing. Examples include signal detection / rectification, modulation, harmonic generation, smoothing circuits to protect against power surges ....].

Superposition.

As the name suggests, we can "superimpose" voltages and currents by addition (taking into account sign, of course). So, the combined effect of several sources can be obtained in a straightforward way.

Specifically, the principle of superposition means that the current flowing in a particular branch of a linear circuit is the algebraic sum of the currents due to each source acting separately.

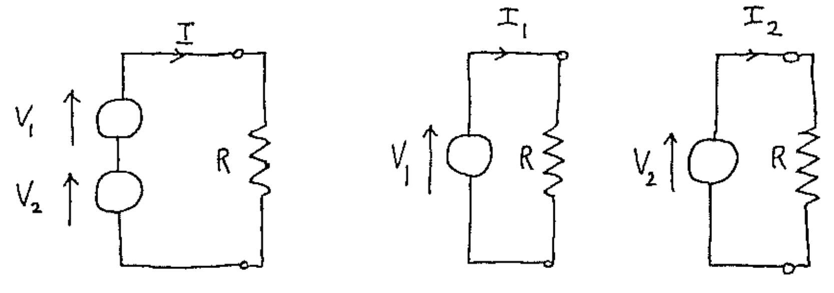


The circular symbols indicate ideal voltage sources. If you are given no other information in a tutorial or exam question, then you should assume that the voltage or current source is ideal, i.e. that it has either zero internal resistance (if a voltage source) or infinite int. res. (if a current source).

$$\left.\begin{aligned} I &= \frac{V_1 + V_2}{R} = \frac{V_1}{R} + \frac{V_2}{R} \\ I &= I_1 + I_2 \end{aligned}\right\} \cdots\cdots \cdots (11)$$

where we can think of I_1 and I_2 as the currents that would flow due to V_1 and V_2 acting separately

This is straightforward; one small point still needs emphasis, though, and that is best shown in the form of a sketch:



$$I = I_1 + I_2$$

Notice that in drawing the two simplified circuits for calculating I_1 and I_2, the voltage sources V_2 and V_1 were not simply removed: they were replaced by their internal resistances — which happened to be zero in this example.

We can now re-state the method of superposition more precisely:

The current flowing in a particular branch of a linear circuit is the algebraic sum of the currents due to each source acting separately, all other sources being replaced by their internal resistances.

The importance of superposition

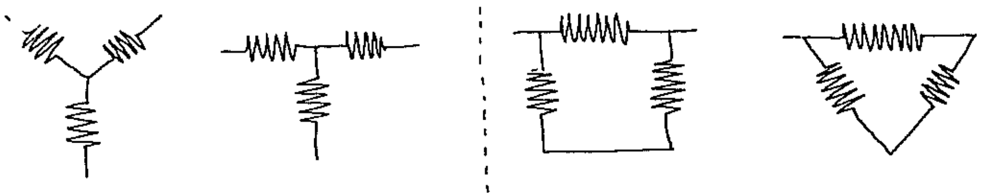
The linear combination of signals allows us to analyse and synthesize the waveforms used in

communication systems. One of the most important examples is the Fourier analysis of periodic waves which gives the harmonic components. These harmonics can be treated separately and their effects combined in linear systems. This in turn allows calculations of bandwidth and waveform distortion — crucial considerations in telecommunications

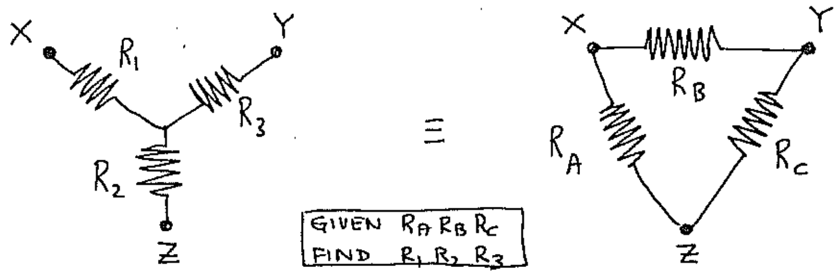
The "T–π" or "Star – Delta" transformation

It looks like hard work, but it is intended to simplify network analysis. Occasionally it does.... though I have my suspicions. It can certainly be used to set some tricky exam questions.

The "T" or "Star" | The "π" or "Delta"



Method of deriving the "delta→star" transformation



Measurements made externally must give identical results :

Measure resistance at XY $\qquad R_1 + R_3 = \dfrac{R_B\,(R_A + R_C)}{R_A + R_B + R_C}$

Similarly, we progress round the sequence of terminals...

Measure resistance at YZ $\qquad R_2 + R_3 = \dfrac{R_C\,(R_A + R_B)}{R_A + R_B + R_C}$

Measure resistance at ZX $\qquad R_1 + R_2 = \dfrac{R_A\,(R_B + R_C)}{R_A + R_B + R_C}$

You should solve for R_1 R_2 and R_3 to verify that:

$$R_1 = \frac{R_A R_B}{\Sigma R} \qquad R_3 = \frac{R_B R_C}{\Sigma R}$$

$$R_2 = \frac{R_A R_C}{\Sigma R}$$

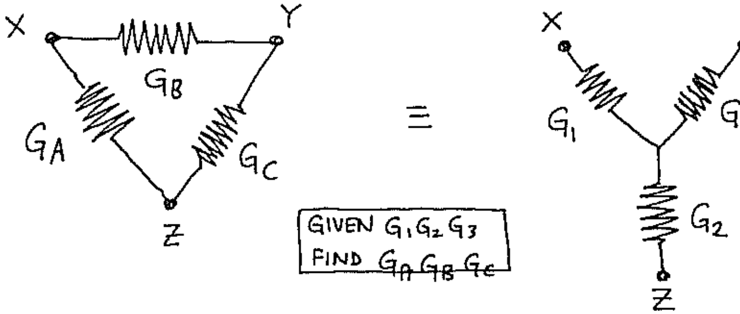
where $\Sigma R = R_A + R_B + R_C$.

Notice the sequence, and the method of setting up the equations.

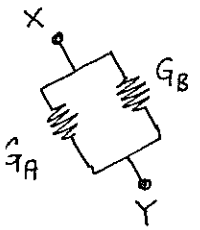
Next, we work the transformation the other way using conductances.

In deriving the results, remember to get the unknowns in simple additive pairs on the LHS and the more complex expressions on the RHS.

Deriving the "star → delta" transformation

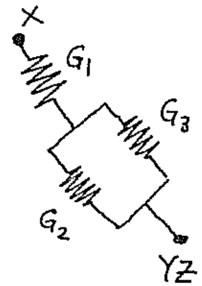


Measure conductance at XY with YZ short-circuited:



$$G_A + G_B = \frac{G_1 (G_2 + G_3)}{\Sigma G}$$

where $\Sigma G = G_1 + G_2 + G_3$



Measure conductance at YZ with XZ on s/c:

$$G_B + G_C = \frac{G_3 (G_1 + G_2)}{\Sigma G}$$

Measure conductance at ZX with XY on s/c:

$$G_A + G_C = \frac{G_2 (G_1 + G_3)}{\Sigma G}$$

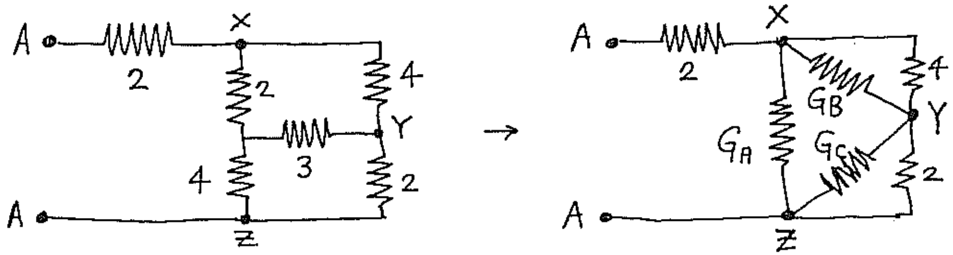
You should verify that the result is:

$$G_A = \frac{G_1 G_2}{\Sigma G}, \quad G_B = \frac{G_1 G_3}{\Sigma G}, \quad G_C = \frac{G_2 G_3}{\Sigma G}$$

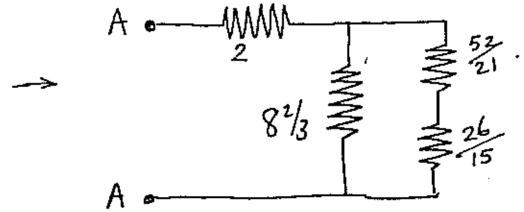
Again, notice the pattern in the results and remember the method.

Circuit simplification using star/delta

Find the resistance measured at AA:



Result is 4·83 Ω
(I think!)



$G_A = \frac{3}{26} \quad \therefore R_A = 8\frac{2}{3}\ \Omega$

$G_B = \frac{2}{13} \quad \therefore R_B = 6\frac{1}{2}\ \Omega$

$G_C = \frac{1}{13} \quad \therefore R_C = 13\ \Omega$

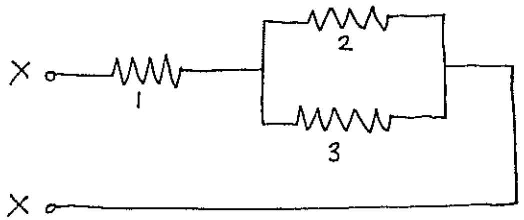
Check please

① Given that the density of free electrons in copper is $5 \times 10^{28}\ m^{-3}$ and the charge on an electron is $1{\cdot}6 \times 10^{-19}$ C, calculate the average drift velocity of electrons in a copper wire of 2·5 mm diameter carrying a current of 1 ampere. Comment on the magnitude of the result.

② Three resistors R_1 R_2 and R_3 are connected (a) in series and (b) in parallel. Derive the standard results for the equivalent single resistor in each case.

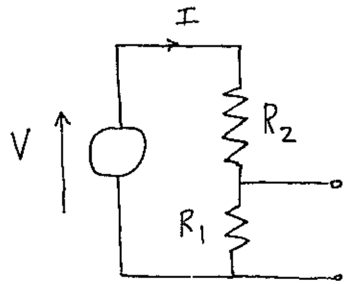
③ Use the results from question ② to obtain the corresponding equivalent conductance for three conductances G_1 G_2 and G_3 connected (a) in parallel and (b) in series.

④ What is (a) the resistance and (b) the conductance of the circuit below?



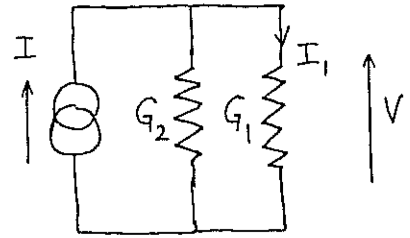
(Numbers are resistances in Ohms). Give the general results as well as the numerical answers.

⑤ For the voltage divider and current divider circuits below, find (a) the voltage ratio V_1/V in terms of resistances and (b) the current ratio I_1/I in terms of conductances.



(a)

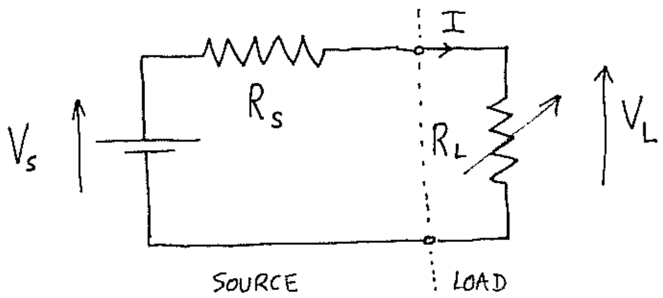
VOLTAGE DIVIDER



(b)

CURRENT DIVIDER

⑥ In the circuit below, the load resistance R_L is varied from s/c to o/c. The source voltage V_s and the source's internal resistance are constant. Which value of R_L gives maximum power dissipation in the load? Express the maximum power in terms of V_s and R_s.

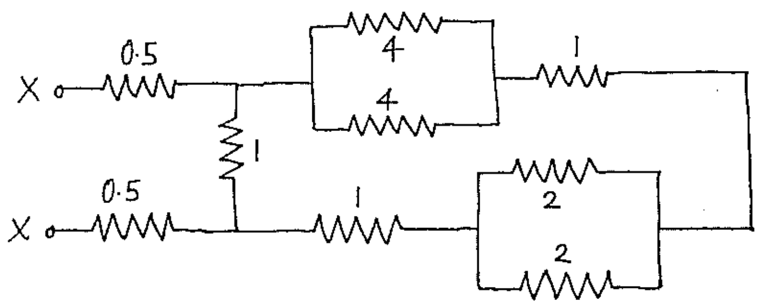


$$P_L = I^2 R_L$$

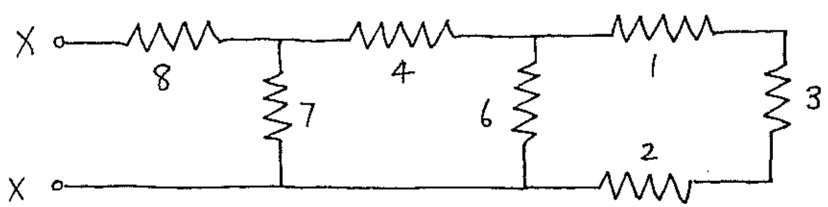
$$I = \frac{V_s}{(R_s + R_L)}$$

(7) Given the characteristics of ideal voltage and current sources, what value of load resistance presents a problem for (a) the voltage source and (b) the current source?

(8) Find the equivalent resistance at the terminals XX for the following network:



(9) Find the resistance measured at XX:



Hint: start from the RHS.

(10) In the circuit of Qu. 6, show that
(a) when $R_S << R_L$ the source approximates to an ideal constant-voltage source, and
(b) when $R_S >> R_L$ it approximates to an ideal

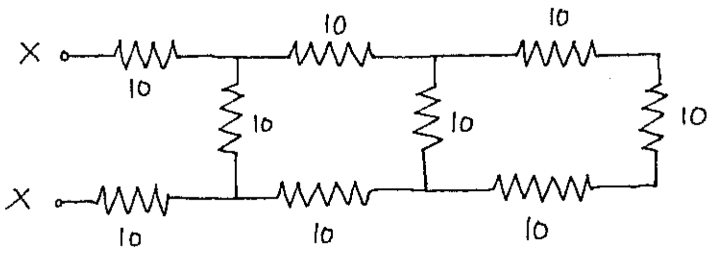
constant-current source.

Hence justify the statements that

(a) A perfect voltage source has zero internal resistance;

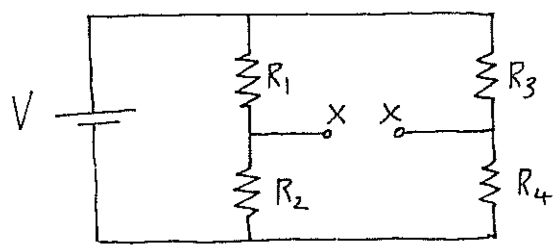
(b) A perfect current source has infinite internal resistance.

(11) Find the equivalent resistance at XX for the following network:



Hint: start from the RHS.

(12) The circuit below is known as a Wheatstone bridge:



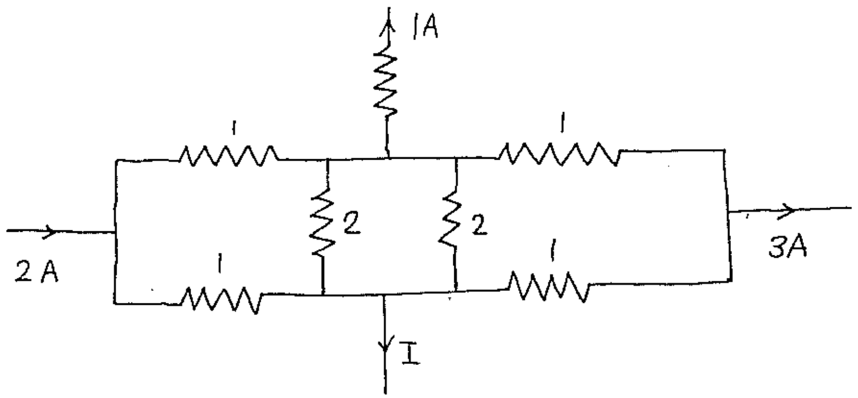
[It is often drawn in a form in which the resistors $R_1 R_2 R_3 R_4$ form a diamond shape]

Show that $V_{xx} = 0$ when

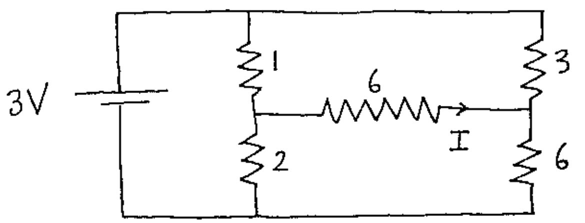
$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$ (the "balance" condition)

In which application would this result be useful ?

(13) Find the current I in the following network :

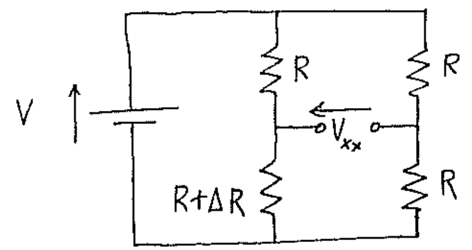


(14) Find the current I in the network below :



(15) The basic Wheatstone bridge configuration can be used in the unbalanced mode also. In this mode, the out-of-balance voltage V_{xx} is a measure of how the resistor(s) have altered. For example, one resistor could be a strain gauge; at balance, $V_{xx} = 0$ and there is no strain. The bridge can be calibrated to relate the strain to V_{xx}.

The bridge shown is initially balanced with all four arms (one of which is a strain gauge) of resistance R.



When strain is applied, the gauge arm resistance is increased to R+ΔR.

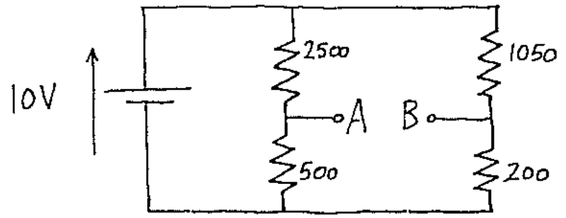
The sensitivity of the bridge is defined as S where:

$$S = \frac{V_{xx}}{\Delta R}$$

(a) Find S in terms of V and R. Assume $\Delta R \ll R$

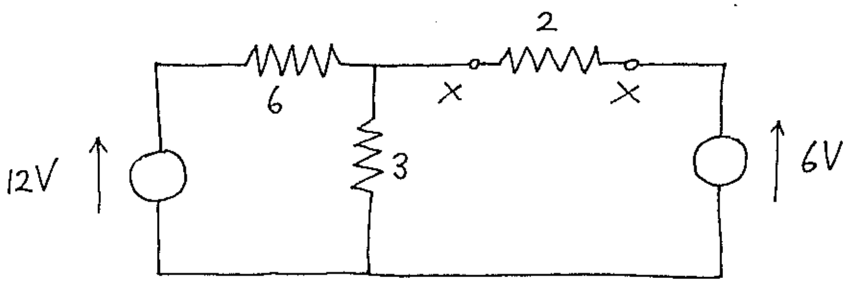
(b) Show that the sensitivity is increased by a factor of 4 if all arms are strain gauges.

(16) In the circuit below, find the current in a 100 Ω resistor connected between A and B:



What would the current be if the 100 Ω were replaced by (a) 1 kΩ (b) 10 kΩ ?

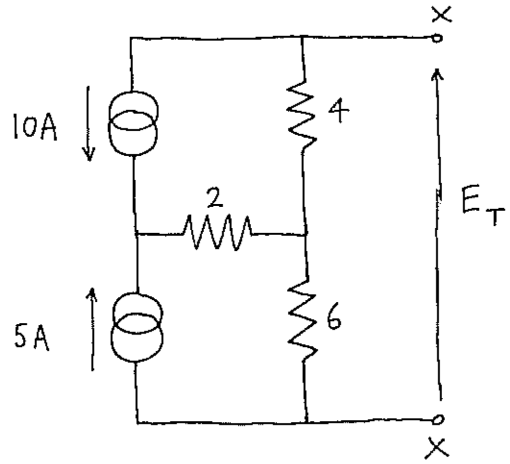
(17) Use Thévenin's theorem to find the current in the 2 Ω resistance:



(18) Solve the previous question using a Norton equivalent circuit.

Use the principle of superposition (linearity) to calculate the currents due to the 6V and 12V sources acting separately.

(19) Find the Thévenin circuit for the network below:

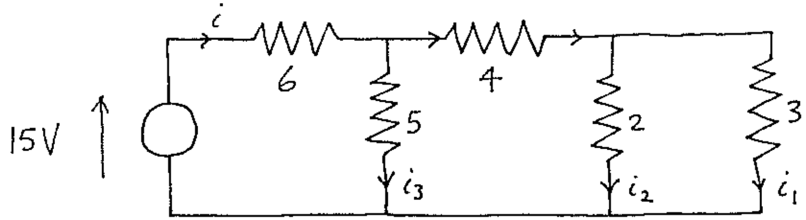


N.B. Ideal current generators (sources) have infinite internal resistances.

