Brunel University
Queen Mary, University of London
Royal Holloway, University of London
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Intercollegiate post-graduate course in High Energy Physics

Paper 1: The Standard Model

Monday, 10 February 2014

Time allowed for Examination: 3 hours

Answer **ALL** questions

Books and notes may be consulted

The Standard Model and beyond part 2

1. Elastic electron-proton scattering.

   (a) Draw the Feynman diagram for the lowest order (electromagnetic) process contributing to electron-proton scattering. **[3]**

The matrix element squared for the lowest order electromagnetic electron-proton scattering (under the assumption of the proton being a structureless, point-like Dirac particle) with given four-momenta,

$$e^-(p_i) + p(P_i) \to e^-(p_f) + p(P_f) \ ,$$

can be written

$$|\mathcal{M}|^2 = \left(\frac{e^2}{q^2}\right)^2 \frac{1}{4}\,\mathrm{Tr}\left[\frac{\not{p}_f + m}{2m}\gamma^\mu \frac{\not{p}_i + m}{2m}\gamma^\nu\right] \mathrm{Tr}\left[\frac{\not{P}_f + M}{2M}\gamma_\mu \frac{\not{P}_i + M}{2M}\gamma_\nu\right] ,$$

where $q = p_f - p_i = P_i - P_f$, with m denoting the electron's mass, and M that of the proton.

   (b) Evaluate the Dirac traces here to give

$$|\mathcal{M}|^2 = \frac{e^4}{2m^2M^2q^4}\left[(p_f.P_f)(p_i.P_i) + (p_f.P_i)(p_i.P_f) - M^2(p_f.p_i) - m^2(P_f.P_i) + 2m^2M^2\right] .$$

[8]

   (c) Assuming four-vectors

$$p_i = (E, \boldsymbol{p}), \quad p_f = (E', \boldsymbol{p}'), \quad P_i = (M, \mathbf{0}), \quad P_f = (E_f, \boldsymbol{P}_f),$$

   show that, in the limit $m \ll E$, energy-momentum conservation implies

$$\frac{E - E'}{M} = -\frac{q^2}{2M^2}.$$

[5]

   (d) Hence show that the cross section,

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2}\frac{E'/E}{1 + (2E/M)\sin^2\theta/2}\,|\mathcal{M}|^2 \ ,$$

where

$$|\mathcal{M}|^2 = \frac{16\pi^2\alpha^2 EE'}{m^2 q^4} \left[1 + \frac{q^2}{4EE'}\left(1 + \frac{E'-E}{M}\right) + \frac{m^2}{2EE'}\left(\frac{E'-E}{M}\right)\right]$$

can be simplified, in the limit $E \gg m$ but $E \ll M$, to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \cos^2 \frac{\theta}{2}\,,$$

where θ is the angle between the outgoing and incoming electron. **[4]**

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2. Collinear factorisation of matrix elements.

Throughout this question one should take the *on-shell* quark and gluon masses to be zero: $p^2 = g^2 = 0$.

Consider an $n+1$ particle process, with amplitude $\mathcal{M}^{(n+1)}$, in which a quark and a gluon are produced; the momenta of the quark and the gluon are denoted p and g respectively, likewise their colour indices are i and a. In the limit that the quark and gluon momenta are collinear ($p.g = E_p E_g (1 - \cos\theta_{pg}) \to 0$) the amplitude is dominated by diagrams involving propagators of the form $1/(p+g)^2$, i.e. it is dominated by graphs in which the gluon is radiated by the quark leg coming out of the n particle process. This is depicted in figure 1, where the right-hand side represents the sum of all graphs involving a quark of momentum $P = p + g$ and colour j branching to the collinear quark-gluon pair.

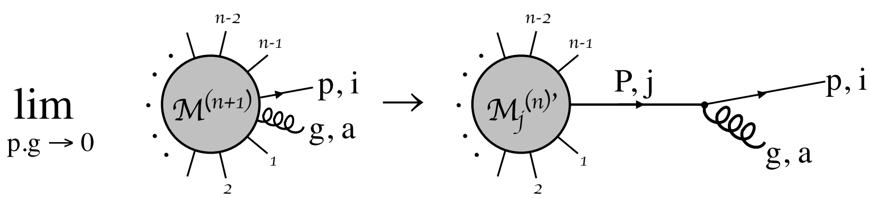

Figure 1: Collinear limit $(p+g)^2 \to 0$ for an arbitrary $n+1$ particle process involving the production of a quark and gluon.

