Queen Mary, University of London
Royal Holloway, University of London
University College London

Intercollegiate post-graduate course in High Energy Physics

Paper 1: Standard Model I

Monday, 22 January 2018 – 10:00

Time allowed for Examination: 3 hours

Answer any four questions

Books and notes may be consulted

Question 1 (20 marks)

(a) Determine the centre of mass energies for the following processes

(i) The highest energy proton-proton collision at the LHC in 2017

(ii) A 10 PeV neutrino interaction inside the IceCube detector (you mass assume the target was a free proton)

(iii) A 10^{21}eV cosmic-ray interaction in the atmosphere (you mass assume the target was a free proton)

[3]

(b) Particle A of mass m_A decays into two particles, B and C, with masses m_B and m_C. Determine the energies of the decay products in the rest frame of particle A. **[4]**

(c) Still considering the case of $A \to B + C$, show that in the rest frame of particle A the outgoing momentum, $|p|$, can be written in terms of the masses of the particles as

$$|p| = \frac{(m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_B^2 m_C^2 - 2m_A^2 m_C^2)^{\frac{1}{2}}}{2m_A}.$$

[3]

(d) Discuss the advantages and disadvantages of future electron-positron and proton-proton colliders. Give an example historical measurement from each type of collider to illustrate the differences. **[5]**

(e) The ratio of the total cross-sections for the processes $e^+e^- \to hadrons$ to $e^+e^- \to \mu^+\mu^-$ is called R. Explain in as much detail as you can how R varies between threshold and 150 GeV total energy. **[5]**

Question 2 (20 marks)

(a) Consider the Bhabha scattering process, $e^-(P_1)e^+(P_2) \to e^-(P_3)e^+(P_4)$ with four-momenta as labelled. In the centre of mass frame the energy of the electron and positron is E and the scattering angle of the outgoing electron (relative to the incoming electron) is θ, in terms of these quantities determine the Mandelstam variables s, t and u. **[5]**

(b) Draw the lowest-order Feynman diagram(s) for Bhabha scattering and label the four-momenta transfer in terms of s, t or u. **[3]**

(c) At lowest order the cross-section for Bhabha scattering can be written as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\left(\frac{\mathrm{Tr}_1}{(P_1 - P_3)^4} + \frac{\mathrm{Tr}_2}{(P_1 + P_2)^4} - \frac{2\,\mathrm{Tr}_3}{(P_1 - P_3)^2\,(P_1 + P_2)^2}\right),$$

where α is the fine structure constant and

$$\mathrm{Tr}_1 = \mathrm{Tr}\left[\frac{\not{P}_3 + m}{2}\gamma_\mu\frac{\not{P}_1 + m}{2}\gamma_\nu\right]\mathrm{Tr}\left[\frac{-\not{P}_2 + m}{2}\gamma^\mu\frac{-\not{P}_4 + m}{2}\gamma^\nu\right],$$

$$\mathrm{Tr}_2 = \mathrm{Tr}\left[\frac{-\not{P}_2 + m}{2}\gamma_\mu\frac{\not{P}_1 + m}{2}\gamma_\nu\right]\mathrm{Tr}\left[\frac{\not{P}_3 + m}{2}\gamma^\mu\frac{-\not{P}_4 + m}{2}\gamma^\nu\right],$$

$$\mathrm{Tr}_3 = \mathrm{Tr}\left[\frac{\not{P}_3 + m}{2}\gamma_\mu\frac{\not{P}_1 + m}{2}\gamma_\nu\frac{-\not{P}_2 + m}{2}\gamma^\mu\frac{-\not{P}_4 + m}{2}\gamma^\nu\right].$$

Hence show that in the relativistic limit the cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2}\left(\frac{1 + \cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} - \frac{2\cos^4\frac{\theta}{2}}{sin^2\frac{\theta}{2}} + \frac{1 + \cos^2\theta}{2}\right).$$

Trace theorems and identities involving gamma-matrices do not need to be derived, but they should be clearly stated. **[12]**

Question 3 (20 marks)

(a) Draw the lowest-order Feynman diagram(s) for electron-proton scattering. **[3]**

(b) If we treat the proton as a fundamental (point-like) particle and label the four-momenta as $e^-(p_i)p(P_i) \to e^-(p_f)p(P_f)$ then the matrix-element squared for this process is

$$|T_{fi}|^2 = \left(\frac{e^2}{q^2}\right)^2 \frac{1}{4} \mathrm{Tr}\left[\frac{\not{p}_f + m}{2m}\gamma^\mu \frac{\not{p}_i + m}{2m}\gamma^\nu\right] \mathrm{Tr}\left[\frac{\not{P}_f + M}{2M}\gamma_\mu \frac{\not{P}_i + M}{2M}\gamma_\nu\right],$$

where m is the mass of the electron, M is the mass of the proton and $q = p_f - p_i$.