They are therefore "replaced" by open-circuits for analytical purposes.

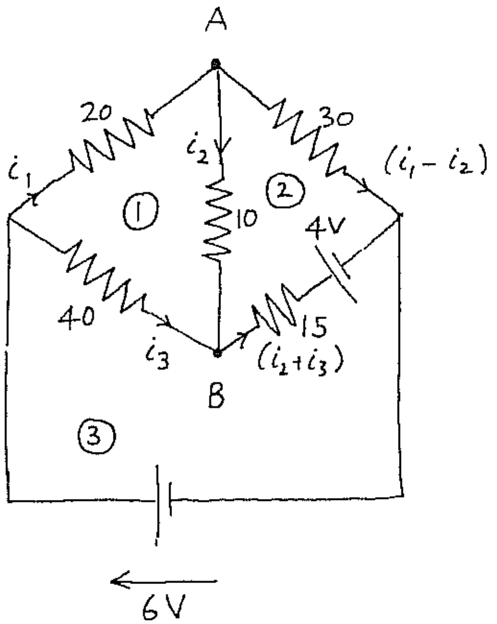
(20) Write down the equations for the three voltage loops in the following network.

(NOTE: This is simply an exercise in getting all the signs correct. No need to solve the equations).



NOTE: Work systematically along the network so that each new equation includes a new component.

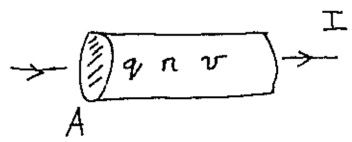
(21) Another exercise in getting the signs right: write down the loop equations for the bridge circuit below:



Circled numbers indicate the three loops required by the question.

Loop (3) is the outermost loop for the purpose of this question.

(1)



I = current $C s^{-1}$
A = cross section area m^2
q = charge on an electron C
n = electron density m^{-3}
v = velocity of electrons $m s^{-1}$

Volume of charge passing any point per second $= vA$.

$\therefore$ Number per second $= nvA$

$\therefore$ Charge per second $= qnvA = I$.

$$\therefore \quad v = \frac{I}{q\,n\,A}$$

$I = 1A \qquad q = 1.6 \times 10^{-19}$ C

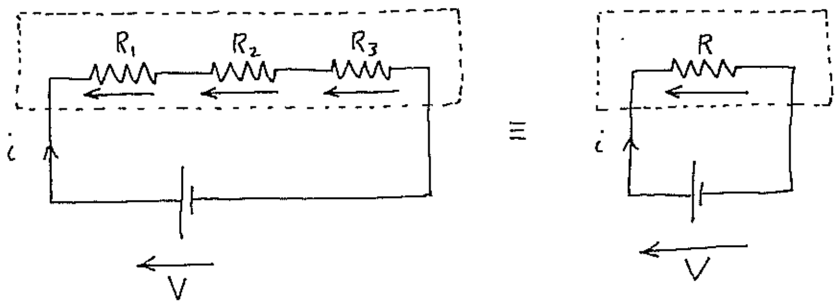
$n = 5 \times 10^{28} \,/m^3$

$A = \frac{\pi}{4}\left(2.5 \times 10^{-3}\right)^2$

$$\therefore \quad v = 2.5 \times 10^{-5} \text{ m/s}$$

which is considerably less than a snail's pace.

(2) (a) Resistors R_1 R_2 and R_3 in series. Find the equivalent single resistor R.



The equivalent resistor R must be indistinguishable from the combined series resistors.

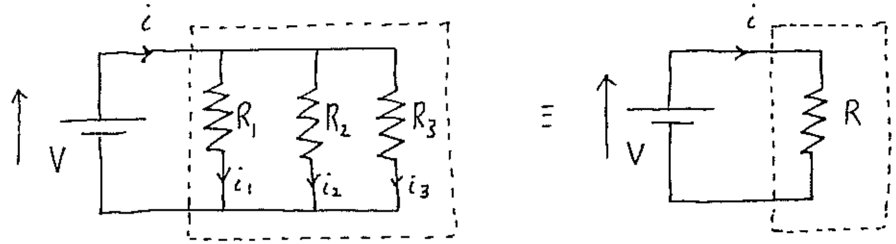
For the LH circuit $\quad V = i\,(R_1 + R_2 + R_3)$

For the RH circuit $\quad V = i\,R$

(Kirchhoff's voltage law)

Hence $\quad R = R_1 + R_2 + R_3$

(b) Resistors R_1 R_2 and R_3 in parallel.



LH: $i = i_1 + i_2 + i_3 = \dfrac{V}{R_1} + \dfrac{V}{R_2} + \dfrac{V}{R_3}$ and this $= \dfrac{V}{R}$

in the RH circuit, which must be indistinguishable. Hence by using Kirchhoff's current law we have established that

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

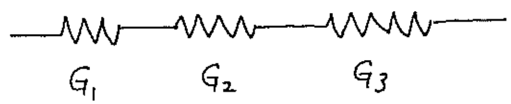
NOTE: it is useful to remember the result for two resistors R_1 and R_2 in parallel:

$$R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

(3) (a) Conductance $G = 1/R$ by definition. From the previous question part (b) it follows that for conductances in parallel:

$$G = G_1 + G_2 + G_3$$

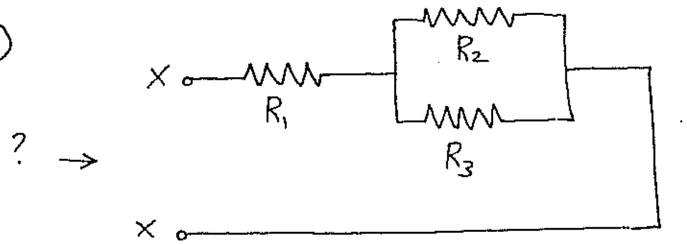
(b) Remembering that "conductances" are physically exactly the same as resistances:



$$\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3}$$

$$\left[\, R = R_1 + R_2 + R_3 \,\right]$$

(4)



Resistances first: $R = R_1 + \dfrac{R_2 R_3}{R_2 + R_3}$

$$= 1 + \frac{6}{5} = \frac{11}{5} \text{ ohms}$$

Conductances: write G_1 G_2 and G_3 in place of the resistances:

$$\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2 + G_3}$$

$$\therefore G = \frac{G_1 (G_2 + G_3)}{G_1 + G_2 + G_3} = \frac{(5/6)}{1\,5/6} = \frac{5}{11}$$

and the units are reciprocal ohms: siemens or mhos.

(5) (a) The voltage divider:

$$V_1 = I R_1 = \left(\frac{V}{R_2 + R_1} \right) R_1$$

$$\therefore \frac{V_1}{V} = \frac{R_1}{R_2 + R_1}$$

(b) The current divider:

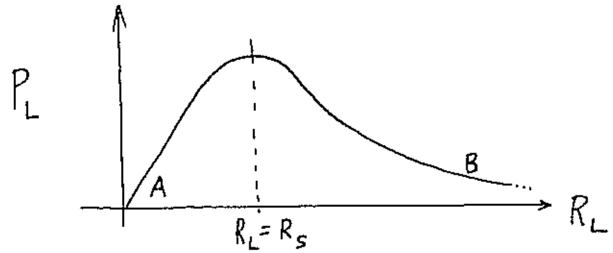
$$I = V \, (G_2 + G_1)$$

$$I_1 = V \, G_1$$

$$\therefore \frac{I_1}{I} = \frac{G_1}{G_2 + G_1}$$

NOTE the symmetry / duality of these results.

(6) First sketch the form of P_L vs R_L :



A: when $R_L \ll R_S$

B: when $R_L \gg R_S$

$$P_L = \frac{V_S^2 R_L}{(R_S + R_L)^2}$$

$$\frac{dP_L}{dR_L} = \frac{(R_S + R_L)^2 V_S^2 - V_S^2 R_L \cdot 2 (R_S + R_L)}{(R_S + R_L)^4}$$

$$= 0 \quad \text{when} \quad R_L = R_S$$

and $P_L \text{ (max)} = V_S^2 / 4 R_S$

Note on the shape of the graph:

REGION A: $R_L \ll R_s$ means that

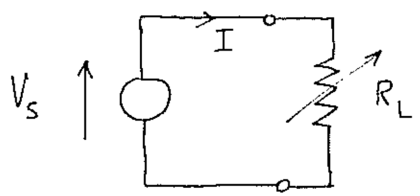
$$P_L \approx \frac{V_s^2 R_L}{R_s^2} \propto R_L$$

REGION B: $R_L \gg R_s$ means that

$$P_L \approx \frac{V_s^2}{R_L} \propto \frac{1}{R_L}$$

These help to make a good sketch of the P_L / R_L function.

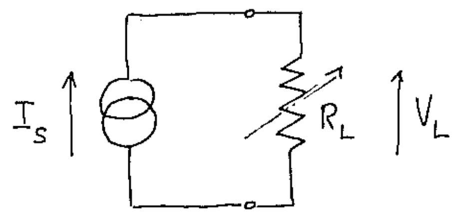
7 (a) the voltage source:



By definition, the voltage source will keep a constant level V_s across any load resistance. As $R_L \to 0$ this implies an increasing current, becoming <u>infinite</u> when $R_L = 0$ i.e., a short-circuit.

(b) the current source. By definition, this will supply a fixed current into any load resistance no matter what its value. As $R_L \to \infty$

this implies an increasing voltage:

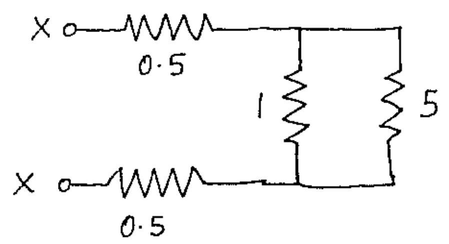


$V_L = I_s R_L$

As R_L approaches an open-circuit, the voltage approaches <u>infinity</u>.

CONCLUSION: don't s/c an ideal voltage source or o/c an ideal current source – you will get silly answers. Check for errors in the circuit you are analysing.

8 Start by simplifying the parallel-resistor combinations. The circuit then becomes:



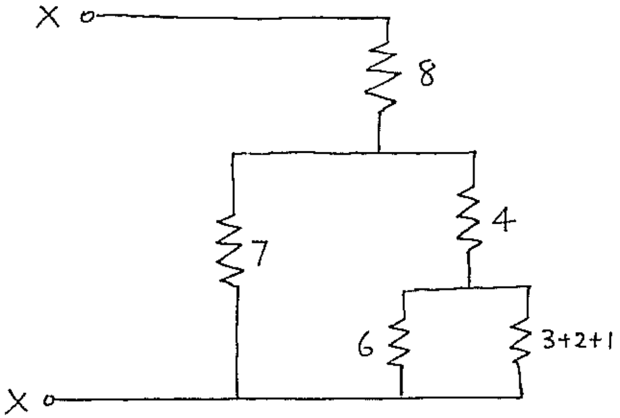
and 1 // 5 gives 5/6 Ω

∴ total = 1 5/6 Ω

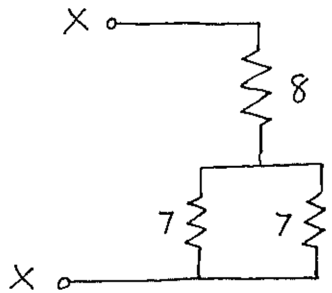
9 It is often helpful to re-draw a circuit before starting to simplify it.

In this case the ladder network of the question can be represented as series-parallel combinations.

Thus:



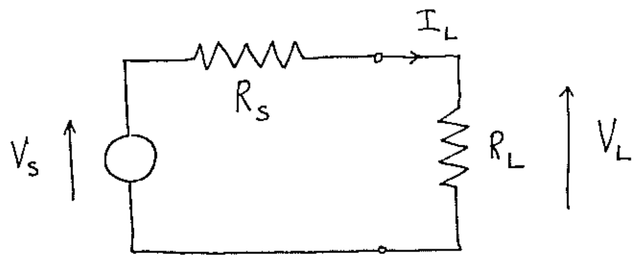
Next:



Giving finally

$8 + 3\tfrac{1}{2} = 11\tfrac{1}{2}\ \Omega.$

(10)



Load voltage $V_L = V_s \cdot \left[\dfrac{R_L}{R_L + R_s} \right]$

and

$$I_L = \frac{V_s}{R_s + R_L}$$

(a) When $R_s \ll R_L$ the expression for V_L becomes:

$$V_L \approx V_s$$

i.e., as long as the inequality is satisfied, V_L is constant i.e. equivalent to an ideal source. To guarantee that the inequality is always true, we would need $R_s = 0$, the condition for the "perfect" voltage source.

(b) When $R_s \gg R_L$ the expression for I_L becomes:

$$I_L \approx \frac{V_s}{R_s} \quad \text{i.e. constant.}$$

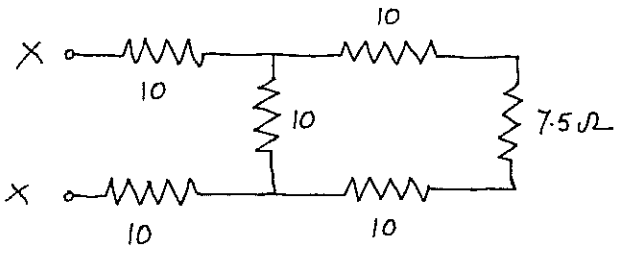
To guarantee that the inequality is always true, we would need $R_s \to \infty$, the condition for the "perfect" current source.

(11) Combining the three 10's on the RHS into 30Ω and putting them in parallel with the adjacent $10\ \Omega$:

$$\frac{10 \times 30}{10 + 30} = 7.5\ \Omega$$

(contd)

This leaves:

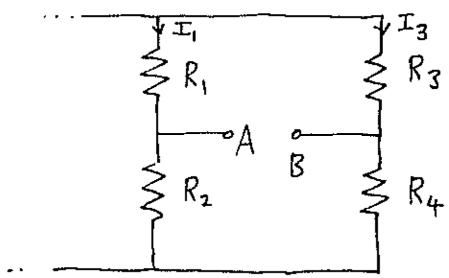


Continuing to walk from R → L along the network we obtain 27.5 Ω in parallel with 10 Ω:

$$\frac{27.5 \times 10}{27.5 + 10} = 7.33\ \Omega$$

giving finally $R_{xx} = 10 + 10 + 7.33 = 27.33\ \Omega$

(12)



At balance, I_1 flows through R_1 and R_2.

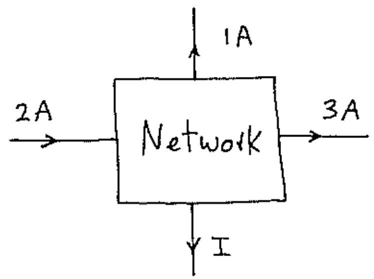
Also, I_3 flows through R_3 and R_4.

Therefore:

$$\left.\begin{aligned} I_1 R_1 &= I_3 R_3 \\ I_1 R_2 &= I_3 R_4 \end{aligned}\right\} \therefore \frac{R_1}{R_2} = \frac{R_3}{R_4}$$

The balance condition is used to find the value of (say) R_1 when the values of the other resistors are known.

(13) If we draw an imaginary boundary around the components, we can apply Kirchhoff's current law:



Choose (say) current entering as ⊕ve, and then apply $\Sigma I = 0$:

$$2 - 1 - 3 - I = 0$$

$$\therefore I = -2A$$

i.e. 2 A in the opposite direction to that shown.

(14) The vertical "arms" in the diagram form a balanced Wheatstone bridge, since

$$\frac{1}{2} = \frac{3}{6} \qquad \therefore \quad I = 0$$

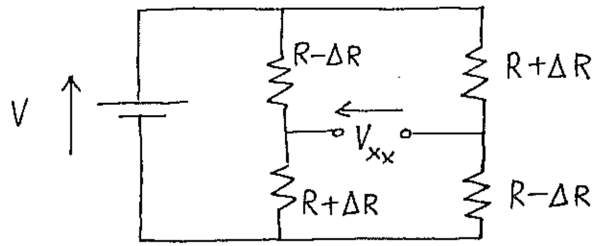
and this applies for any resistor in place of the 6 Ω shown.

(15)
(a) $$V_{xx} = V\left[\frac{R+\Delta R}{2R+\Delta R} - \frac{R}{2R}\right]$$

$$= V\left[\frac{R\,\Delta R}{2R\,(2R+\Delta R)}\right]$$. Now use $\Delta R \ll R$

$$V_{xx} \approx \frac{V.\,\Delta R}{4R} \qquad \therefore\ S = \frac{V}{4R}$$

(b) The strain gauges must be physically arranged so that two increase and two decrease by ΔR :



NOTE the arrangement of gauges !

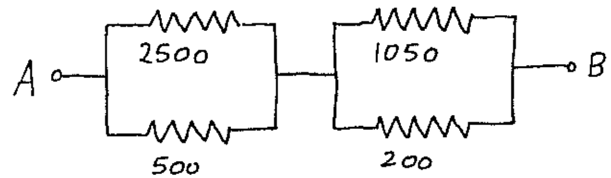
$$V_{xx} = V\left[\frac{R+\Delta R}{2R} - \frac{R-\Delta R}{2R}\right] = \frac{V.\,\Delta R}{R}$$

$$\therefore\ S = \frac{V_{xx}}{\Delta R} = \frac{V}{R}$$, a factor of 4 higher.

(16) Because the question asks for the current with 3 different resistances between A and B, it is worth using the Thévenin circuit.

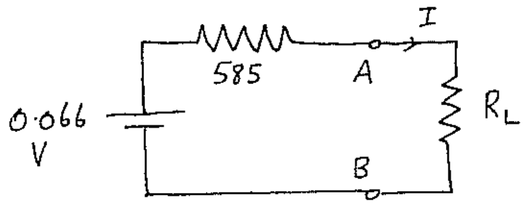
$$E_T = 10\left[\frac{500}{500+2500} - \frac{200}{200+1050}\right] = 0{\cdot}066\,V$$

$$R_T = \left[\frac{2500\times 500}{2500+500} + \frac{1050\times 200}{1050+200}\right] = 585\,\Omega$$



10V is replaced by a s/c to find R_T.

The Thévenin circuit is therefore:

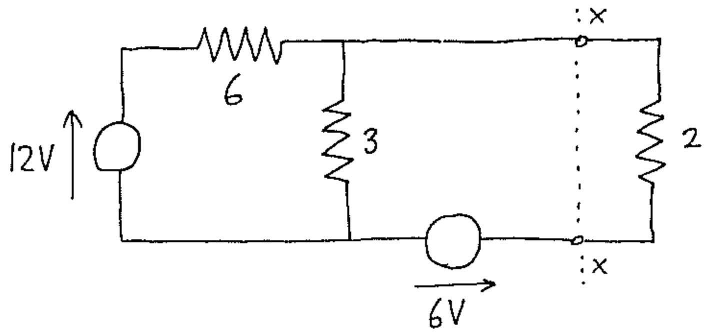


$$I = \frac{0{\cdot}066}{R_L + 585}$$

For $R_L = 100\,\Omega$, $I = 96\,\mu A$
$1\,k\Omega$ $I = 42\,\mu A$
$10\,k\Omega$ $I = 6{\cdot}2\,\mu A$

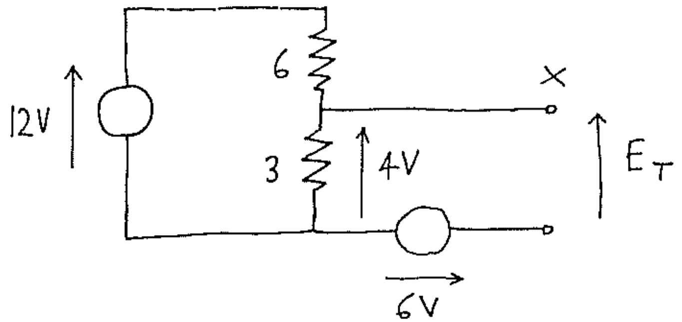
The equivalent circuit approach is useful when repeated calculations are required.

(17) It may be helpful to re-draw the circuit into standard form:



Looking into XX the resistance is $6 // 3 = 2\,\Omega$, where the 12V and 6V sources have been replaced by their internal resistances (zero).

To find E_T:



$$E_T - 4 + 6 = 0$$

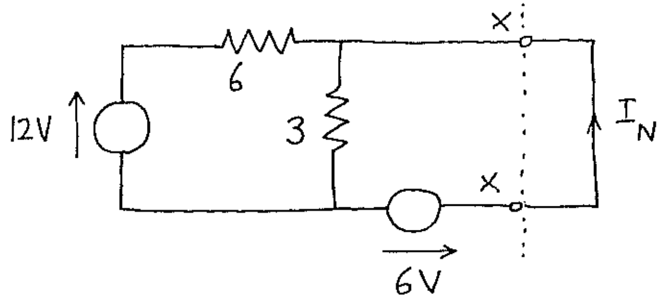
$$\therefore E_T = -2V \text{ i.e.}$$

2V in the opposite direction to that shown.

The Thévenin circuit is therefore $R_T = 2\,\Omega$ and $E_T = 2V$. The current through a $2\,\Omega$ load resistor is therefore $2/4 = \frac{1}{2}$ A.