Neglecting terms that are finite as $p.g \to 0$, using standard Feynman rules, in the collinear limit, the amplitude may then be written

$$\mathcal{M}^{(n+1)} = \epsilon^*(g)^\mu \, \overline{u(p)} \left(-ig_s T^a_{ij} \gamma_\mu\right) \frac{i\left(\not{p} + \not{g}\right)}{(p+g)^2} \mathcal{M}_j^{(n)\prime},$$

where $\mathcal{M}_j^{(n)\prime}$ denotes all contributions to the $n+1$ particle amplitude, except the $q(p+g,j) \to q(p,i) + g(g,a)$ branching, g_s is the strong coupling constant and T^a_{ij} is a Gell-Mann matrix.

(a) Derive the complex conjugate amplitude:

$$\mathcal{M}^{(n+1)\dagger} = \tfrac{g_s}{2p.g} T^a_{j'i} \mathcal{M}_{j'}^{(n)\prime\dagger} \gamma_0 \left(\not{p} + \not{g}\right) \gamma_\nu u(p) \, \epsilon(g)^\nu \ .$$

[5]

(b) Summing over gluon polarizations and colour indices (a and i) gives

$$\begin{aligned}\sum\nolimits_{\text{pol,col}} \mathcal{M}^{(n+1)}\mathcal{M}^{(n+1)\dagger} &= \tfrac{g_s^2 C_F}{(2p.g)^2}\left(-\eta^{\mu\nu} + \tfrac{g^\mu n^\nu + n^\mu g^\nu}{n.g}\right)\delta_{jj'} \\ &\times \ \overline{u\,(p)}\gamma_\mu\left(\not{p}+\not{g}\right)\mathcal{M}_j^{(n)\prime}\mathcal{M}_{j'}^{(n)\prime\dagger}\gamma_0\left(\not{p}+\not{g}\right)\gamma_\nu u\,(p)\ ,\end{aligned}$$

where $\eta_{\mu\nu}$ here denotes the usual metric tensor, and n is an unphysical gauge vector arising in the sum over gluon polarizations: n is a light-like four-vector, $n^2 = 0$, which is arbitrary except for the constraints $n.p \neq 0$ and $n.g \neq 0$. Perform a further sum over the external quark spins in this expression to give the full spin-polarization- and colour-summed squared amplitude as

$$\begin{aligned}\sum \mathcal{M}^{(n+1)}\mathcal{M}^{(n+1)\dagger} &= \tfrac{g_s^2 C_F}{(2p.g)^2}\left(-\eta^{\mu\nu} + \tfrac{g^\mu n^\nu + n^\mu g^\nu}{n.g}\right)\delta_{jj'} \\ &\times\ \mathrm{Tr}\left[\mathcal{M}_{j'}^{(n)\prime\dagger}\gamma_0\left(\not{p}+\not{g}\right)\gamma_\nu\not{p}\gamma_\mu\left(\not{p}+\not{g}\right)\mathcal{M}_j^{(n)\prime}\right]\ .\end{aligned}$$

[5]

(c) After straightforward Dirac algebra this matrix element simplifies further to yield

$$\sum \mathcal{M}^{(n+1)}\mathcal{M}^{(n+1)\dagger} = \tfrac{g_s^2 C_F \delta_{jj'}}{(p.g)(n.g)}\,\mathrm{Tr}\left[\mathcal{M}_{j'}^{(n)\prime\dagger}\gamma_0\left(n.\left(p+g\right)\left(\not{p}+\not{g}\right) + n.p\not{p} - p.g\not{n}\right)\mathcal{M}_j^{(n)\prime}\right]\ .$$