(18) The Norton equivalent resistance is the same as for the Thévenin circuit, which we have found to be $2\,\Omega$ in the previous question.

The Norton current is the s/c current between terminals XX: the current due to the 6V acting alone is added to that due to the 12V alone

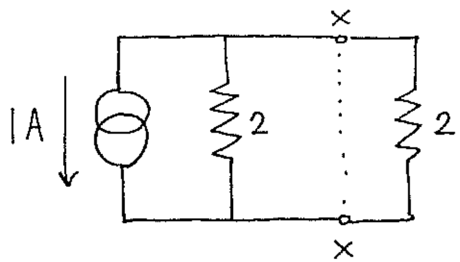


Due to the 6V source, $I_N = \frac{6}{2} = 3A$

Due to the 12V source, $I_N = \frac{-12}{6} = -2A$

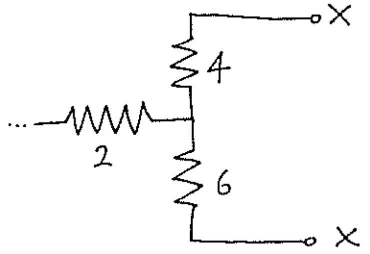
$\therefore I_N = 1A$

To check the current in the $2\,\Omega$ load:



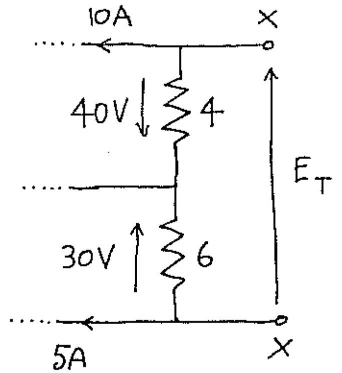
The current divides equally between the two resistors, i.e. ½ A flows through the $2\,\Omega$ load as found earlier.

(19) To find R_T we look into XX and replace all sources by their internal resistances:



$$R_T = 4+6 = 10\,\Omega.$$

To find E_T we need the o/c voltage at XX:

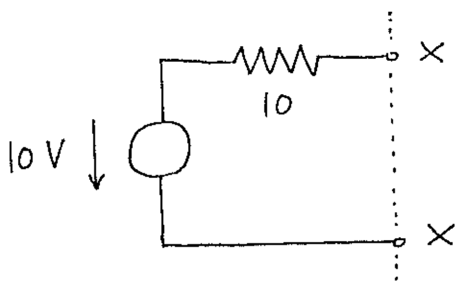


$$V_T + 40 - 30 = 0$$

$$\therefore V_T = -10\text{V}, \quad \text{i.e.}$$

10V in the direction opposite to that shown.

The Thévenin circuit is thus:



(20) Applying the Ohm's law sign convention consistently together with Kirchhoff's voltage law to each loop in turn:

Note first that $i = i_1 + i_2 + i_3$

Choose (say) to take the +ve direction around a loop to be the counterclockwise direction:

LH loop : $5i_3 + 6(i_3 + i_2 + i_1) - 15 = 0$

Middle : $2i_2 + 4(i_2 + i_1) - 5i_3 = 0$

RH loop : $3i_1 - 2i_2 = 0$

} 3 eqns, 3 unknowns

NOTE: Taking a "new" loop which encloses all three inner loops would not provide a new independent equation, since all the information it would contain has already been used once.

(21) First check the currents at the junctions:

Junction A : $i_1 - i_2 - (i_1 - i_2) = 0$ ✓OK

Junction B : $i_3 + i_2 - (i_2 + i_3) = 0$ ✓OK

Choose (say) anticlockwise directions round the loops to be the positive directions....

Loop 1 : $20i_1 - 40i_3 + 10i_2 = 0$

Loop 2 : $30(i_1 - i_2) - 10i_2 - 15(i_2 + i_3) - 4 = 0$

Loop 3 : $20i_1 - 6 + 30(i_1 - i_2) = 0$

NOTE that the third equation requires the extra component – the 6V battery – to make it independent of the other two loops.

The essential ground rules and basic theorems of circuit analysis have been covered in the notes on DC. It is now time to generalize them.

Introduction to AC signals and circuits

Now we are getting to the central theme of communications and data transmission: the processing of time-varying signals. The work we cover in this section lays the foundation for understanding filters, bandwidth, and amplification.

Two questions arise before we start:

(1) Why is AC circuit analysis so preoccupied with sinusoidal waveforms? Why not square waves or triangular waves?

(2) How are sinusoidal voltages generated?

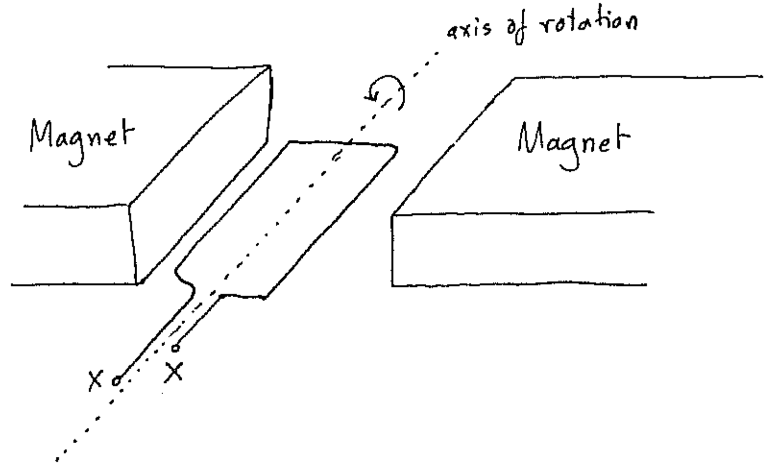
Firstly, sinusoids are, like exponentials, easily dealt with analytically: the derivative and the integral of a sinusoidal wave are themselves both sinusoids.

Any periodic waveform (e.g. a square wave or a triangular wave) can be regarded as a sum of sine waves (this is the Fourier series) at multiples of its fundamental frequency. This means that once

we have analysed a circuit or signal in terms of its Fourier components, its response to any waveform is known, provided that the circuit is linear and the waveform is periodic in time.

Secondly, sine waves can be generated in many ways. One way is considered in the notes on Transients: a zero or very low-loss LC circuit is given an electrical impulse and then allowed to oscillate at its natural frequency. Similarly, a tuning-fork or an empty wine-glass will resonate with a single frequency.

Another way is to use an alternator — a simple form of AC generator consisting of a rectangular coil rotating in a uniform magnetic field:



This should remind you of your school physics. The voltage V_{XX} is a sinusoidal function of time:

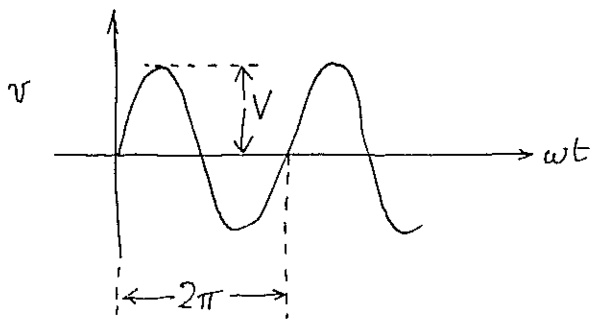
$$V_{XX} = NAB\omega \sin \omega t \qquad \cdots\cdots (1)$$

In equation (1),

N = Nº of turns in the coil
A = cross sectional area m^2
B = (uniform) flux density Wb/m^2
ω = rotation speed radians /second
ωt = angle through which coil rotates in time t

Sine waves, phase angles, and definitions

It is convenient to give the definitions for a voltage that is varying sinusoidally; it could equally be a current.



$$v = V \sin \omega t \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad (2)$$

and V is the amplitude of the waveform

A complete rotation of the loop in the magnetic field corresponds to a 2π increase in ωt, the angle through which the coil has rotated i.e., one complete revolution.

In the waveform this 2π increase is

called a cycle. If the time taken for a complete cycle is T, then:

$$\left.\begin{aligned} \omega T &= 2\pi \\ \text{or} \quad \omega &= 2\pi\left(\tfrac{1}{T}\right) = 2\pi f \end{aligned}\right\} \ldots\ldots \quad \ldots\ldots \quad (3)$$

where f is the frequency, i.e., the number of complete cycles per second. (Unit: the Hertz).

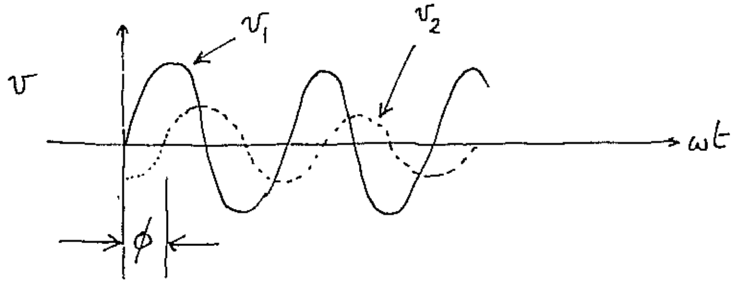
In the UK, the frequency of the mains electricity supply is 50 Hz, the waveform of the voltage is sinusoidal, and it is indeed generated by rotating coils in a magnetic field (well, near enough for our purposes).

The amplitude of the mains voltage in the UK is about 340V by the way. This may seem wrong: the "mains voltage" is generally quoted as being about 240V. The reason for this apparent discrepancy will become clear when we have covered more theory.

Equation (2) is consistent with the diagram in the sense that $v = 0$ when $\omega t = 0$. The waveform is no less sinusoidal if we move it along the axis: it still has the same essential shape.

If we move it exactly a quarter of a cycle to the left, for example, it is a cosine function because it now has its maximum at $\omega t = 0$.

Phase angle between two sinusoids



I shall call the v_1 waveform our "reference" waveform since it corresponds to equation (2). More generally we can now compare the v_2 waveform with it:

$$\left.\begin{aligned} v_1 &= V_1 \sin \omega t \\ v_2 &= V_2 \sin(\omega t - \phi) \end{aligned}\right\} \quad \ldots \ldots \ldots \ (4)$$

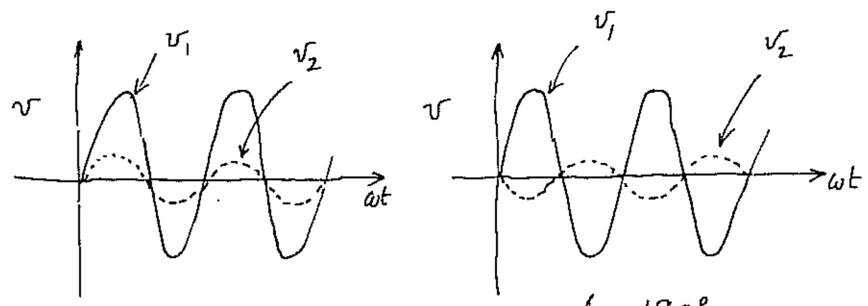
where ϕ is the <u>phase angle</u> of v_2 relative to v_1

($v_2 = 0$ when $\omega t = \phi$ or $\phi + \pi$, etc.)

There are several phase angles that are of sufficient importance to have special names:

If $\phi = 0$ then v_1 and v_2 are <u>in phase</u>
If $\phi = \pm 90°$ " " " " " " <u>quadrature</u>
If $\phi = 180°$ " " " " " " in <u>antiphase</u>

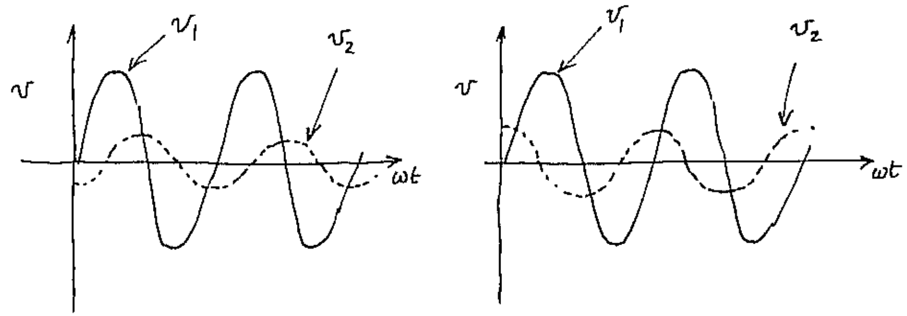
These fundamental phase relationships are illustrated by sketches:



$\phi = 0°$
<u>IN PHASE</u>

$\phi = 180°$
<u>IN ANTIPHASE</u>

These (above) are the most easily remembered; the quadrature waveforms are sometimes confused:



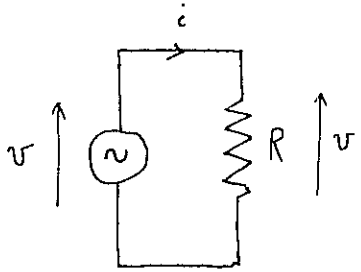
$\phi = 90°$
<u>v_2 LAGS v_1</u>

$\phi = -90°$
<u>v_2 LEADS v_1</u>

The rule: look at two <u>adjacent</u> peaks. The peak that occurs <u>earlier in time</u> is part of the <u>leading</u> waveform.

Response of a resistor to $v = V \sin \omega t$

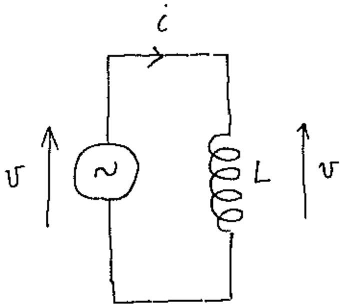
We shall apply our standard reference voltage in turn to the three basic circuit elements. For R:



$$\left.\begin{aligned} v &= V \sin \omega t \\ v &= i R \\ \therefore i &= \frac{V}{R} \sin \omega t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5)$$

$\therefore \phi = 0$ and the current is in phase with the voltage

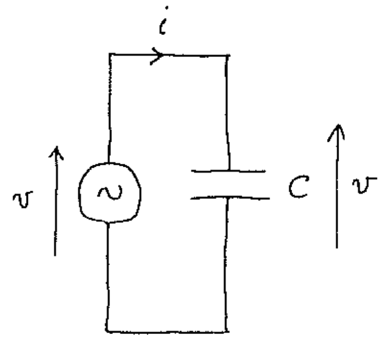
Response of an inductor to $v = V \sin \omega t$



$$\left.\begin{aligned} v &= V \sin \omega t \\ v &= L \frac{di}{dt} \\ \therefore i &= \frac{-V}{\omega L} \cos \omega t \\ \text{or} \quad i &= \frac{V}{\omega L} \sin (\omega t - 90^\circ) \end{aligned}\right\} \quad \ldots\ldots (6)$$

So that current <u>LAGS</u> voltage by 90° for the inductor.

Response of a capacitor to $v = V \sin \omega t$



$$\left.\begin{aligned} v &= V \sin \omega t \\ i &= C \frac{dv}{dt} \\ \therefore i &= \omega C V \cos \omega t \end{aligned}\right\} \quad \ldots\ldots (7)$$

$$\text{or} \quad i = \omega C V \sin (\omega t + 90^\circ)$$

So that current <u>LEADS</u> voltage by 90° for the capacitor.

To help remember these results

The acronym <u>C</u>I<u>V</u>I<u>L</u> is often mentioned.

My own preference is for the more flexible expression "ELI likes ICE." This allows me, on a bad day, to use "ELI <u>dislikes</u> ICE."

NOTE: The concepts of leading, lagging etc. are applicable to any sinusoidal waveforms having the same frequency.

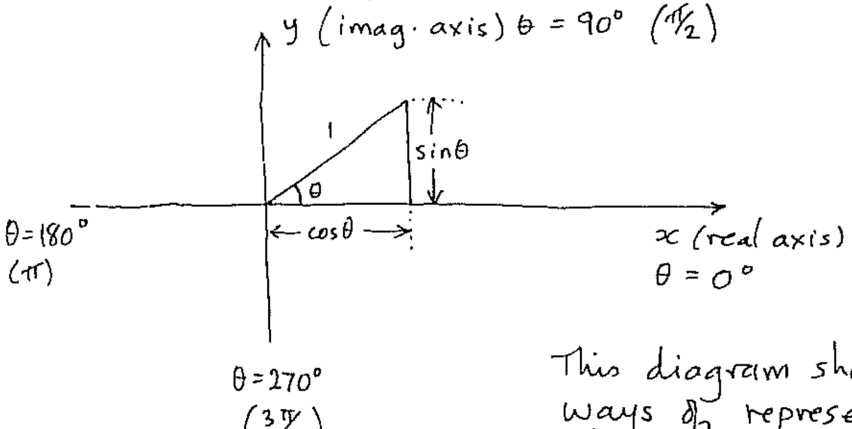
Analysis of AC circuits using complex numbers

The good news is that we can analyse AC circuits using exactly the same rules as for DC. To do this, we generalize the concept of resistance and give it an impressive new name: impedance. But first, some revision.

Interpretation of Euler's theorem

$$e^{j\theta} = \underbrace{\cos\theta + j\sin\theta}_{\substack{\text{cartesian}\\ \text{co-ordinates}\\ x+jy}} = \underbrace{1\angle\theta}_{\substack{\text{polar}\\ \text{co-ordinates}\\ r\angle\theta}} \quad \cdots\cdots \quad (8)$$

[In EE we use j for $\sqrt{-1}$ since i is reserved for current.]



This diagram shows two ways of representing $e^{j\theta}$

From the theorem:

$$\left.\begin{aligned} \sin\theta &= \Im\left[e^{j\theta}\right] \\ \cos\theta &= \Re\left[e^{j\theta}\right] \end{aligned}\right\} \cdots \cdots \cdots \cdots (9)$$

$\Im \equiv$ "imaginary part of"

$\Re \equiv$ "real part of"

Using this notation we could write

$$V_1 \sin\omega t = \Im\left[V_1 e^{j\omega t}\right] \cdots \cdots \cdots (10)$$

though most books simply use:

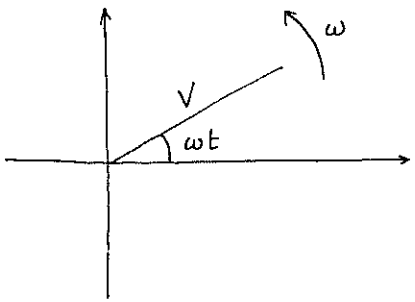
$$V_1 \sin\omega t = V_1 e^{j\omega t} \cdots \cdots \cdots \cdots (11)$$

HEALTH WARNING: equation (11) is more convenient than (10) but it can lead to confusion and even errors if you forget that it is (10) which is strictly correct.

This is especially important when calculating power using complex numbers for voltage and current.

Phasor diagrams

A variation on the basic diagram allows us to build up a shorthand notation:



$\theta = \omega t$

same co-ordinate system as before

The difference is that the angle θ is now a function of time. Physically, we can visualize this as meaning that the line of length V is rotating anticlockwise at ω radians per second – its "angular velocity" or "angular frequency" as it is called in electro-speak.

Note that this is the same ω as in equation (3), so $\omega = 2\pi f$ still applies. The reason for this correspondence will become clearer shortly.

In our new shorthand notation, the rotating line (called a phasor) is represented by:

$$V e^{j\omega t} \quad \dots\dots \quad \dots\dots \quad \dots\dots \quad (12)$$

Using eqn. (11) we can now write v_1 and v_2 in

the compact form:

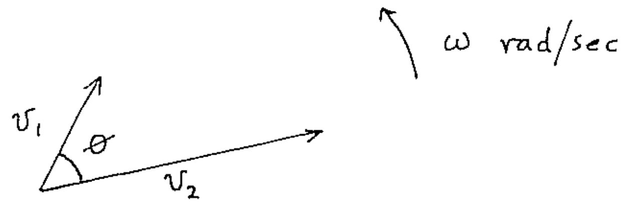
$$\left. \begin{aligned} v_1 &= V_1 e^{j\omega t} \\ v_2 &= V_2 e^{j(\omega t - \phi)} \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad (13)$$

[in place of:

$$v_1 = V_1 \sin \omega t$$

$$v_2 = V_2 \sin(\omega t - \phi) \ ]$$

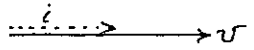
and the next step is to represent v_1 and v_2 in a kind of vector diagram (called a phasor diagram in EE):



This is excellent: it gives all the essential information about the voltages v_1 and v_2 without having to sketch tedious waveforms like those on pages 5 and 6. That's real progress.

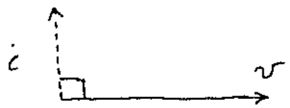
Phasors are like vectors in that the rules of addition apply.

The economy of notation is emphasized by representing current and voltage for R, L and C using phasor diagrams :



Resistor: in phase

Inductor: i lags v by 90°



Capacitor: i leads v by 90°

The concept of j as an "operator"

It is very helpful in AC circuit analysis to have a simple physical picture of what j represents.

Start with the representation of v_1 by a phasor:

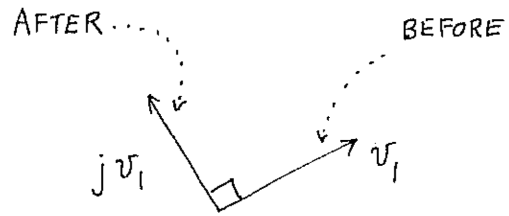
$$v_1 = V_1 e^{j\omega t}$$

Now: what does jv_1 look like?!

In terms of algebra,

$$\left.\begin{aligned} jv_1 &= jV_1 e^{j\omega t} \\ &= V_1 e^{j(\omega t + 90°)} \end{aligned}\right\} \quad \ldots\ldots \quad (14)$$

and so the phasor diagram becomes:



So: <u>multiplication</u> by j is equivalent to <u>rotation</u> <u>anticlockwise</u> <u>by</u> <u>90°</u> without changing the magnitude of the phasor quantity.

Similarly, $j \times jv_1 = -v_1$ which is a 180° rotation.

The concept of impedance

We are now ready to generalize the concept of resistance to include AC voltages and currents, provided that they are varying sinusoidally and at the same frequency.

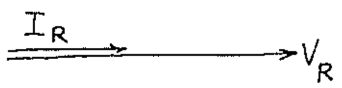
In AC theory, the ratio volts/current is still measured in ohms but is called <u>impedance</u>, and it is represented by a complex number, Z.

The notation in AC circuit analysis is to return to upper case V and I for voltage and current. This has the advantages:

- We used V and I in the DC theory, and all the AC methods are the same: familiar notation reminds us of this.
- It should prevent any confusion with instantaneous values v and i, which we have dealt with separately.

The power of the complex number approach is that it carries all the essential phase information as well as magnitude, with no extra effort on our part. (Refer back to the first paragraph in the DC analysis notes).

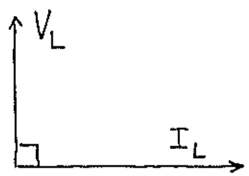
Impedance of a resistor



$$\left.\begin{aligned} V_R &= I_R \cdot R \\ Z_R &= \frac{V_R}{I_R} \end{aligned}\right\} \cdots\cdots (15)$$

exactly as with DC. Regarded in phasor terms, we say that to obtain the voltage phasor from the current phasor for a resistor, multiply the magnitude by R and leave the direction unaltered.

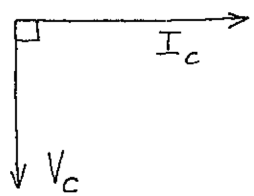
Impedance of an inductor



$$\left.\begin{aligned} V_L &= I_L \cdot j\omega L \\ Z_L &= j\omega L \end{aligned}\right\} \cdots\cdots \cdots\cdots (16)$$

We now interpret this as saying that to get the voltage phasor V_L from the current phasor I_L we rotate it anticlockwise by 90° and multiply its magnitude by ωL. To see where ωL comes from, refer to equation (6) on page 7.

Impedance of a capacitor



$$\left.\begin{aligned} V_c &= I_c \cdot \left(\frac{1}{j\omega c}\right) \\ &= I_c \cdot \left(\frac{-j}{\omega c}\right) \\ Z_c &= \frac{1}{j\omega C} \text{ or } \frac{-j}{\omega C} \end{aligned}\right\} \cdots\cdots (17)$$

In this case, to get the voltage phasor from the current phasor I_L we rotate the current phasor clockwise by 90° and multiply its magnitude by $1/\omega C$.

To see where $1/\omega C$ comes from, refer to equation (7) on page 8.

A note on units

Z is the ratio of volts / amps and is still measured in ohms.

The V and I as used on pages 7 and 8 refer to the amplitudes of the respective waveforms. Any voltages or currents calculated using impedances will also come out as numbers representing amplitudes.

Another option for V and I is to use the RMS values of the sinusoids, where

$$\text{RMS value} = \frac{1}{\sqrt{2}} \times \text{amplitude} \quad \cdots \quad (18)$$

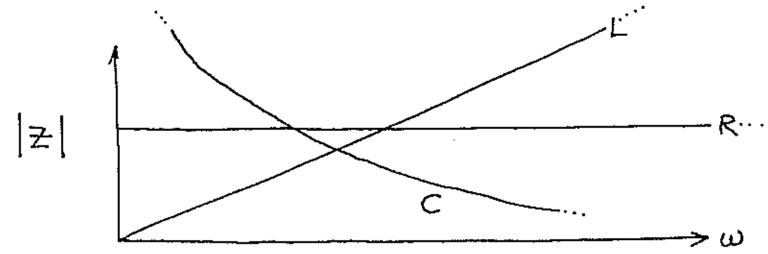
Hence, if RMS values of V and I are used in calculations, then the resulting voltages or currents will be their RMS values.

[RMS = Root Mean Square and will be explained when we deal with power calculations in AC circuits].

Impedance can depend on frequency, and usually does

After the simplicity of the constant resistor, R, the behaviour of L and C will take some getting used to... an overview is helpful here.

Here is a sketch showing how the impedance (magnitude) varies with frequency :



Resistor : $|Z_R| = R$ and is independent of frequency;

Inductor : $|Z_L| = \omega L$ and is directly proportional to frequency;

Capacitor : $|Z_C| = \frac{1}{\omega C}$ and is inversely proportional to frequency.

Bear these in mind, and note the constant 90° phase shifts for L and C (not shown in the diagram).

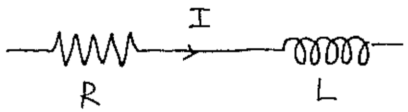
Combining impedances in series and parallel

The rules (as promised) are exactly the same as those for combining resistances. See DC notes pages 14 - 16.

Impedance and admittance

In the DC notes the reciprocal of resistance was defined as conductance. Its symbol is G. We can now generalize both concepts.

First, note that Z is a complex number in general i.e. it has finite real and imaginary components. For example, consider R and L in series:



$$Z_R = R$$

$$Z_L = j\omega L$$

$$\therefore Z_{total} = R + j\omega L.$$

In general,

$$\left.\begin{array}{l} Z = R + jX \\ \text{where } R = \text{resistance, } X = \underline{\text{reactance}} \end{array}\right\} \quad \ldots (19)$$

Admittance (the counterpart of DC conductance) is defined as

$$\left.\begin{array}{l} Y = 1/Z \\ \text{and } Y = G + jB \end{array}\right\} \quad \ldots\ldots\ldots\ldots (20)$$

where G = conductance, B = susceptance

From equations (19) and (20) it follows that

$$G + jB = \frac{1}{R + jX} \quad \dots\dots \quad \dots\dots \quad \dots\dots \quad (21)$$

which means that if we are given R and X we can find G and B, and vice versa.

Thus:

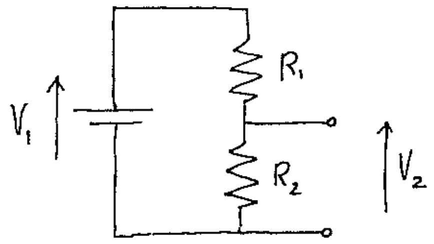
$$\left.\begin{aligned} G &= \frac{R}{R^2 + X^2} \\ B &= \frac{-X}{R^2 + X^2} \end{aligned}\right\} \quad \dots\dots \quad \dots\dots \quad \dots\dots \quad (22)$$

NOTE the minus sign and make sure you understand why it is there. Minus signs are like road signs in that you ignore them at your peril...

Applications of frequency-dependent Z

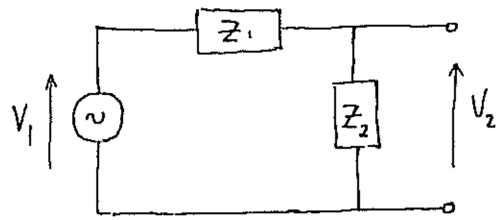
We have now covered enough of the basic material to catch a glimpse of the powerful methods that are available. The fact that inductance and capacitance behave like different "resistances" at different frequencies opens up possibilities for filtering signals: designing circuits that will pass or block particular ranges of frequencies.

First, consider this "potential divider" made from two resistors R_1 and R_2:



$$\frac{V_2}{V_1} = \frac{R_2}{R_1 + R_2} \quad \dots\dots \quad \dots\dots \quad (23)$$

Now look at its AC counterpart:



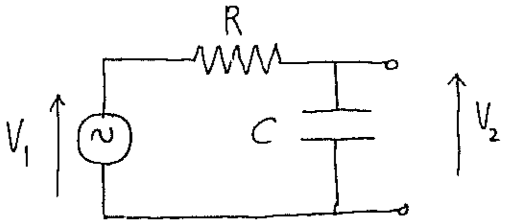
$$\frac{V_2}{V_1} = \frac{Z_2}{Z_1 + Z_2} \quad \dots\dots \quad (24)$$

In this latter form, V_2 can be regarded as an output voltage and V_1 as the input to the circuit!

With the resistors R_1 and R_2, the ratio V_2/V_1 is the same with a battery or an AC signal generator as input, and is independent of frequency.

Suppose, though, that we make $Z_1 = R$ and replace R_2 by a capacitor C; then

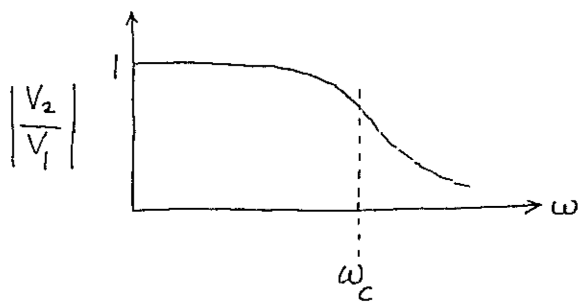
$$Z_2 = Z_c = 1/j\omega C$$



$$\frac{V_2}{V_1} = \frac{1}{1 + j\omega CR} \quad \cdots (25)$$

This ratio is a complex number containing both the magnitude of the ratio and the phase of V_2 relative to V_1. To see the filtering action of the circuit, plot the variation of $\left|\frac{V_2}{V_1}\right|$ with frequency:

$$\left|\frac{V_2}{V_1}\right| = \frac{1}{(1 + \omega^2 C^2 R^2)^{1/2}} \quad \cdots\cdots\cdots \quad (26)$$



This is the response of a LOW-PASS FILTER

At very low frequencies ($\omega \to 0$) the output V_2 approaches the input V_1. As the frequency is increased there is a turning-point (called the cut-off frequency) beyond which higher frequencies are rapidly attenuated. In fact as $\omega \to \infty$, the output voltage V_2 approaches zero.

Of course there isn't really a sharp cut-off,

but it is useful to have a rough idea of where the filter starts to have an effect.

Looking at equation (26), it depends on the term $\omega^2 C^2 R^2$ compared with 1.

If $\omega^2 C^2 R^2 \ll 1$ then the filter behaves like a resistive network and lets all frequencies through unattenuated.

If $\omega^2 C^2 R^2 \gg 1$ then increasing frequencies are heavily attenuated.

Arbitrarily, the frequency $\omega = \omega_c$ is defined by the condition:

$$\left.\begin{aligned} \omega_c^2 C^2 R^2 &= 1 \\ \text{that is,} \quad \omega_c &= \frac{1}{CR} \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (27)$$

and this is the frequency for which $\left|\frac{V_2}{V_1}\right|$ falls to $\frac{1}{\sqrt{2}}$, also known as the "3dB" frequency.

Here is a link between the time and frequency domains: the <u>frequency</u> at which this low-pass filter cuts off is the inverse of the <u>time-constant</u> of the filter components C and R.

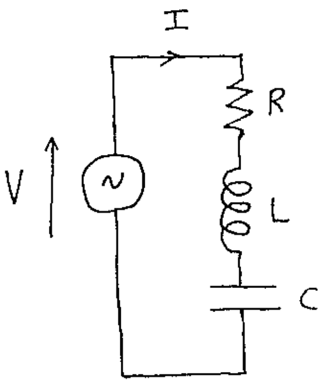
AC signals : low-pass and high-pass filters

By interchanging C and R, we produce a high-pass filter. If we use combinations of R, L and C we can make filters that pass or strongly attenuate bands (ranges) of frequencies.

One important example is the band-pass RLC filter. It is considered in the notes on Transients, pages 10-12.

The difference here is that the voltage source (which can be thought of as the input to the filter) is a continuous sinewave rather than a single pulse.

The series-resonant RLC circuit : a natural filter



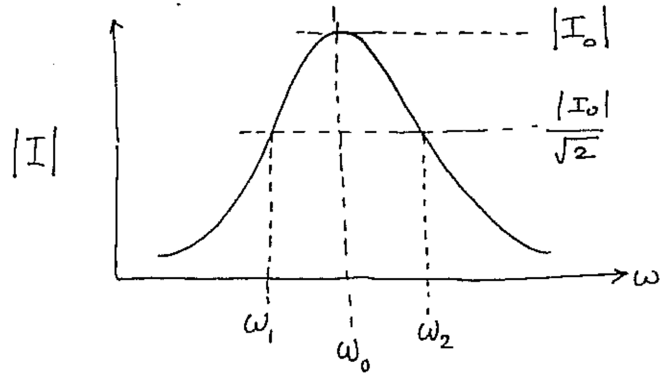
The voltage source is of constant amplitude and variable frequency; what is the variation of current magnitude with frequency?

$$Z = R + j\left[\omega L - \frac{1}{\omega C}\right] \quad \cdots\cdots \quad (28)$$

$$\therefore \; |I| = \frac{|V|}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} \quad \cdots\cdots \quad (29)$$

and this is sketched on the next page.

AC signals : RLC resonant-circuit filter



The current peaks at a frequency ω_0 defined as the resonance frequency:

ω_0 = frequency at which the impedance is a pure resistance

From equation (28) we see that

$$\left.\begin{aligned} Z = R \quad \text{when} \quad \omega L &= \frac{1}{\omega C} \\ \therefore \; \omega_0 &= \frac{1}{\sqrt{LC}} \\ \text{and} \quad f_0 &= \frac{1}{2\pi\sqrt{LC}} \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (30)$$

The frequency range $(\omega_2 - \omega_1)$ is defined as the 3dB or "half power" bandwidth and refers to the frequencies at which $|I|$ falls to $\frac{1}{\sqrt{2}}$ of its peak value at resonance.

The "sharpness" of this resonance curve is a measure of the frequency-selectivity of the circuit as a filter.

This idea is used in defining the Q-factor (quality factor) of the circuit around resonance:

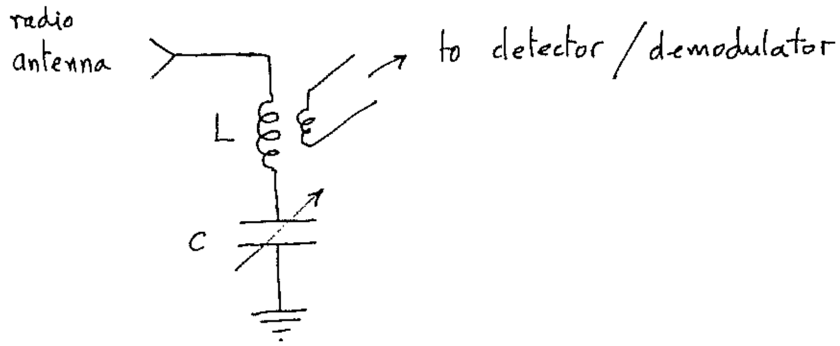
$$Q_0 = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{f_2 - f_1} = \frac{f_0}{\Delta f} \qquad \ldots\ldots (31)$$

See also p.12 in Transients notes. These definitions of Q-factor can be shown to be equivalent to one another and to the more general definition:

$$Q = 2\pi \left[\frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}\right] \ldots\ldots (32)$$

<u>NOTE</u> that this definition is not restricted to any particular frequency or even to a resonance condition.

The origin of the "quality factor" idea can be seen if we look at the circuit in a slightly different way. In the early days of radio, the sets were "tuned" to a particular station by adjusting the capacitor C until resonance was found...

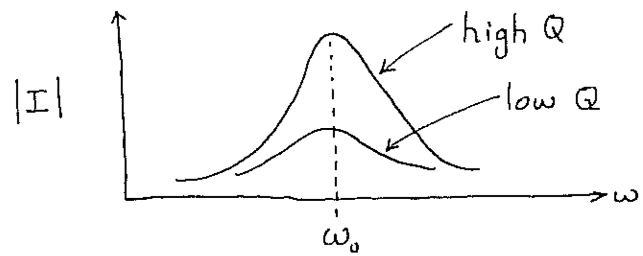


It was found that some coils (the inductor L) were much better than others at "tuning" even though they had the same inductance. Eventually it was realized that it was the <u>resistance</u> of the wire used to make the coil that was the important factor.

Now return to the series resonant circuit: we can regard the L and the R together as representing a practical inductor. Then

$$Q_0 = \frac{\omega_0 L}{R}$$ (See p.12 of notes on Transients)

is telling us that the larger the inductance and the smaller the resistance, the higher the Q-factor of the coil – the sharper the tuning:



The high-Q circuit discriminates more effectively against unwanted frequencies close to the desired frequency ω_0.

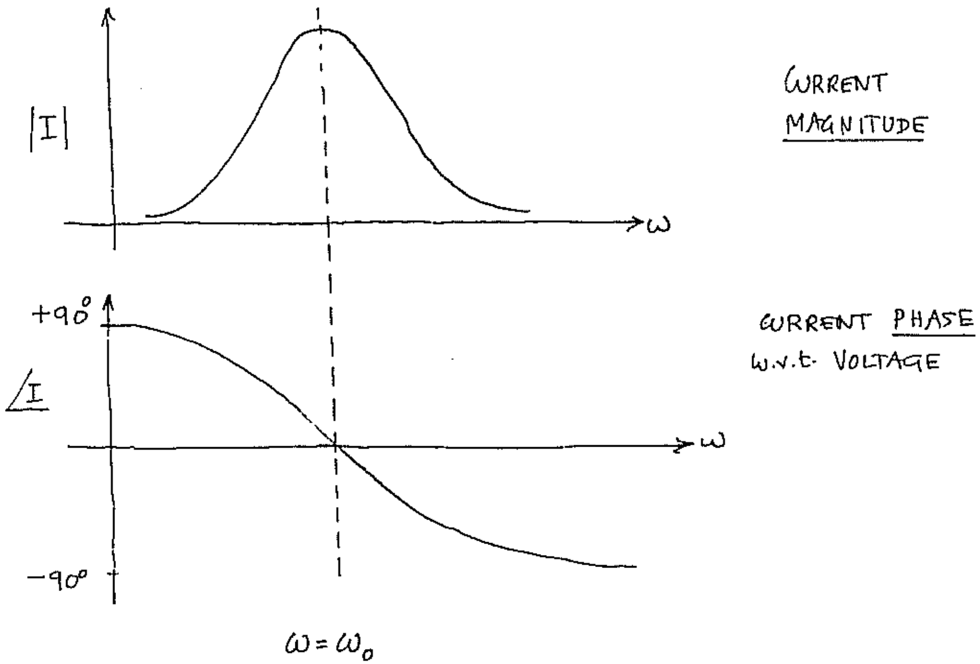
This type of circuit functions as a BAND-PASS filter. Such filters are fundamental to telecommunications using radio waves, and they have their counterparts in optical communications also.

NOTE In the notes on Transients, the concept is considered in time-domain (eqn. (11) measures the decay rate of free oscillations in terms of L/R) and now in the frequency-domain (eqn. (31) in particular). It gives me an opportunity to illustrate a compromise in the design of filters.

The compromise becomes necessary when we realize that the signal to be filtered may be changing its amplitude rapidly. In that case, although a high-Q circuit will have a narrower pass-band, it will also have a longer response time to changes in amplitude. If the Q is too high... what do you think will happen? Suppose it is an AM transmission?

Phase shift of current w.r.t. voltage in a series resonant circuit

So far we have only considered the amplitude response of the current in the series resonant RLC circuit. The filtering process usually involves changes of phase as well as of magnitude in the output (in this case we can think of the current as the "output" quantity of interest).



To understand the phase shift curve:

$$I = \frac{V}{R + j(\omega L - 1/\omega C)} \quad \cdots\cdots \quad (33)$$

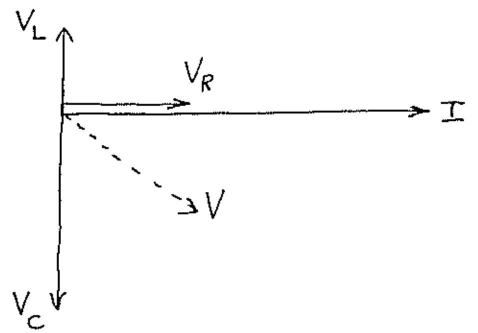
Define the phase angle of V to be zero for convenience; then:

$$\angle I = -\tan^{-1}\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right] \quad \cdots\cdots \quad (34)$$

Make sure you understand how equation (34) defines the $\angle I$ sketch above. We can sketch the phasor diagrams to check that the diagram is correct.

At low frequencies the capacitive reactance dominates the impedance, so that current leads voltage. At resonance the phase angle is zero by definition. At high frequencies, inductance dominates and so current lags voltage.

In constructing a phasor diagram showing all the voltages, bear in mind that the current is the same through components in series. It is therefore easier to draw the I phasor first and relate all the others to it in turn.