Keeping only the dominant $\mathcal{O}\left(1/p.g\right)$ terms, one can replace in the trace and the $1/n.g$ part of the denominator

$$p = zP, \qquad g = \left(1-z\right)P\,,$$

where z is the *momentum fraction* of the daughter quark with respect to the parent with momentum P (i.e. z is just a scalar number). Using these momentum relations write $\sum \mathcal{M}^{(n+1)}\mathcal{M}^{(n+1)\dagger}$ in the form

$$\sum \mathcal{M}^{(n+1)}\mathcal{M}^{(n+1)\dagger} = \tfrac{g_s^2}{p.g}\,\widehat{P}_{qq}\left(z\right)\,\mathrm{Tr}\left[\mathcal{M}_{j'}^{(n)\prime\dagger}\gamma_0\not{P}\delta_{jj'}\mathcal{M}_j^{(n)\prime}\right]\ ,$$

recording explicitly the form you obtain for the function $\widehat{P}_{qq}\left(z\right)$. **[5]**

(d) Using the completeness relation for a fermion with (light-like) momentum P, $\sum u_{j'}\left(P\right)\overline{u}_j\left(P\right) = \not{P}\delta_{jj'}$, derive the factorized form of the spin- and colour-summed squared matrix element for the $n+1$ particle process, in terms of the n particle one:

$$\sum \mathcal{M}^{(n+1)}\mathcal{M}^{(n+1)\dagger} = \frac{2g_s^2}{P^2}\,\widehat{P}_{qq}\left(z\right)\sum \mathcal{M}^{(n)\dagger}\mathcal{M}^{(n)}$$

where $\mathcal{M}^{(n)}$ is the amplitude for the $n-$particle process, related to $\mathcal{M}^{(n)\prime}$ by

$$\mathcal{M}_j^{(n)} = \overline{u_j\,(P)}\,\mathcal{M}_j^{(n)\prime}\,, \quad \text{and (hence)} \quad \mathcal{M}_{j'}^{(n)\dagger} = \mathcal{M}_{j'}^{(n)\prime\dagger}\gamma_0 u_{j'}\,(P)\ .$$

Comment on whether this result is interesting. **[5]**

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3. Spontaneous breaking of global symmetry in a complex scalar field theory.

The Lagrangian density for a complex scalar (ϕ) field theory is given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \ .$$

(a) Show that the ground/vacuum state field configuration ϕ_0 satisfies i) $|\phi_0| = 0$ for $m^2 > 0$ and ii) $|\phi_0| = \sqrt{\frac{-m^2}{2\lambda}}$ for $m^2 < 0$. **[6]**

(b) Comment briefly on the difference in the vacuum state obtained for $m^2 < 0$ with respect to that found for $m^2 > 0$. **[2]**

(c) Assuming $m^2 < 0$ and taking as the vacuum state for ϕ

$$\phi_0 = |\phi_0| \ , \qquad |\phi_0| = \sqrt{\frac{-m^2}{2\lambda}} \ ,$$

determine the Lagrangian density in terms of two real scalar fields, ϕ_1 and ϕ_2, reparameterizing ϕ as

$$\phi = |\phi_0| + \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \ .$$

[8]

(d) Comment on the nature of the various terms in the Lagrangian that results in terms of ϕ_1 and ϕ_2, in particular, comment on the masses of ϕ_1 and ϕ_2 and whether or not these are what you might have expected them to be, based on what you know of spontaneous symmetry breaking. **[4]**

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4. Abelian gauge invariance for a complex scalar field theory.

The Lagrangian density for a complex scalar (ϕ) field theory is given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - V(\phi, \phi^*)$$

$$V(\phi, \phi^*) = -m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \ .$$

(a) Determine how the potential $V(\phi, \phi^*)$ changes under a local $U(1)$ symmetry transformations $\phi \to U\phi$, $U = e^{iq\Lambda}$, $\Lambda = \Lambda(x)$. **[3]**

(b) Determine how the derivative term $\partial_\mu \phi$ changes under the same $U(1)$ transformation. **[4]**

(c) Defining the covariant derivative as

$$D_\mu = \partial_\mu + iqA_\mu \, ,$$

with A^μ transforming as

$$A^\mu \to A'^\mu = U A^\mu U^\dagger + \frac{i}{q} (\partial^\mu U) U^\dagger = A^\mu - \partial^\mu \Lambda \, ,$$

determine the result of the same $U(1)$ transformation applied to $D_\mu \phi$. **[6]**

(d) Hence show that

$$\mathcal{L}_{\text{gauged}} = (D_\mu \phi) \left(D^{\dagger \mu} \phi^* \right) - V(\phi, \phi^*) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu \, .$$

is $U(1)$ gauge invariant. **[7]**

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5. Euler-Lagrange field equations and symmetry transformations.