In general, $|V_L| \neq |V_C|$ but at resonance these two voltages are

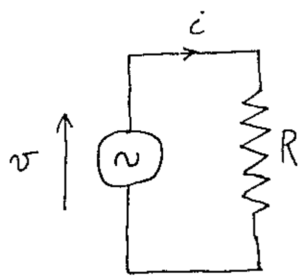
- equal in magnitude
- 180° out of phase (in antiphase)

leaving only V_R which is in phase with I, as it must be, and $V = V_R$ at resonance when the whole circuit behaves as a pure resistor at that particular frequency: $Z = R$.

Power in an AC circuit

It is easy to work out the power in a DC circuit – equation 2. It is simply the product of voltage and current. How convenient it would be if the same rule could apply to AC as to DC.

By using "RMS" values of AC voltage and AC current we can ensure that this is the case – for a resistor at least.

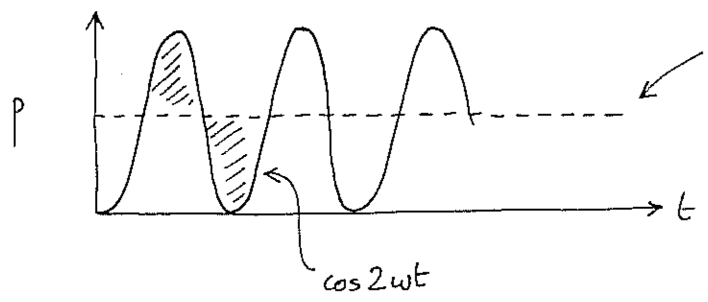


$$\left.\begin{aligned} v &= V_m \sin \omega t \\ \therefore i &= \frac{V_m}{R} \sin \omega t \end{aligned}\right\} \quad \cdots\cdots (35)$$

The instantaneous power is given by vi and we want to work out the average power over a complete cycle or an integer number of complete cycles:

$$\left.\begin{aligned} p = vi &= \frac{V_m^2}{R} \sin^2 \omega t \\ \text{or,} \quad p &= \frac{V_m^2}{2R}\left[1 - \cos 2\omega t\right] \end{aligned}\right\} \quad \cdots (36)$$

The average of the time-varying term, $\cos 2\omega t$, is zero over a complete cycle as illustrated by the waveform sketch on the next page.



NOTE that:

- the shaded areas are equal
- the frequency at which the power varies is twice the supply frequency.

The RMS voltage

We need a single number to characterize an alternating voltage that varies between $+V_m$ and $-V_m$. The average is no good because it is always zero. How about the amplitude, V_m?

$$\text{Average power} \quad \bar{P} = \frac{V_m^2}{2R} \quad \cdots\cdots \quad \cdots\cdots \quad (37)$$

$$\text{Compare with DC} \quad \bar{P} = \frac{V_{dc}^2}{R}.$$

If we choose to characterize the AC voltage by its amplitude, then there is always the annoying factor of 2 to remember.

By general agreement, we use a useful dodge to make the DC and AC formulae look exactly the same:

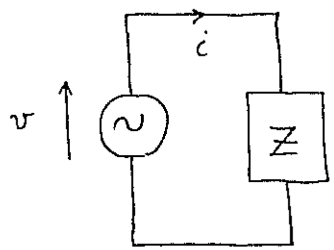
$$\left.\begin{aligned} \bar{P} &= \frac{V_m^2}{2R} = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{V_m}{\sqrt{2}}\right)\cdot\frac{1}{R} \\ &= \frac{V_{rms}^2}{R} \end{aligned}\right\} \quad \dots\dots\dots \quad (38)$$

The "trick" is just to choose $\frac{1}{\sqrt{2}} V_m$ rather than V_m to define the AC quantity.

RMS = root-mean-square.

Now, AC power calculations are just like DC. When the mains voltage is given as 240V it means that this is the RMS value. The amplitude is $240\sqrt{2} = 339$ V approx.

Power in a circuit containing reactance



$$\left.\begin{aligned} v &= V_m \sin\omega t \\ i &= I_m \sin(\omega t + \phi) \end{aligned}\right\} \dots \quad (39)$$

for maximum generality (any Z).

The variables I_m and ϕ allow any magnitude and phase angle for Z, which could be (say) a pure inductor or a pure capacitor.

Proceeding exactly as before,

$$p = vi = V_m \sin\omega t \cdot I_m \sin(\omega t + \phi)$$

or $$p = \frac{V_m I_m}{2}\left[\cos\phi - \cos(2\omega t + \phi)\right]$$

$$\dots \quad \dots \quad \dots \quad (40)$$

Again the time-varying term averages to zero giving:

$$\bar{p} = \frac{V_m I_m}{2}\cos\phi$$

$$= \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\phi \quad \dots \quad \dots \quad (41)$$

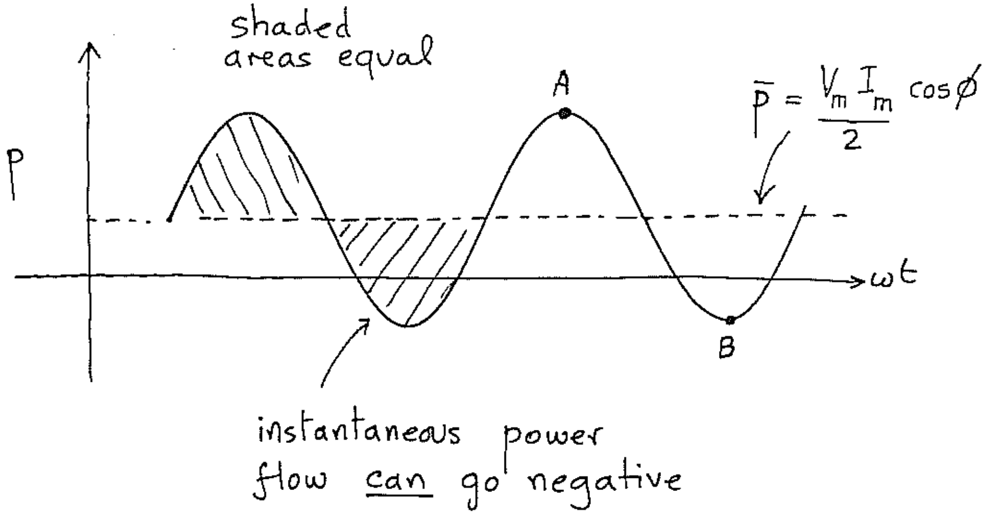
$$= V_{rms} \cdot I_{rms} \cdot \cos\phi$$

and $\cos\phi$ is called the <u>power factor.</u>

AC circuits: power and power factor

It is instructive to sketch the waveforms for the general impedance and also for the special case of pure reactance.

In the general case of an impedance Z:



A: $\quad p = \dfrac{V_m I_m}{2}\left[\cos\phi + 1\right]$

B: $\quad p = \dfrac{V_m I_m}{2}\left[\cos\phi - 1\right]$

When p is negative, energy is flowing back to the source from the load.

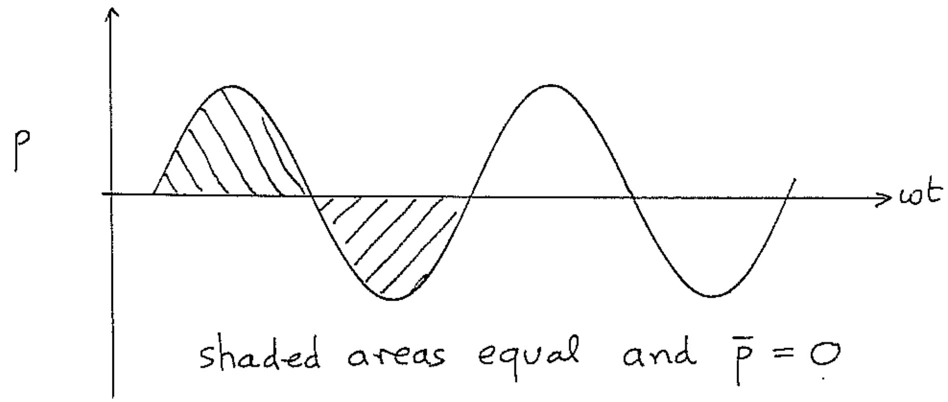
AC circuits: power and power factor

NOTE:

If $Z = R$, $\quad \phi = 0$ and $\bar{p} = \dfrac{V_m I_m}{2}$

If $Z = L$ or C, $\quad \phi = 90^\circ (\pm)$ and $\bar{p} = 0$

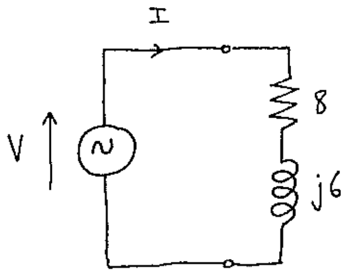
For the inductor and capacitor, no energy is dissipated:



Energy is stored in either the electric field or the magnetic field (C or L) and then periodically returned to the source.

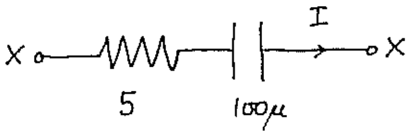
In this special case the Camden students would indeed not have to pay.

1. Use complex algebra to calculate the current I in the circuit below:



$V = \frac{10}{\sqrt{2}} \angle 0^\circ$

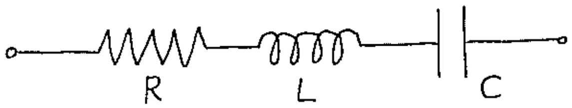
2.



I = 5A RMS.

Find the voltage across XX. (Since we are told that the current is RMS, the voltage will also be RMS). Frequency f = 318 Hz.

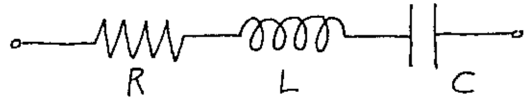
3. Find the impedance of the circuit below at f = 500 Hz.



R = 8Ω ; L = 2·38 mH and C = 14·1 µF

4. Find the impedance of the circuit below at f = 1 MHz.

(a)



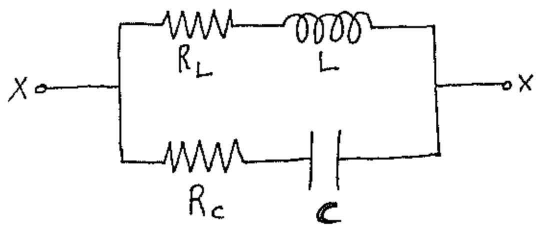
R = 1 kΩ L = 10 mH C = 0·1 µF

(b) Find the impedance when they are connected in parallel.

5. An AC source of fixed internal impedance $Z_s = R_s + jX_s$ supplies power to a variable load $Z_L = R_L + jX_L$. The source frequency is constant.

Find the load impedance for maximum power transfer from the source.

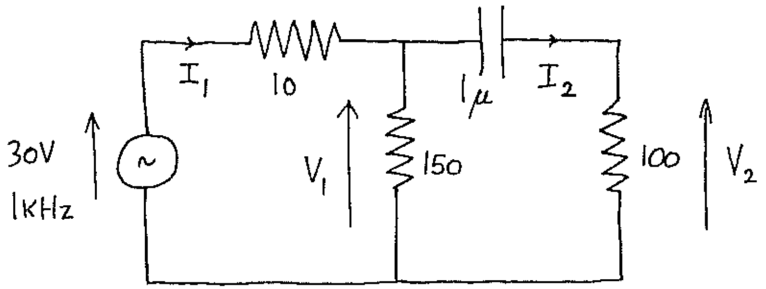
6. The circuit below was given in question 6 of the "transients" and solved using time-domain analysis:



(contd)

The requirement is to find the relation between the components such that Z_{XX} is a constant resistance at <u>all frequencies</u>.

(7) In the circuit below, resistances are in ohms and the capacitance is in farads.



Taking the voltage source as the phase reference, calculate:

(a) The total impedance seen by the 30V source;

(b) The current I_1;

(c) The voltage V_1;

(d) The current I_2;

(e) The voltage V_2.

(8) A series RLC circuit is driven by a sinusoidal voltage source at its resonant

frequency; show that, at resonance, the total stored energy is constant (i.e., independent of time).

(9) Use the analysis of question 8 to show that the Q-factor of a series RLC circuit at resonance can be expressed as

$$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

Using the general definition:

$$Q = 2\pi \left[\frac{\text{maximum stored energy}}{\text{energy dissipated per cycle}} \right]$$

(10) Yet another definition of Q-factor at resonance, based on 3dB ("half-power") bandwidth, is

$$Q_0 = \frac{\omega_0}{\omega_2 - \omega_1}$$

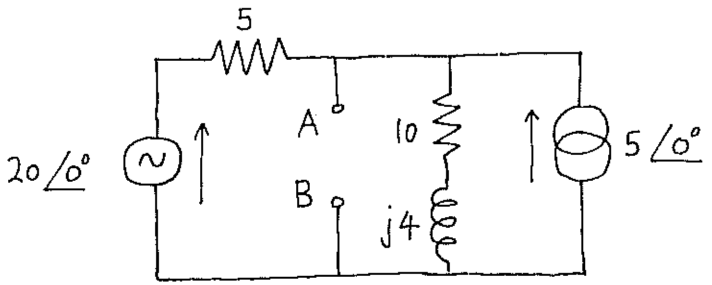
[see notes]. Verify this result for a series RLC circuit by applying the condition that ω_2 and ω_1 are the frequencies at which the current magnitude falls to $1/\sqrt{2}$ of its maximum value (at resonance).

(11) Show that if a series RLC circuit is given an electrical impulse and then allowed to oscillate at its natural frequency,

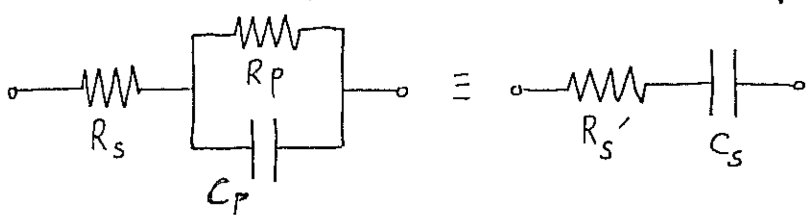
$$Q_0 \approx N\pi \quad \text{to within } 1\% \text{ if } Q_0 \geq 5$$

where N is the number of oscillation cycles for the current amplitude to fall to 1/e of its initial value.

(12) Find the Thévenin and Norton equivalent circuits at the terminals AB of the circuit below:



(13) Sometimes it is useful to convert a parallel circuit into an "equivalent series circuit" which has the same impedance at the same frequency:

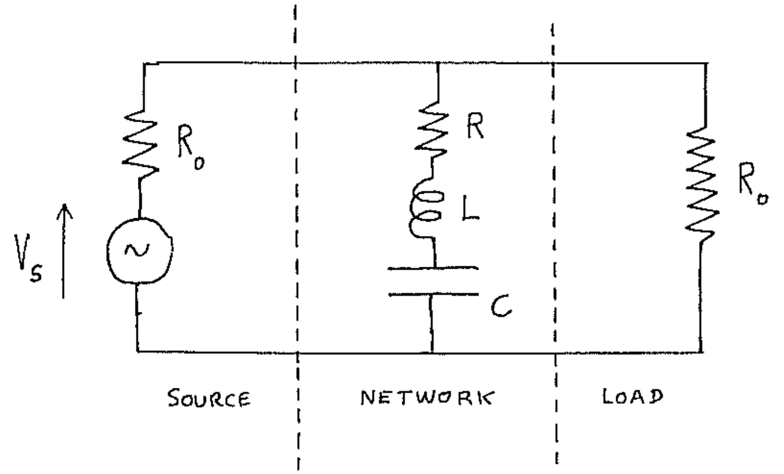


If the values of R_S, R_P and C_P are given (plus the frequency ω), find R_S' and C_S.

(14) The diagram shows a (genuine!) test circuit in which V_S has constant amplitude and variable frequency. All three components L, C and R have to be determined indirectly from measurement of the power absorbed by the load resistance R_0.

The transmission ratio T is defined by:

$$T = \frac{\text{Power in } R_0 \text{ when LCR absent}}{\text{Power in } R_0 \text{ when LCR present}}.$$

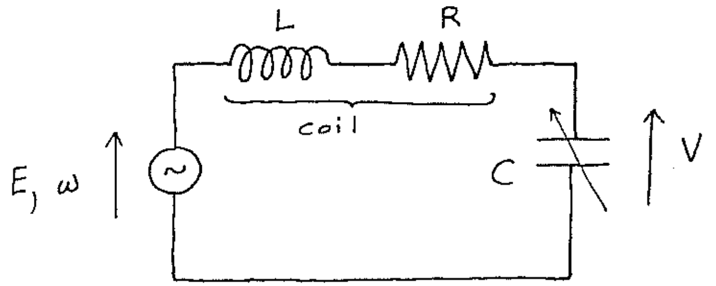


(a) Sketch the variation of T with frequency in the vicinity of the LCR resonance;

(b) Show that R can be found from R_0 and the maximum value of T ($= T_m$, say);

(c) If ω_1 and ω_2 are the (angular) frequencies at which T falls to half its maximum value, obtain an expression for the capacitance C in terms of R_0, T_m and these two frequencies.

(15) The circuit below is driven from a constant-amplitude, constant-frequency sinusoidal voltage E.



The capacitor setting for resonance is C_0. When the setting is increased by δC, it is found that V falls to $1/K$ times its value at resonance.

Show that the Q-factor of the coil can be determined from these observations and is given by:

$$Q = \frac{C_0}{\delta C}\sqrt{K^2 - 1}$$

provided that $\delta C \ll C_0$.

(16) In AC analysis, the ratio of two quantities varying sinusoidally at the same frequency can be expressed in the general form $a + jb$.

What can you deduce about a and b if you are told that the two quantities are:

(a) In phase;

(b) in quadrature;

(c) in antiphase;

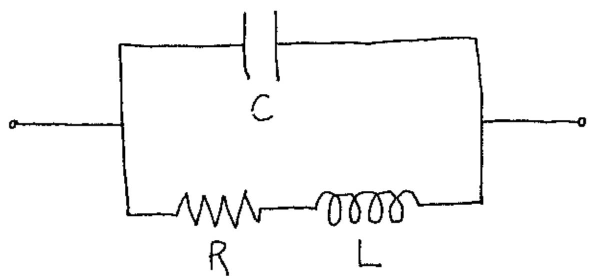
(d) 45° out of phase?

(17) The series/parallel circuit in the diagram is driven by a sinusoidal voltage source. Find

(a) The resonant frequency;

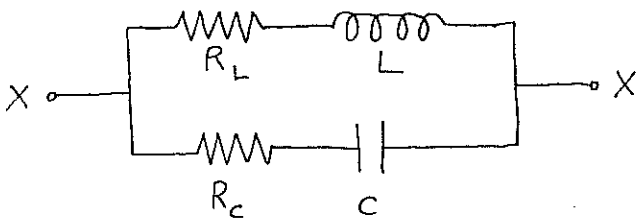
(b) The impedance at resonance.

Sketch the variation of the impedance magnitude from very low to very high frequencies.



(18) Show that if a series RLC circuit is driven at its resonant frequency, the capacitor voltage is Q times the supply voltage.

(19) The circuit below has been analysed before. Now that the concept of Q has been discussed, consider it again:

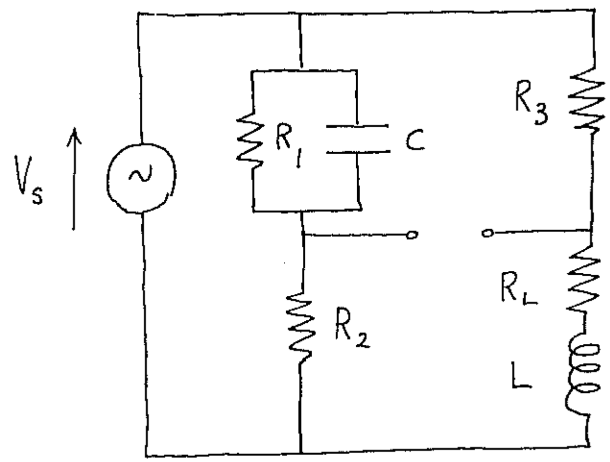


Discuss the following contrary arguments:

(a) When $R_L = R_c = \sqrt{L/C}$ the circuit behaves as a pure resistor at all frequencies. A pure resistor only dissipates energy; it does not store energy. Therefore the Q-factor of the circuit is zero according to the general definition.

(b) When a sinewave (or any other) source is connected to XX, finite currents flow in both arms and there must be stored energy. Therefore the Q-factor of the circuit is finite according to the general definition.

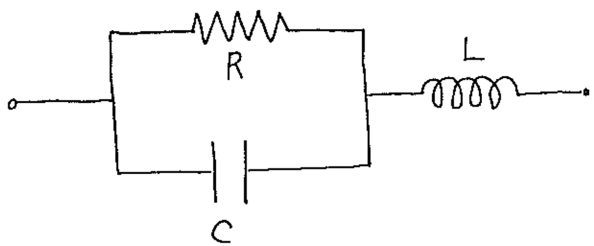
(20) The Maxwell bridge shown in the diagram is used to find the inductance L and series resistance R_L of a coil. Solve the balance equation



Show that the balance condition for L and R_L are independent of frequency.

In such a test, it was found that the result for R_L did increase with frequency. Discuss possible explanations for this.

(21) (a) Sketch the variation of impedance magnitude with frequency for the circuit below.

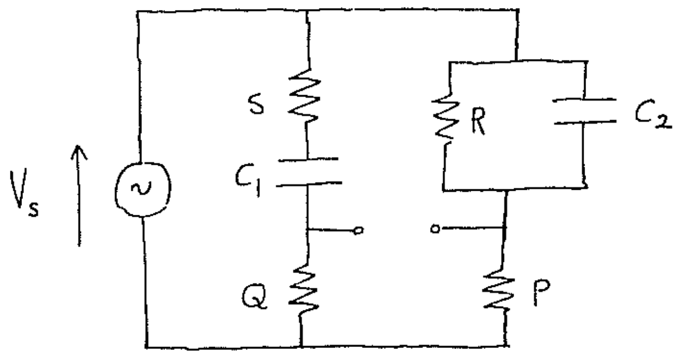


(b) Derive an expression for the resonance frequency and the impedance at resonance;

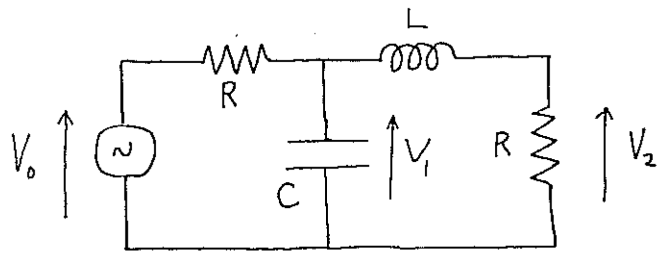
(c) Use your result to sketch a second curve showing the effect of a decrease in the resistance R.

(d) How is the sketch modified if the resistance increases indefinitely ?

(22) For the bridge circuit shown, derive expressions for C_1 and C_2 in terms of the bridge frequency ω and the values of the other components.



(23)



V_0 is a sinusoidal source of frequency ω.

Find the frequency at which V_2 and V_0 are in quadrature, and the magnitude of the ratio V_2/V_0 when the quadrature condition is satisfied.

You may find it helpful to use the intermediate voltage V_1 in the analysis.

(1) First, notice that the question contains mixed notation: the voltage V is given in polar form but the impedance of the inductor ($=j\omega L$) has been worked out for you in ohms and is given in rectangula co-ordinates.

It does not matter whether we work in polar or rectangular co-ordinates, but it is a good idea to convert all data to the same form.

The question does not say whether the voltage is the RMS, peak, or what. In a case such as that it is usual to assume that voltages and currents are RMS values. If you assume that V is RMS and divide by the impedance, the resulting current magnitude will also be RMS.

The impedance of the load is $8+j6\ \Omega$. The voltage is $7.07+j0$

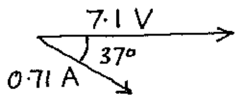
$$\therefore I = \frac{7.07+j0}{8+j6} = 0.71\angle{-37^\circ}$$

$$= 0.57-j0.42 \text{ amps}$$

Check: convert $8+j6$ to polar form:

$$I = \frac{7.07\angle 0^\circ}{10\angle 37^\circ} = 0.71\angle{-37^\circ} \quad \text{as before.}$$

Phasor diagram:



(2) Take I as $5\angle 0^\circ = 5+j0$ as a convenient reference.

$$Z = 5 + \frac{1}{j\omega C}$$

$$\omega C = 2\pi \times 318 \times 100 \times 10^{-6} = 0.2\ \Omega$$

$$\therefore Z = 5-j5\ \Omega$$

$$\therefore V = (5+j0)(5-j5) = 25-j25 \text{ volts}$$

$$= 35.4\angle{-45^\circ}$$

(3)

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

$$= 8 + j\left[7.5 - 22.5\right] = 8-j15$$

$$= 17.0\angle{-62^\circ}$$

So that the impedance is predominantly capacitive at this frequency.

(4) This question is mainly to make you pay close attention to units and exponents.

(a) $R = 10^3\ \Omega$ $L = 10\times10^{-3}\,H = 10^{-2}\,H$

$C = 0.1\times10^{-6}\,F = 10^{-7}\,F$

$f = 1\ \text{MHz} = 10^6\ \text{Hz}$

In series: $Z = R + j\left[\omega L - \frac{1}{\omega C}\right]$

$\omega L = 2\pi\times10^6\times10^{-2} = 6.28\times10^4\ \Omega$

$1/\omega C = \dfrac{1}{2\pi\times10^6\times10^{-7}} = 1.59\ \Omega$

$\therefore Z = 1000 + j\,6.28.10^4$ very nearly

(b) $Y = \frac{1}{R} + j\left[\omega C - \frac{1}{\omega L}\right]$

$\omega C = 6.28\times10^{-1}$ $1/\omega L = 1.59\times10^{-5}$

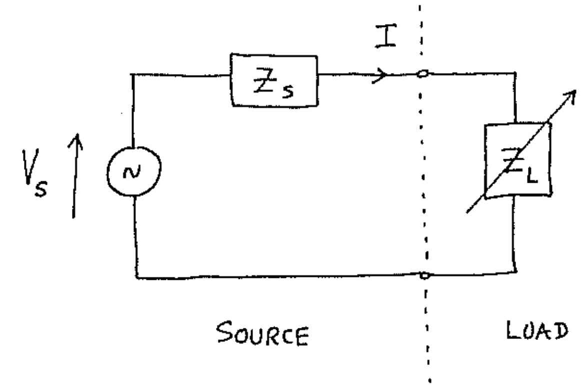
$Y = 10^{-3} + j\left(0.628 - 1.59\times10^{-5}\right)$

$\approx 10^{-3} + j\,0.628\ \ \Omega^{-1}$

$\therefore Z \approx 2.53\times10^{-3} - j1.59\ \ \Omega$

Most of the errors in exam answers arise because powers of 10 have become lost somewhere.

⑤



$Z_s = R_s + jX_s$

$Z_L = R_L + jX_L$

Z_s is constant, Z_L is variable.

Power is dissipated only in the resistive part of the load, R_L.

$$P_L = |I|^2 R_L$$

$$= \frac{|V_s|^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

First consider varying X_L, which is physically equivalent to "tuning" the load for maximum current (resonance).

For a given value of R_L the current will maximize when

$$X_s + X_L = 0 \quad \text{i.e. } X_L = -X_s$$

Next,

$$\frac{dP_L}{dR_L} = \frac{|V_s|^2 (R_s+R_L)^2 - 2R_L(R_s+R_L)|V_s|^2}{(R_s+R_L)^4} = 0$$

giving $R_L = R_S$.

This is the generalized Maximum Power Transfer theorem (a useful example of electro-speak) and in summary:

$$X_L = -X_S$$

$$R_L = R_S$$

and under this condition the maximum power is

$$P_L\,(max) = \frac{|V_S|^2}{4R_S}$$

⑥ I shall use admittances in this case. For the given circuit,

$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C + \frac{1}{j\omega C}} = G + jB$$

$$= \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C - \frac{1}{j\omega C}}{R_C^2 + \frac{1}{\omega^2 C^2}}$$

The condition for zero susceptance will be considered first. It then has to be shown that G is independent of frequency.

The zero-susceptance condition reduces to:

$$\omega^2 L \left[C R_C^2 - L \right] = R_L^2 - \frac{L}{C}$$

and the RHS is independent of frequency. We can make the LHS independent also by making

$$\left.\begin{aligned} R_C^2 &= \frac{L}{C} \\ \text{in which case} \quad R_L^2 &= \frac{L}{C} \end{aligned}\right\} \quad \text{i.e., } R_C = R_L = \sqrt{\frac{L}{C}} = R \text{ say.}$$

Now we must check whether G is really constant, i.e. independent of frequency.

$$G = \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + 1/\omega^2 C^2} \quad \text{and } R_L = R_C = R$$

$$= \frac{1}{R}\left[\frac{1}{1+x} + \frac{1}{1 + 1/x} \right] \quad \text{where } x = \omega^2 LC$$

and the bracketed expression comes to unity.

This confirms that if $R_C = R_L = \sqrt{\frac{L}{C}}$ then the admittance (and hence impedance) is a pure conductance (and hence resistance) which is constant i.e. independent of frequency.

(7) (a)
The capacitive reactance will be needed, so work it out first:

$$\frac{1}{\omega C} = \frac{1}{2\pi . 10^3 . 10^{-6}} = -j\,159\ \Omega$$

In series with 100 Ω this gives

$$Z = 100 - j\,159 \quad \text{or} \quad 188\ \angle -58^\circ \quad \text{ohms}$$

Now combine this with 150 Ω in parallel:

$$\frac{150\,[100 - j\,159]}{250 - j\,159} \quad \text{ohms}$$

$$= 86 - j\,41\ \Omega \quad \text{approx} \quad \text{or} \quad 95\ \angle -25.4^\circ$$

Finally, add the 10 Ω resistance in series to find the total impedance seen by the 30V source:

$$Z_{tot} = 96 - j\,41\ \Omega \quad \text{or} \quad 104\ \angle -23^\circ$$

(b) Now we can work out the current I_1

$$I_1 = \frac{30 + j\,0}{96 - j\,41}$$

$$= 0.27 + j\,0.112\ \text{A} \quad \text{or} \quad 0.29\ \angle 23^\circ$$

(c) The voltage V_1 can now be found:

$$V_1 = I_1\,[86 - j\,41]$$

$$= 27.4 - j\,1.25 \quad \text{or} \quad 27.4\ \angle -2.4^\circ$$

(d) The current I_2 is thus:

$$I_2 = \frac{V_1}{100 - j\,159}$$

$$= 0.083 + j\,0.120 \quad \text{or} \quad 0.146\ \angle 55^\circ$$

(e) The voltage V_2 is just $100\,I_2$:

$$V_2 = 8.3 + j\,12 \quad \text{or} \quad 14.6\ \angle 55^\circ$$

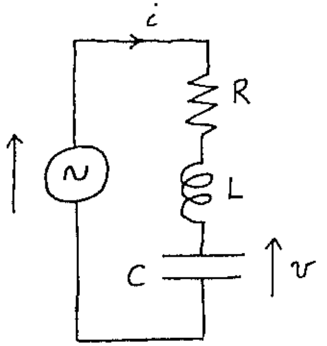
(8) The question refers to time-dependence, so we should use the fundamental equations discussed in the AC theory notes.

Using the results derived in question 5 of the "Transients" tutorial:

$$E_{tot} = \tfrac{1}{2} L i^2 + \tfrac{1}{2} C v^2$$

We have the interesting challenge of having to first express E_{tot} as a function of time, in order to show that it isn't — if you see what I mean. (Another nice piece of electro-speak for you).

The thing to remember is that the stored energy in the inductor and capacitor will not vary in phase with one another; we must get the relative timing right.



We are concerned with the current through the inductor and the voltage across the capacitor.

The current can be expressed as:

$$i = I_m \sin \omega t$$

and this same current flows through the capacitor, whose voltage v we can find from:

$$v = \frac{1}{C}\int i\,dt = -\frac{I_m}{\omega C}\cos \omega t$$

$$\therefore \quad E_{tot} = \frac{1}{2} L I_m^2 \sin^2 \omega t + \frac{1}{2}\frac{I_m^2}{\omega^2 C}\cos^2 \omega t$$

This looks complicated, and (in me at least) produces an almost irresistible urge to be out to lunch for as long as possible.

Fortunately, at resonance the result is not difficult to obtain:

since $\omega_o L = \dfrac{1}{\omega_o C}$

$\therefore \quad E_{tot} = \frac{1}{2} I_m^2 L$ which is independent of time.

(9) The maximum stored energy at resonance is, from the previous solution:

$$E_{sto} = \frac{1}{2} L I_m^2$$

The energy dissipated per cycle at resonance is:

$$E_{dis} = \int_0^T \left[I_m \sin \omega_o t \right]^2 R\,dt$$

where $T = \dfrac{2\pi}{\omega_o}$

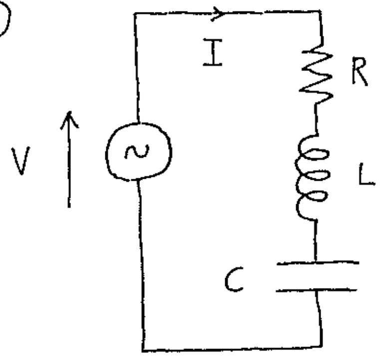
$$\therefore \quad E_{dis} = \frac{\pi R I_m^2}{\omega_o}$$

From these results we obtain

$$Q_0 = 2\pi \left[\frac{\frac{1}{2} \cdot L \cdot I_m^{\,2} \cdot \omega_0}{\pi R I_m^{\,2}} \right]$$

$$= \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

(10)



$$I = \frac{V}{R + j\left(\omega L - 1/\omega C\right)}$$

$$|I| = \frac{|V|}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}}$$

and $\omega = \omega_0$ when $\omega L = 1/\omega C$, at resonance, when $I = I_0 = \dfrac{|V|}{R}$

$$\therefore \left|\frac{I}{I_0}\right|^2 = \frac{R^2}{R^2 + \left(\omega L - 1/\omega C\right)^2} = \frac{1}{2}$$

at the frequencies ω_2 and ω_1.

Therefore the equation to be solved for ω is:

$$\omega L - \frac{1}{\omega C} = \pm R$$

and we can use $Q_0 = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 C R}$.

The two positive solutions for ω are then expressible as

$$\omega_2 = \omega_0 \left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} + \frac{1}{2Q_0} \right]$$

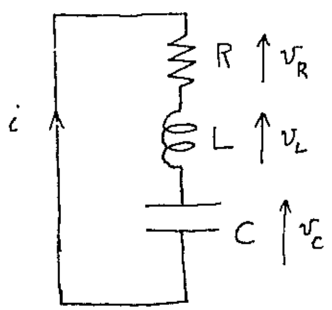
$$\omega_1 = \omega_0 \left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} - \frac{1}{2Q_0} \right]$$

$$\therefore \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$$

and $Q_0 = \dfrac{\omega_0}{\omega_2 - \omega_1}$ as required.

NOTE: the "half power" reference is to the power dissipated in R, which is halved when the current falls by a factor of $\sqrt{2}$ at frequencies ω_1 and ω_2.

(11) After the impulse has been applied, the circuit becomes:



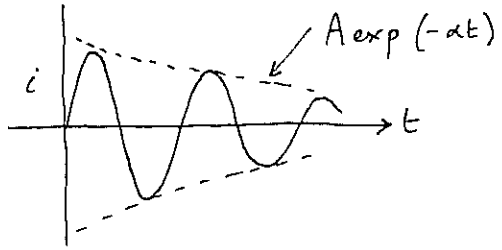
$$v_R + v_L + v_c = 0$$

$$\therefore\ iR + L\frac{di}{dt} + v_c = 0$$

$$\therefore\ L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

which is a standard second-order linear differential equation, the solution of which can be expressed in the form:

$$i = A\exp(-\alpha t).\ \sin(\beta t + B)$$



It is a decaying sinusoid, and the dotted line (the "envelope" of the waveform)

where A and B are constants, and

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{1}{LC} - \alpha^2}$$

Using $\omega_0^2 = \frac{1}{LC}$ and $Q_0 = \frac{\omega_0 L}{R}$ we can

now write:

$$\beta = \omega_0 \sqrt{1 - \left(\frac{1}{2Q_0}\right)^2}$$

and this helps to see the physical significance of the parameters α and β.

α is a measure of the rate of decay of the oscillations;

β is the (angular) frequency and is very close to ω_0 if $Q_0 \geqslant 5$. This is where we make the assumption that

$$\beta \approx \omega_0 \quad \text{if} \quad Q_0 \gg 1.$$

The envelope "decays" with a time-constant $\tau = 1/\alpha = 2L/R$ and this is the time we want since it is the time taken for the amplitude to decay by a factor of e.

Now we need to work out how many cycles this time τ represents.

At a frequency ω_0 the time T for a cycle is:

$$T = \frac{2\pi}{\omega_0} \text{ seconds}$$

assuming $\beta = \omega_0$

(contd)

Therefore N, which is τ / T can now be expressed as :

$$N = \frac{2L}{R} \cdot \frac{\omega_0}{2\pi} = \frac{Q_0}{\pi}$$

$$\therefore \; Q_0 = N\pi$$

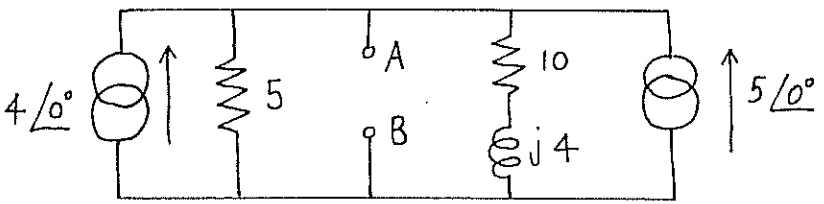
subject to the assumption $\beta = \omega_0$.

12 Since the circuit contains both voltage and current generators, we could begin by converting both to (say) current generators. It is then straightforward to interchange the two equivalent circuits by remembering that

$$\left.\begin{aligned} Z_N &= Z_T = Z \\ \text{and} \quad V_T &= I_N Z \end{aligned}\right\} \text{generalized from the DC}$$

To the left of AB, the Norton circuit is still 5Ω and it is in parallel with a current generator I_N where

$$I_N = \frac{20 \angle 0^\circ}{5} = 4 \angle 0^\circ$$



For the Norton circuit,

$$I_N = 4 \angle 0^\circ + 5 \angle 0^\circ = 9 \angle 0^\circ \text{ amps}$$

$$\text{and} \quad Z_N = \frac{5\,(10 + j4)}{5 + (10 + j4)} = 3{\cdot}47 \angle 6{\cdot}9^\circ \text{ ohms}$$

For the Thévenin circuit,

$$Z_T = 3{\cdot}47 \angle 6{\cdot}9^\circ \text{ as before (that was easy)}$$

$$V_T = I_N Z = 9 \angle 0^\circ \times 3{\cdot}47 \angle 6{\cdot}9^\circ$$

$$= 31{\cdot}2 \angle 6{\cdot}9^\circ$$

NOTE : The complex quantities could just as well have been expressed in rectangular co-ordinate form. Since the question did not ask for one in particular, it is up to you.

13 The approach is to write down the impedances of the two circuits and equate real & imaginary parts separately.

For the LH circuit :

$$Z_{LH} = R_s + \frac{R_p \left(\frac{1}{j\omega C_p}\right)}{R_p + \left(\frac{1}{j\omega C_p}\right)}$$

(contd)

Collecting terms:

$$Z_{LH} = \left[R_s + \frac{R_p}{1+(\omega C_p R_p)^2} \right] + j \left[\frac{-\omega C_p R_p^2}{1+(\omega C_p R_p)^2} \right]$$

Compare this with the RH circuit impedance:

$$Z_{RH} = R_s' + j \left[-\frac{1}{\omega C_s} \right]$$

Equate real parts:

$$R_s' = R_s + \frac{R_p}{1+(\omega C_p R_p)^2}$$

Equate imaginary parts:

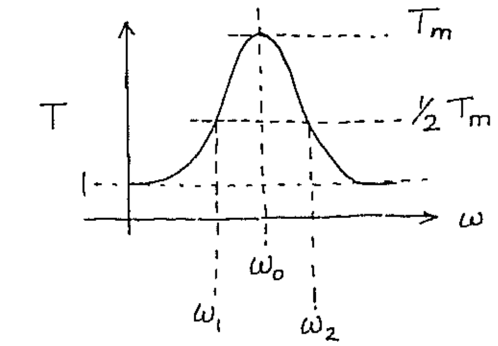
$$\frac{1}{\omega C_s} = \frac{\omega^2 C_p R_p^2}{1+(\omega C_p R_p)^2}$$

$$\therefore \quad C_s = C_p \left[\frac{1+(\omega C_p R_p)^2}{(\omega C_p R_p)^2} \right]$$

NOTE: It is often helpful to put the results in the form of the "original" component followed by (or multiplied by) a conversion factor.

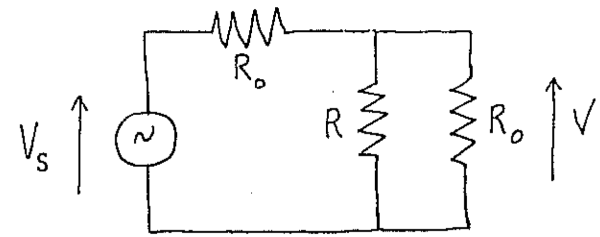
(14) (a) In the absence of the LCR, the source is connected directly to a matched load, so the power P_o in R_o will be at its maximum. This means that T will always be > 1.

Further, the shunting effect of LCR means that T passes through a peak at the series resonant frequency. Remote from resonance, the LCR will have a high parallel (shunt) impedance, so $T \to 1$ as $\omega \to 0$ or ∞.



NOTE: T is a power ratio, not voltage or current as with all previous questions.

(b) At LCR series resonance the circuit simplifies to:



With R absent:

$$P_o = \frac{|V_s|^2}{4 R_o}$$

and when R is present:

$$P_o = \frac{|V|^2}{R_o}$$ where V has to be found.