(a) Consider a Lagrangian density $\mathcal{L}\left(\phi^{(i)}, \partial_\mu \phi^{(i)}\right)$ and the corresponding action,

$$S = \int d^4x \, \mathcal{L}\left(\phi^{(i)}, \partial_\mu \phi^{(i)}\right) ,$$

where (i) labels various fields; $i = 1, ..., N$. Consider small variations of the fields

$$\phi^{(i)}(x,t) \to \phi^{(i)}(x,t) + \delta\phi^{(i)}(x,t) ,$$

the variations $\delta\phi^{(i)}$ all being zero at space-time infinity (the boundary of the action integral). Applying the variational principle, in particular, by imposing the action be extremised with respect to the field variations ($\delta S = 0$) derive the Euler-Lagrange differential equations obeyed by the fields $\phi^{(i)}$. **[10]**

(b) Suppose that $\mathcal{L}$, the Lagrangian density itself, is invariant under some symmetry transformation group. Under an infinitesimal transformation associated with the 'ath' generator of this symmetry group we denote the change in the fields and their derivatives naturally as

$$\phi^{(i)} \to \phi^{(i)} + \delta_a \phi^{(i)}, \quad \partial_\nu \phi^{(i)} \to \partial_\nu \phi^{(i)} + \partial_\nu \delta_a \phi^{(i)} ,$$

where the subscript a on δ_a is simply there to clarify that the infinitesimal change δ is to be associated with a 'rotation' by the 'ath' generator of the group only. Assuming that the fields $\phi^{(i)}$ and their derivatives $\partial_\nu \phi^{(i)}$ satisfy the Euler-Lagrange field equations, compute the change in the Lagrangian density $\delta\mathcal{L}$ and show that invariance of the Lagrangian implies the conservation of four-vector *currents*:

$$\partial_\nu J_a^\nu = 0 \quad \text{where} \quad J_a^\nu = \sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi^{(i)}\right)} \delta_a \phi^{(i)} .$$

[10]

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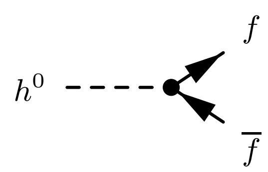
6. Computation of the width for Higgs boson decay to fermion-antifermion pairs.

Since the Higgs boson is a scalar particle, rather than associating a polarization vector or spinor to its presence as an external particle in amplitudes (as would be the case if it were a vector boson or a fermion), instead one simply associates a trivial factor '1' to each external Higgs boson in the amplitude.

The vertex Feynman rule for a Higgs boson coupling to a fermion is depicted in Fig. 2.



$$= -i\frac{m_f}{v} = -\frac{ie}{2\sin\theta_w}\frac{m_f}{m_W}$$

Figure 2: The vertex Feynman rule for a Higgs boson coupling to a fermion; e is the electric charge and θ_w the Weinberg angle, while m_f and m_W are, respectively, the mass of the fermion and the W boson.

(a) Denoting the fermion momentum by p and the anti-fermion momentum by k, write down the amplitude for a Higgs boson decaying into a fermion anti-fermion pair. **[5]**

(b) Compute the amplitude squared for a Higgs boson decaying into a fermion pair, summed over final-state fermion spins and colours, averaged over incoming polarizations, eliminating all momenta in terms of m_h (the Higgs boson mass) and m_f. Do not neglect the fermion mass. **[8]**

(c) Using the expression for the two-body Lorentz invariant phase space

$$d\text{LIPS} = \frac{1}{4\pi^2}\frac{|\vec{p}|}{4m_h}\,d\Omega\,,$$

where $\vec{p}$ is the three-momentum of either decay product in the Higgs boson rest frame, and $d\Omega$ is the solid angle, compute the width for a Higgs boson decaying into a fermion anti-fermion pair, again, eliminating all momenta in terms of m_h and m_f. Do not neglect the fermion mass. **[7]**

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