From the diagram:

$$V = \frac{V_s}{2\left(1+\frac{R_o}{2R}\right)}$$

So, when R is present, the power P_o in R_o is given by:

$$P_o = \frac{|V_s|^2}{4R_o\left(1+\frac{R_o}{2R}\right)^2}$$

from which T_m, the maximum value of T,

$$T_m = \left(1+\frac{R_o}{2R}\right)^2$$

so that, for a measurement at resonance,

$$R = \frac{R_o}{2\left(\sqrt{T_m}-1\right)}$$

(c) For the more general off-resonance condition we replace R by Z, where

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

(contd)

and the expression for T becomes:

$$T = \left|1+\frac{R_o}{2Z}\right|^2$$

we require the two frequencies ω_1 and ω_2 for which T becomes $\frac{1}{2}T_m$.

As with the analysis for question 10 on AC circuits, it leads to a quadratic in ω and eventually to:

$$C = \left(\frac{\omega_2-\omega_1}{\omega_2\omega_1}\right)\cdot\frac{2\left(\sqrt{T_m}-1\right)\sqrt{1-2/T_m}}{R_o}$$

[and, incidentally,

$$L = \frac{1}{\omega_2\,\omega_1\,C} \quad \text{if needed.}]$$

(15)

$$V = E\left[\frac{\frac{1}{j\omega C}}{R+j\left(\omega L - 1/\omega C\right)}\right]$$

At resonance, $C = C_o$ and $V = V_o$

$$\therefore\ V_o = \frac{-jE}{\omega C R} \quad \text{and} \quad \frac{V_o}{V} = \frac{C}{C_o R}\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]$$

When $C = C_0 + \delta C$ we have, putting $Q = \frac{\omega L}{R}$,

$$\frac{V_0}{V} = \frac{C_0 + \delta C}{C_0}\left[1 + jQ\left(1 - \frac{1}{1 + \delta C/C_0}\right)\right] = K$$

Now putting $x = \delta C/C_0$ we can write

$$K = (1+x)\left[1 + j\frac{Qx}{1+x}\right]$$

The voltage refers to magnitude only, so:

$$|K|^2 = (1+x)^2 + Q^2x^2$$

or
$$Q = \frac{1}{x}\sqrt{|K|^2 - (1+x)^2}$$

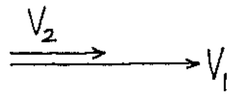
$$= \frac{C_0}{\delta C}\sqrt{K^2 - \left(1 + \frac{\delta C}{C_0}\right)^2}$$

and this contains no approximations. If we now take $\delta C \ll C_0$ it reduces to:

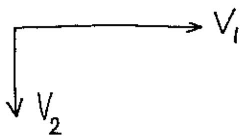
$$Q = \frac{C_0}{\delta C}\sqrt{K^2 - 1}$$

(16) Simplest approach is based on

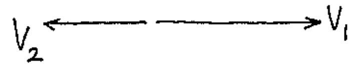
$$\frac{V_2}{V_1} = a + jb$$

(a)


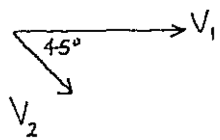
a is real and (+)ve
$b = 0$

(b)


$a = 0$
b can be (+) or (−)

(c)


a is (−)ve.
$b = 0$

(d)


$|a| = |b|$

(17)

(a) The algebra is slightly reduced if we work with admittances:

$$Y = j\omega C + \frac{1}{R + j\omega L}$$

$$= \frac{R}{R^2 + \omega^2 L^2} + j\left[\omega C - \frac{\omega L}{R^2+\omega^2 L^2}\right]$$

At resonance the susceptance is zero:

$$\omega_0 C = \frac{\omega_0 L}{R^2 + \omega_0^2 L^2}$$

$$\therefore \quad \omega_0^2 = \frac{1}{LC} - \left(\frac{R}{L}\right)^2$$

CHECK: when $R \to 0$, $\omega_0 \to \frac{1}{\sqrt{LC}}$ as would be expected.

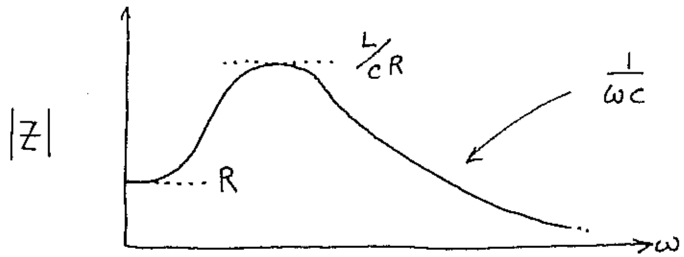
(b) At resonance

$$Y_0 = \frac{R}{\left(\frac{L}{C}\right)} \quad \therefore \quad Z_0 = \frac{L}{CR}$$

(c) To sketch $|Z|$ as a function of ω, note that

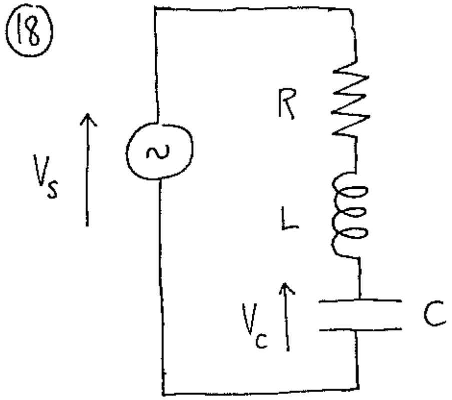
$|Z| \to R$ as $\omega \to 0$

$|Z| \to 1/\omega C$ as $\omega \to \infty$



NOTE: for this circuit the resonant frequency is <u>not</u> the same as the frequency of maximum impedance.

(18)



$$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$\frac{V_c}{V_s} = \frac{\left(\frac{1}{j\omega C}\right)}{R + j\omega L + \frac{1}{j\omega C}}$$

$$\therefore \quad \left(\frac{V_c}{V_s}\right)^2 = \frac{\frac{1}{\omega^2 C^2}}{R^2 + \left(\omega L - 1/\omega C\right)^2}$$

At resonance:

$$\frac{V_c}{V_s} = \frac{1}{\omega_0 C R} = Q_0$$

19. Discuss this with your tutors. Naturally I have my own ideas too...

20.
$$\frac{R_1\left(\frac{1}{j\omega C}\right)}{\left(R_1+\frac{1}{j\omega C}\right)R_2} = \frac{R_3}{R_L + j\omega L}$$

leading to:

$$R_2 R_3 \left(1 + j\omega C R_1\right) = R_1 \left(R_L + j\omega L\right)$$

Equate imaginary parts:

$$\omega C R_1 R_2 R_3 = \omega L R_1$$

$$\therefore L = C R_2 R_3$$

Equate real parts:

$$R_2 R_3 = R_1 R_L$$

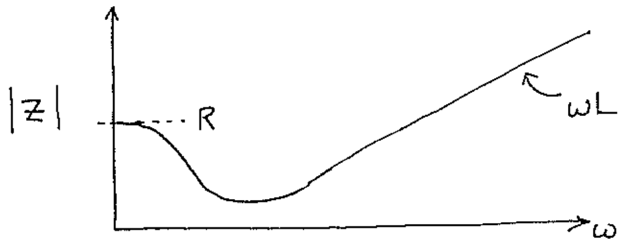
$$\therefore R_L = \frac{R_2 R_3}{R_1}$$

independent of frequency

If R_L increases with drive frequency, then assuming that the other resistors are reliable, it is probably due to the phenomenon known in the trade as "the skin effect". It is the tendency of current (i.e., AC current) to concentrate towards the surface of a conductor as the frequency is increased. This reduces the effective cross-sectional area of the wire and hence the resistance.

21. (a) As $\omega \to 0$ $\quad |Z| \to R$

As $\omega \to \infty$ $\quad |Z| \to \omega L$



(b)
$$Z = j\omega L + \frac{R\left(\frac{1}{j\omega C}\right)}{R + \left(1/j\omega C\right)}$$

$$= \frac{R}{1+\omega^2 C^2 R^2} + j\left[\omega L - \frac{\omega C R^2}{1+\omega^2 C^2 R^2}\right]$$

and for resonance, the reactance is zero.

At resonance $\omega = \omega_0$ and

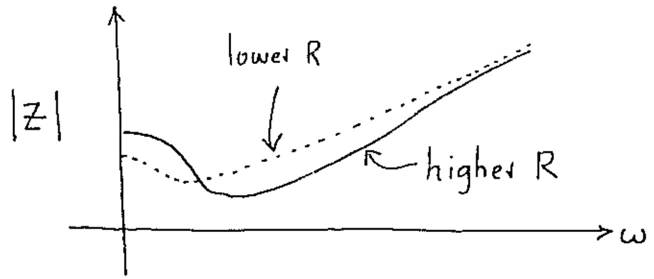
$$\omega_0^2 = \frac{1}{LC} - \left(\frac{1}{CR}\right)^2$$

CHECK: as $R \to \infty$ we approach $\omega_0 = \frac{1}{\sqrt{LC}}$ as expected.

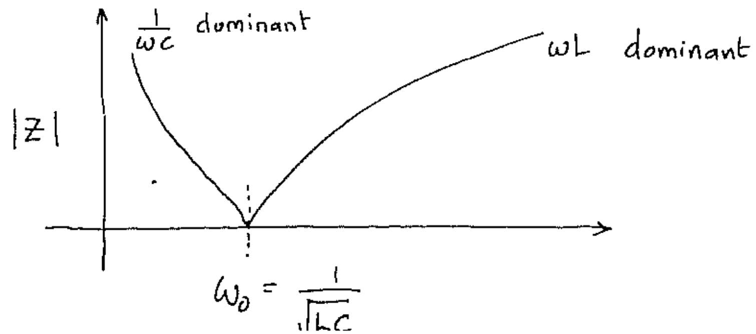
The impedance at resonance is thus:

$$Z_0 = \frac{L}{CR}$$

(c)



(d)



(22) The balance equation is:

$$\frac{S + \frac{1}{j\omega C_1}}{Q} = \frac{\left[\frac{R\left(\frac{1}{j\omega C_2}\right)}{R + 1/j\omega C_2}\right]}{P}$$

and leads to:

$$\left[S + \frac{C_2 R}{C_1}\right] + j\left[\omega C_2 R S - \frac{1}{\omega C_1}\right] = \frac{QR}{P}$$

Equate imaginary parts:

$$\omega C_2 R S - \frac{1}{\omega C_1} = 0$$

$$\therefore \quad C_1 C_2 = \frac{1}{\omega^2 R S}$$

Equate real parts:

$$S + \frac{C_2 R}{C_1} = \frac{QR}{P}$$

$$\therefore \quad \frac{C_2}{C_1} = \frac{Q}{P} - \frac{S}{R}$$

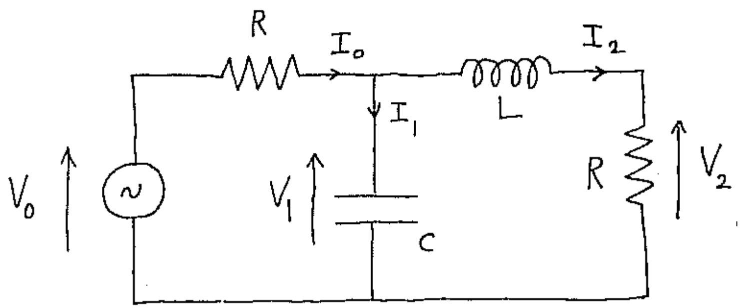
and we can now isolate the two capacitances ...

Thus:

$$C_2^2 = \frac{1}{\omega^2 R S}\left[\frac{Q}{P} - \frac{S}{R}\right]$$

$$C_1^2 = \frac{1}{\omega^2 R S\left[\frac{Q}{P} - \frac{S}{R}\right]}$$

23



At the top junction: $I_0 = I_1 + I_2$

and we can substitute in turn for each of these three currents:

$$\frac{V_0 - V_1}{R} = j\omega C V_1 + \frac{V_2}{R} \quad \ldots\ldots\ldots \quad (1)$$

also: $$V_1 = V_2 + j\frac{\omega L V_2}{R} \quad \ldots\ldots\ldots\ldots \quad (2)$$

From equation (1):

$$V_0 - V_1 = V_2 + j\omega C R V_1$$

$$\therefore \; V_0 - V_2 = V_1\left(1 + j\omega CR\right) \ldots\ldots\ldots\ldots \quad (3)$$

From equation (2):

$$V_1 = V_2\left(1 + j\frac{\omega L}{R}\right) \ldots\ldots\ldots\ldots \quad (4)$$

Substitute (4) into (3):

$$V_0 - V_2 = V_2\left(1 + j\frac{\omega L}{R}\right)\left(1 + j\omega CR\right)$$

$$\therefore \; \frac{V_0}{V_2} - 1 = 1 - \omega^2 LC + j\omega CR + j\frac{\omega L}{R}$$

$$\therefore \; \frac{V_0}{V_2} = \left(2 - \omega^2 LC\right) + j\left(\omega CR + \frac{\omega L}{R}\right)$$

$$= a + jb.$$

For V_0 and V_2 to be in quadrature, $a = 0$

$\therefore \; \omega^2 LC = 2$ is the condition.

The final result for the ratio is then:

$$\left|\frac{V_0}{V_2}\right| = \omega CR + \frac{\omega L}{R}$$

Introducing "transient" signals

So far, we have considered only voltages and currents that remain constant, i.e. they don't vary with time. This has enabled us to concentrate on the key concepts and rules for dealing with circuits.

Time-varying signals, for the purpose of this introductory course, divide into two groups:

- signals (waveforms) that eventually die away or are suddenly switched on or off;
- periodic waveforms, especially sinusoids, that are "continuous" in the sense that we analyse them without reference to when they are switched on or off.

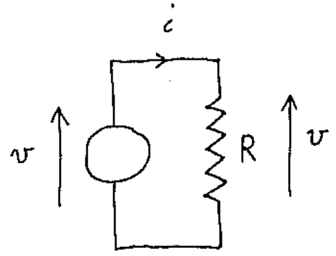
The first group I shall call "transients" and the second is known as AC analysis.

Transient response of resistance, R

First, a change of notation to remind us that the currents and voltages we are dealing with are, in general, time-varying.

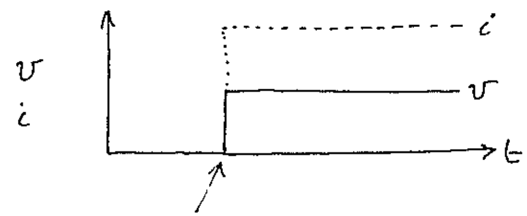
Ohm's law is now $v = i R$ .... (1)

where v, i are "instantaneous" values.



RESISTOR

We shall make v a voltage that is a step function in time:

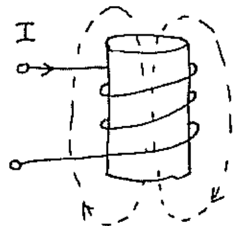


Voltage switched on

There is nothing in equation (1) to suggest that the current should do anything other than follow the voltage – hence the graph.

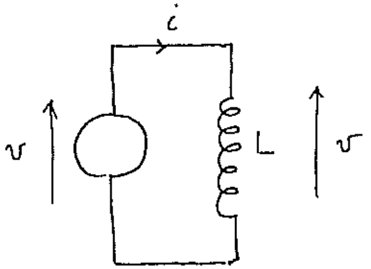
Transient response of inductance, L

First, what is an inductor? Physically, it could be a coil of wire; often, the wire is coiled around magnetic material such as iron:



The current (in this case, a DC current I) produces a magnetic flux and a magnetic field which "links" with the turns of the coil and stores energy in the field.

The fundamental laws of Faraday and Lenz are embodied in the next equation, from which we can deduce everything you need to know about inductors:



$$v = L \frac{di}{dt} \quad \dots\dots \quad \dots. \quad (2)$$

L = inductance in Henrys

How does this new component respond to a step change in voltage?

Rearrange equation (2) slightly to give

$$\frac{v}{L} = \frac{di}{dt} \quad \dots. \quad \dots.. \quad (3)$$

and we have the clue. Because we shall always be dealing with finite v and L in practical circuits, it follows that the LHS of eqn (3) must be finite.

$\therefore$ RHS must also be finite

$\therefore$ $\frac{di}{dt}$ must always be finite

$\therefore$ current in an inductor <u>cannot</u> <u>change</u> <u>in</u> <u>zero</u> <u>time</u>

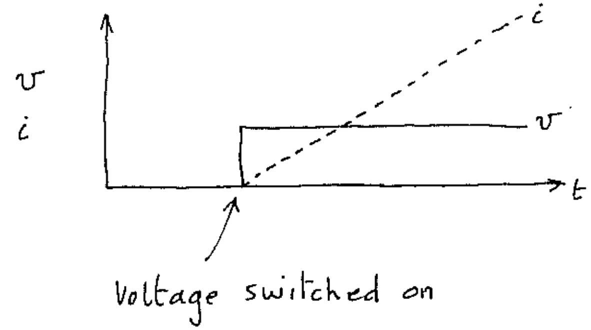
This leads to the essential rule that we need when analysing the behaviour of an inductor when a sudden change occurs in the circuit:

– the current just after a sudden change in the circuit is the same as the current just before the change, for an inductor.

We shall need this boundary condition later.

Suppose we apply a step function of voltage to an inductor:

INDUCTOR



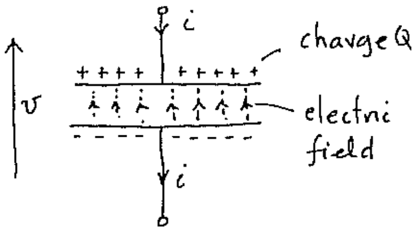
The boundary condition requires that the inductor current must start from zero (its value just before the sudden change); the fundamental equation (3) requires that di/dt is constant.

It looks as though the current would rise indefinitely and in principle that is correct. In practice, the source of voltage will reach its limit, of course.

Make sure you understand this section.

Transient response of a capacitor

First, what is a capacitor? Physically, it can be a pair of parallel metal plates separated by an insulating dielectric such as air. Charges can accumulate on the plates, and energy is stored in the electric field between them.



For a capacitor,

$$Q = vC \quad \cdots\cdots \quad \cdots\cdots \quad (4)$$

where Q = charge in coulombs

C = capacitance in Farads

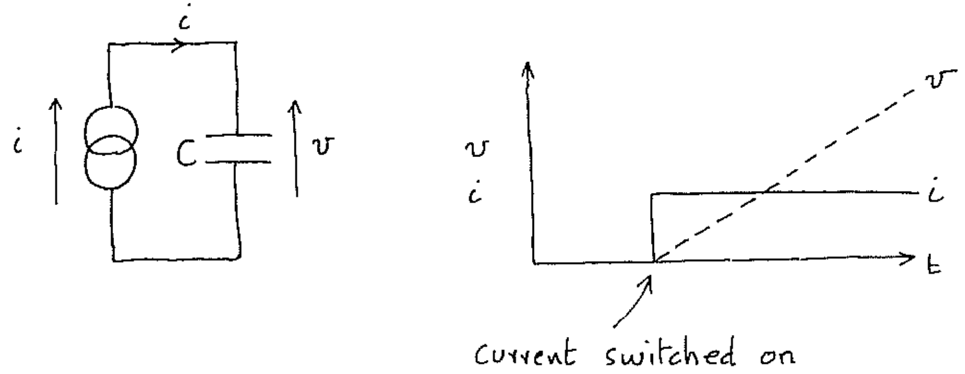
Although the medium between the plates is insulating, when the voltage is time-varying there is current flow. This seeming paradox was resolved by Maxwell, who introduced the concept of displacement current.

The ideas of displacement current and the continuity of current allow us to write:

$$i = \frac{dQ}{dt} = C\frac{dv}{dt} \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad (5)$$

Notice the symmetry between this and equation (2) for the inductor.

To see the fundamental equation governing the transient response of the capacitor:



I have used a step function of current rather than voltage; the reason for this should be clear soon.

Following the same argument as for the inductor, we can re-write eqn (5) slightly as:

$$\frac{i}{C} = \frac{dv}{dt} \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad (6)$$

Since in practical systems i and C must always be finite, it follows that $\frac{dv}{dt}$ must also be finite.

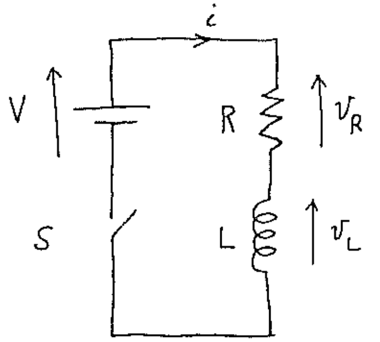
$\therefore$ the voltage across a capacitor <u>cannot change in zero time</u>. This gives the rule:

- the voltage on a capacitor is the same just before and just after a sudden change in the circuit.

For a constant current, the voltage rate of rise dv/dt is constant also in this idealized model.

R and L in series with a battery and switch

We shall now look at combinations of R, L and C which introduce a delay in the response of a circuit to a step function (e.g. a sudden change in the circuit when a switch opens or closes).



Note: upper case V for the battery since it is a constant voltage.

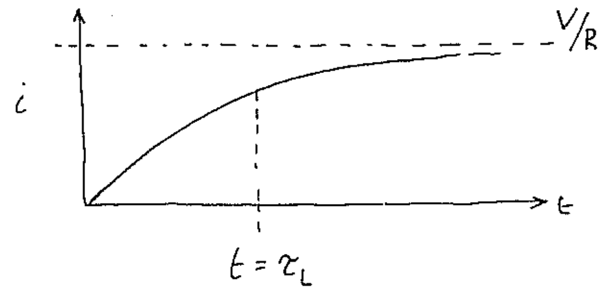
Switch is closed at $t = 0$ and we shall find how the current i varies with time.

When the switch is closed:

$$\left.\begin{aligned} & V - v_R - v_L = 0 \\ \therefore\ & V = iR + L\frac{di}{dt} \quad \text{and } i(t=0) = 0 \\ \therefore\ & i = \frac{V}{R}\left[1 - \exp\left(-t/\tau_L\right)\right] \\ & \text{where } \tau_L = L/R \end{aligned}\right\} \quad \ldots\ldots\ \ldots\ldots \quad (7)$$

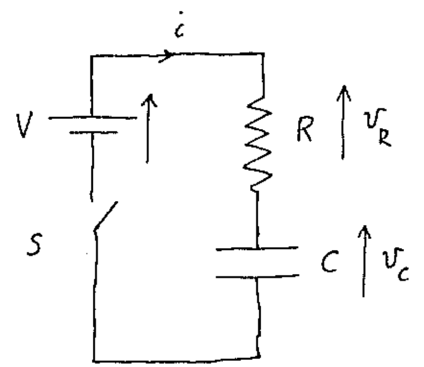
and τ_L is the "time-constant" of the circuit, and gives

a feel for the speed of its response when S is closed



When $t = \tau_L$, the current i has risen to $1 - 1/e \approx 63\%$ of its final value.

R and C in series with a battery and a switch



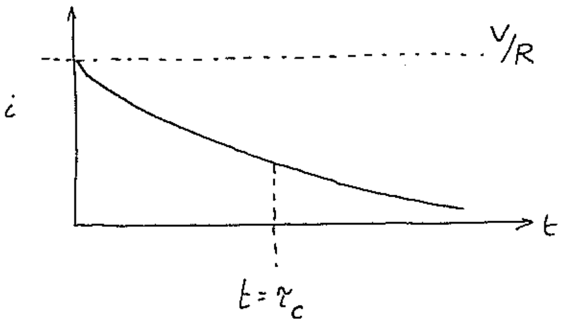
When the switch is closed:

$$\left.\begin{aligned} & V - v_R - v_C = 0 \\ \therefore\ & V = iR + v_C \\ \therefore\ & \frac{dV}{dt} = 0 = R\frac{di}{dt} + \frac{d}{dt}v_C \end{aligned}\right\} \ldots (8)$$

$$\therefore\ i = \frac{V}{R}\exp\left(-t/\tau_C\right) \qquad \text{since } v_C(t=0) = 0$$

where $\tau_C = CR$, the time-constant.

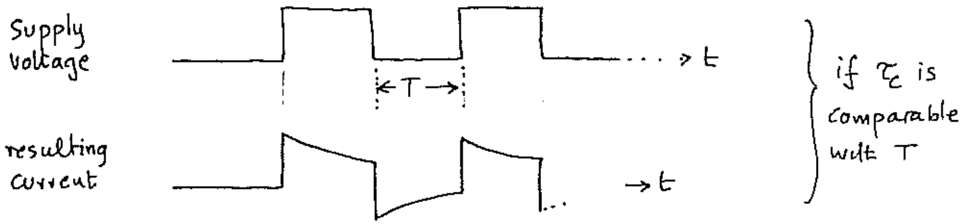
NOTE how the boundary condition has been applied: at $t=0$, $v_C = 0$ (capacitor initially uncharged), hence all voltage across R at $t=0$, hence current through R is V/R at $t=0$.



When $t = \tau_c$ the current has fallen to $\frac{1}{e} \approx 37\%$ of its peak value.

The importance of LR and CR time-constants

In high-speed digital circuits, pulsed waveforms are the carriers of information. Their shapes can be degraded by the presence of unwanted L or C, and the type of analysis that we are applying here can give an indication of how serious this is likely to be. If these "parasitic" effects are not considered, data corruption can result.



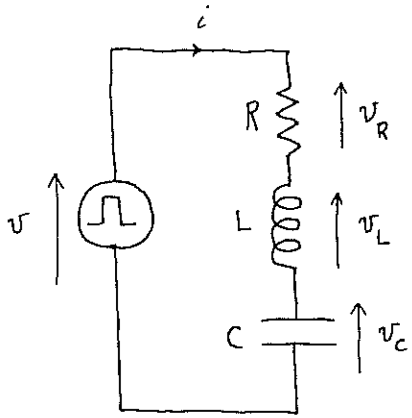
If the resulting waveforms decay too quickly, later voltage waveforms could fall below a critical threshold

and so cause an error, e.g. by a logical "1" being mis-read as a logical "0".

Transient response of R, L, C to an electrical impulse

We are now moving towards one of the central concepts in communications and signal analysis, namely the duality between the time and frequency domains. This topic is treated under headings such as Fourier Series and Fourier transforms.

In looking at the way a series R, L, C circuit responds to an electrical impulse, we can introduce a concept which is usefully considered in both the time and frequency domains – the concept of Q.



I have set up this circuit so that we can concentrate on the oscillatory behaviour. The voltage source delivers a pulse to the system and then returns to zero. This in effect allows the capacitor to acquire a charge and then replaces the voltage source by a short-circuit. Under those conditions:

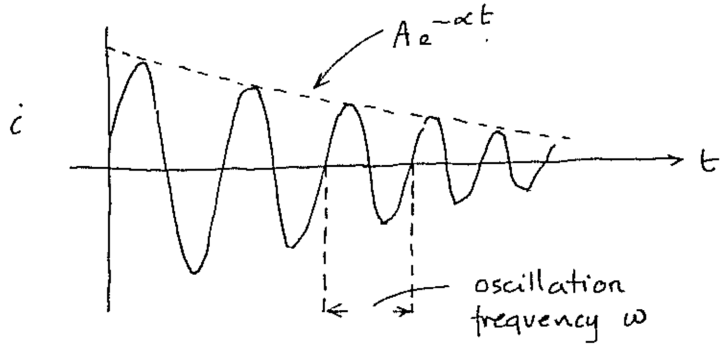
$$v_R + v_L + v_C = 0 \qquad \ldots\ldots \quad (9)$$

Substituting equations (1), (2) and (5) gives:

$$\left.\begin{aligned} &iR + L\frac{di}{dt} + V_c = 0 \\ &L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \\ &\text{from which} \\ &i = A\exp(-\alpha t)\sin(\omega t + \theta) \end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

where A and θ are constants depending on the initial conditions, and

$$\alpha = \frac{R}{2L}, \qquad \omega = \sqrt{\frac{1}{LC} - \alpha^2} \qquad \ldots\ldots \quad (11)$$



Now we can give clear physical interpretations to the terms α and ω in equation (10).

The constant α measures the rate of decay of the envelope. It depends upon R and L and is the counterpart of the time-constant τ that we met before.

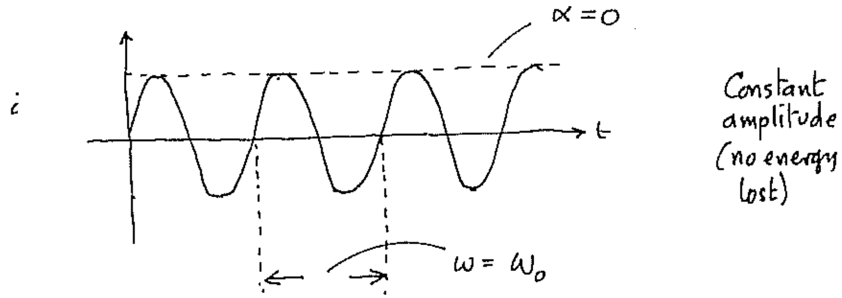
The waveform shows oscillations at frequency ω;

the "frequency" being a term that will be defined properly when we move to AC circuits.

An important special case is the circuit where R=0.

In that case $\alpha = 0$ and $\omega = \omega_0 = \sqrt{\frac{1}{LC}}$ .... (13)

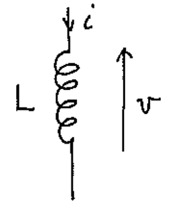
and the oscillations never decay to zero. The interpretation of this is that it is the resistor R which dissipates energy. If it is reduced to zero then the energy is exchanged periodically between the magnetic field of the inductor and the electric field of the capacitor at $\omega = \omega_0$, called the <u>resonance</u> frequency.



The less energy dissipated, the higher the "Q" factor of the circuit. This idea goes back to the early days of radio when the quality of a tuning coil was judged by how "sharply" it could resonate with the capacitor to discriminate against interference from unwanted broadcasting stations.

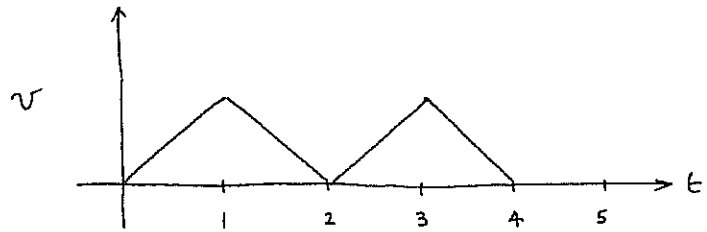
In fact $$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} \qquad \ldots\ldots \quad \ldots\ldots \quad (14)$$

① Lenz's law, applied to an inductor, says that the induced voltage caused by a change in current will oppose the change. Show that the positive sign convention and equation below are consistent with the law:

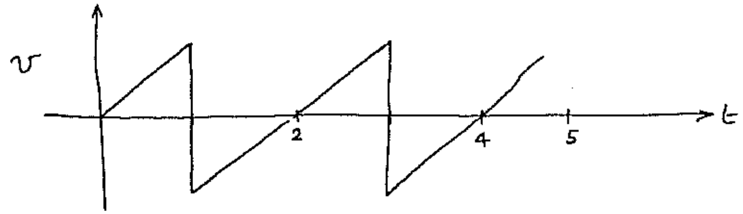


$$v = L\frac{di}{dt}$$

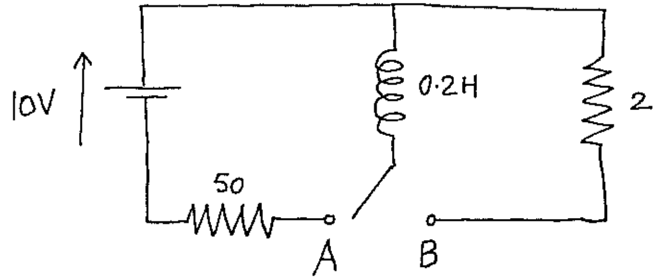
2) The sketch below shows the voltage applied to a capacitor. Using the same time axis, sketch the resulting current waveform.



3) From the voltage waveform applied to an inductor sketched below, draw the current waveform.



④



(a) The switch has been in position A for a long time. What is the current in the inductor?

(b) The position of the switch is suddenly changed to B. Find the current as a function of time through the 2 Ω resistor.

⑤ Show that

(a) An inductor carrying a current i has an energy

$$E_L = \frac{1}{2} L i^2 \text{ joules}$$

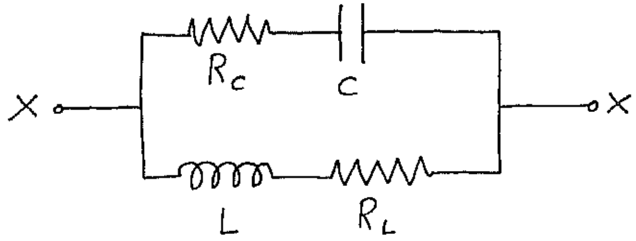
stored in its magnetic field

(b) A capacitor with a voltage v has an energy

$$E_c = \frac{1}{2} C v^2 \text{ joules}$$

stored in its electric field.

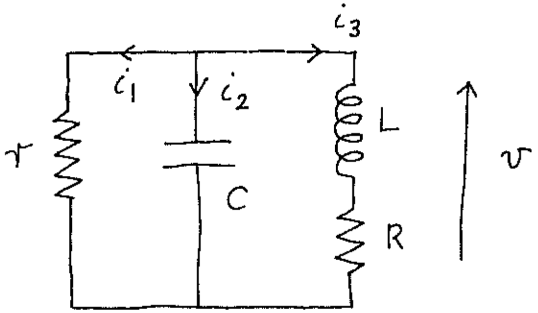
⑥



Find the relationship between the components in the above circuit such that the equivalent circuit at XX is a constant resistance, R.

[Hint: consider each arm separately, and how the current in each will vary with time].

⑦ The circuit below is given an electrical impulse and then allowed to oscillate at its natural frequency. Derive the oscillation frequency in terms of the component values and check your result by considering limiting values of r and R.



Hint: set up the differential equation for the voltage, v.

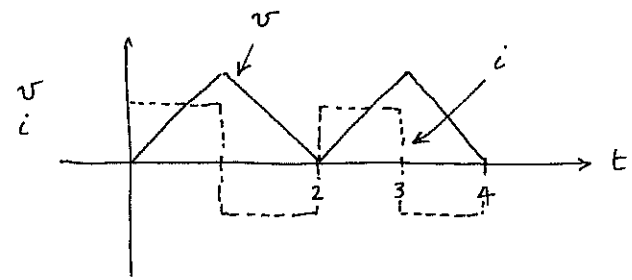
① Suppose that the current i, supplied from an external source such as a battery in series with a resistor, is increasing.

Then $\frac{di}{dt}$ is ⊕ve making v positive also,

which means that v is in the direction shown. The voltage across the inductor is therefore opposing the cause of the increasing current by attempting to counteract the battery.

NOTE: the diagram and the equation, taken together, define the way in which current and voltage are incorporated into circuit equations for an inductor. There is NO NEGATIVE SIGN IN THE EQUATION.

② For a capacitor, $i = C\frac{dv}{dt}$, so:



③ For an inductor, $v = L\frac{di}{dt}$ $\therefore\ i = \frac{1}{L}\int v\,dt$

between appropriate limits.

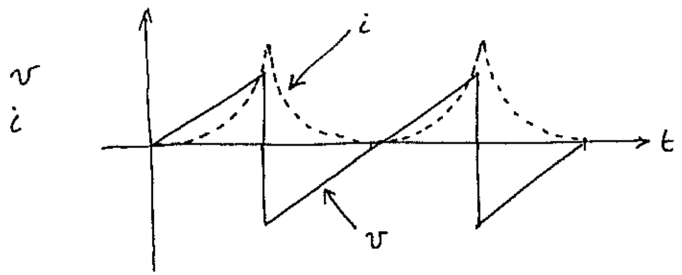
In the given waveform,

$$v = \text{const.} \times t$$

$$\therefore \quad i = \text{const.} \times t^2$$

and $\int v \, dt = 0$ over one cycle, so that current $= 0$ at $t = 2, 4 \ldots$

This enables us to draw the current waveform:



(4) (a) A "long time" means that a steady current is established

$$\therefore \frac{di}{dt} = 0 \quad \text{giving } v_L = 0$$

$$\therefore \quad i = \frac{10}{50} = 0{\cdot}2A$$

(b) The current in the inductor cannot change instantaneously, so it is still 0·2A when the switch is connected to B.

For that loop, therefore,

$$v_L + v_R = 0 \quad \text{i.e.} \quad L\frac{di}{dt} + iR = 0$$

$$\therefore \quad i = I_0 \exp\left(-t/\tau\right)$$

with $I_0 = 0{\cdot}2A$ and $\tau = \frac{L}{R} = \frac{0.2}{2} = 0{\cdot}1$ second.

$$\therefore \quad i = 0{\cdot}2 \exp\left(-10t\right)$$

(5) (a) Imagine that the inductor is supplied with a gradually increasing current; the instantaneous rate at which energy is being supplied is

$$p = vi \quad \text{joules/second} \quad (\text{watts})$$

$$= Li\frac{di}{dt}$$

$$\text{Total energy supplied} = \int_{-\infty}^{t} vi \, dt = \int_{0}^{i} Li \, di$$

$$E_L = \tfrac{1}{2}Li^2$$

So that the total energy stored by an inductor is a function of the instantaneous current and is independent of the voltage.

(b) By a similar argument, suppose that a capacitor is gradually charged from initially zero voltage:

$$p = vi \quad \text{joules/second}$$

$$= Cv\frac{dv}{dt}$$

$$\text{Total energy supplied} = \int_{-\infty}^{t} vi\,dt = \int_{0}^{v} Cv\,dv$$

$$E_C = \frac{1}{2}Cv^2$$

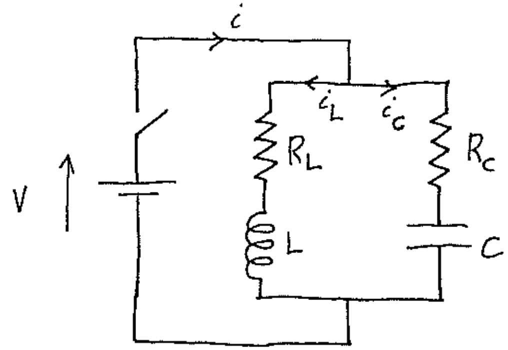
i.e., the total energy stored in a capacitor depends only upon the instantaneous voltage and is independent of the current.

6) Everything depends upon approaching this in the right way. In this section I shall use the time-domain approach; in the next section (AC circuits) I shall solve the same problem using a frequency-domain approach.

If the network is to be equivalent to a constant resistance, then certain conclusions can be drawn immediately.

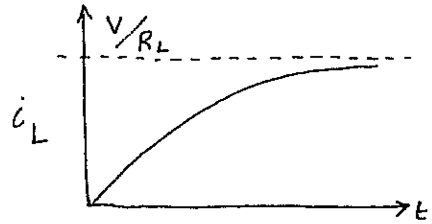
For example: if we suddenly connect a DC voltage source to XX, the current into XX must rise immediately to V/R and remain at that

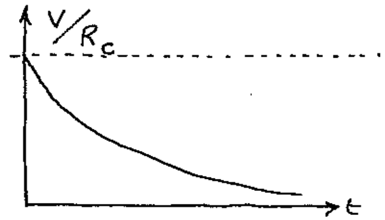
level indefinitely. Looking at the network, this seems highly improbable – yet it can be done.



The two arms are in parallel and each responds to the voltage V independently of the other.

Now draw the two current responses i_L and i_C:





$$i_L = \frac{V}{R_L}\left[1 - \exp\left(-t/\tau_L\right)\right] \qquad i_C = \frac{V}{R_C}\exp\left(-t/\tau_C\right)$$

$$\text{where } \tau_L = L/R_L \qquad \text{where } \tau_C = CR_C$$

This looks encouraging: one current increases, the other decreases. The current $i = i_L + i_C$ and must be constant at all times if the whole network is to be resistive. In fact,

$$i(t=0) \text{ must equal } i(t=\infty) = \text{constant}.$$

From the individual arms of the network we see that:

$$i(0) = \frac{V}{R_C} \quad \text{and} \quad i(\infty) = \frac{V}{R_L}$$

$$\therefore \quad R_C = R_L = R \quad \text{say.}$$

Now we have to make $i = i_L + i_C$ independent of time:

$$i = \frac{V}{R}\left[1 - \exp\left(-t/\tau_L\right) + \exp\left(-t/\tau_C\right)\right]$$

which reduces to $\frac{V}{R}$ if $\tau_L = \tau_C$

i.e. if $\frac{L}{R_L} = CR_C$

Finally, the complete requirement is that

$$R_L = R_C = \sqrt{\frac{L}{C}}$$

7) The equations describing the circuit can be written as:

$$v = i_1 r \qquad i_2 = C\frac{dv}{dt} \qquad v = i_3 R + L\frac{di_3}{dt}$$

and $i_1 + i_2 + i_3 = 0$

These equations combine to give the voltage equation:

$$LC\frac{d^2v}{dt^2} + \left(RC + \frac{L}{r}\right)\frac{dv}{dt} + \left(1 + \frac{R}{r}\right)v = 0$$

which is of standard form and has a general solution of the type:

$$v = A\exp(m_1 t) + B\exp(m_2 t)$$

The oscillatory solution requires that m_1 and m_2 are complex, say

$$\left.\begin{aligned} m_1 &= \alpha + j\beta \\ m_2 &= \alpha - j\beta \end{aligned}\right\} \quad \begin{aligned} &\alpha \text{ is the decay rate} \\ &\beta \text{ is the angular frequency} \end{aligned}$$

and can be expressed in the form:

$$\beta = \left[\left(\frac{1 + R/r}{LC}\right)\left(1 - \frac{(RC + L/r)^{1/2}}{LC\,(1 + R/r)}\right)\right]^{1/2}$$

CHECK: for zero loss $r \to \infty$ and $R \to 0$ giving

$$\beta = \frac{1}{\sqrt{LC}}$$ which is correct for loss-free resonance.