Evaluate the traces to show

$$|T_{fi}|^2 = \frac{e^4}{2m^2M^2q^4}\left[(p_f \cdot P_f)(p_i \cdot P_i) + (p_f \cdot P_i)(p_i \cdot P_f) - M^2(p_f \cdot p_i) - m^2(P_f \cdot P_i) + 2m^2M^2\right].$$

[8]

(c) In the rest frame of the initial state proton the four vectors can be written as

$$p_i = (E, \boldsymbol{p}), \qquad p_f = (E', \boldsymbol{p'}), \qquad P_i = (M, 0), \qquad p_f = (E'_f, \boldsymbol{p'_f}),$$

show that for $m \ll E$,

$$\frac{E - E'}{M} = \frac{-q^2}{2M^2}.$$

[5]

(d) The cross-section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2}\frac{\frac{E'}{E}}{1 + \left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}}|T_{fi}|^2,$$

where θ is the angle between the outgoing electron and incoming electron, and

$$|T_{fi}|^2 = \frac{16\pi^2\alpha^2EE'}{m^2q^4}\left[1 + \frac{q^2}{4EE'}\left(1 + \frac{E' - E}{M}\right) + \frac{m^2}{2EE'}\left(\frac{E' - E}{M}\right)\right].$$

Show in the regime where $E \gg m$ but $E \ll M$ that this can be simplified to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2}.$$

[4]

Question 4 (20 marks)

(a) A possible decay of the W boson is

$$W^+(q) \to \mu^+(p) + \nu_\mu(k),$$

where q, p and k are the four-momenta of the W^+, μ^+ and ν_μ respectively. The transition amplitude squared is:

$$|T_{fi}|^2 = \frac{g_W^2}{3}\left[p\cdot k + \frac{2}{M_W^2}(q\cdot k)(q\cdot p)\right],$$

where g_W is the weak coupling and M_W is the mass of the W^+. In the W rest frame, show that

$$q\cdot k = \frac{M_W^2}{2}\left(1-\frac{m_\mu^2}{M_W^2}\right), \quad q\cdot p = \frac{M_W^2}{2}\left(1+\frac{m_\mu^2}{M_W^2}\right), \quad p\cdot k = \frac{M_W^2}{2}\left(1-\frac{m_\mu^2}{M_W^2}\right),$$

where m_μ is the mass of the μ^+. **[10]**

(b) Use the relationship

$$\Gamma = \frac{1}{16\pi M_W}|T_{fi}|^2,$$

to derive the partial width, Γ, in terms of the masses of the W and μ and show that this can be simplified to

$$\Gamma \simeq \frac{G_F}{\sqrt{2}}\frac{M_W^3}{6\pi}.$$

[10]

Question 5 (20 marks)

(a) In electron-photon scattering $e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')$ the following traces occur.

$$\mathrm{Tr}\left[\gamma^\mu \not{k} \gamma^\nu (\not{p} + m) \gamma_\nu \not{k}' \gamma_\mu (\not{p}' + m)\right],$$

$$\mathrm{Tr}\left[\gamma^\mu \not{k} \gamma^\nu (\not{p} + m) \gamma_\mu \not{k}' \gamma_\nu (\not{p}' + m)\right].$$

Evaluate these traces in terms of the four-vectors (you do not need to convert to Mandelstam variables; trace identities and identities involving gamma-matrices do not need to be derived but they should be clearly stated). **[10]**

(b) In the mass-less limit, the terms in the final squared transition amplitude for Compton scattering are

(i)

$$2e^4 \left(\frac{-u}{s}\right),$$

(ii)

$$2e^4 \left(\frac{-s}{u}\right),$$

(iii)

$$2e^4 \frac{t}{us}(s + u + t).$$

Draw the Feynman diagram(s) which contribute to each term. Hence write down the final squared transition amplitudes when the incoming photon is real and when it is virtual. **[5]**

(c) The Compton condition is

$$\lambda' = \lambda + \frac{2\pi}{m}(1 - \cos\theta)$$

where λ is the wavelength of the incoming photon, λ' is the wavelength of the scattered photon and θ is the scattering angle. The Klein-Nishina cross-section for Compton scattering is

$$\frac{d\sigma}{d\Omega}(\lambda, \lambda') = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon'^{\star} \cdot \epsilon)^2 - 2\right],$$

where ω, ω' are the energies and ϵ, ϵ' are the polarisation vectors of the incoming and outgoing photons. Determine the cross-section in the low-energy limit, where $\omega \to 0$, in terms of the fine structure constant, α, the mass of the electron, m, and the polarisation vectors. **[5]**

Question 6 (20 marks)

(a) The Pauli spin matrices are $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Verify that the Pauli spin matrices are Hermitian and satisfy the commutation relation,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

[3]

(b) Verify the following identity for the Pauli spin matrices,

$$(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b}),$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\boldsymbol{a}$ & $\boldsymbol{b}$ are any two vectors. **[3]**

(c) The Klein-Gordon equation is

$$\left(-\nabla^2 + m^2\right)\psi = -\frac{\partial^2 \psi}{\partial t^2}$$

and the Dirac Equation can be written as

$$H_D\psi = (\alpha_1 p_X + \alpha_2 p_y + \alpha_3 p_Z + \beta m)\psi = i\frac{\partial \psi}{\partial t}.$$

Show that in order for these equations to be equivalent, the α_i and β must satisfy the following

$$\alpha_i\alpha_j + \alpha_j\alpha_i = \delta_{ij}, \qquad \beta\alpha_i + \alpha_i\beta = 0, \qquad \beta^2 = 1.$$

[3]

(d) Show that this requires matrices for α_i, β which are trace-less and have eigen-values ± 1 which must be even-dimensional. **[2]**

(e) Show that the angular momentum operator $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ does not commute with the Dirac Hamiltonian. Defining $\Sigma = -i\alpha_1\alpha_2\alpha_3\boldsymbol{\alpha}$ show that this also does not commute with the Dirac Hamiltonian. Use these results to define a new quantity which is conserved, what is this quantity? **[5]**

(f) A particle with mass m and kinematic energy T strikes a particle of equal mass m which is initially at rest. Show that the kinematic energy T' of the particle scattered elastically at an angle θ is given by

$$T' = \frac{T\cos^2\theta}{1 + \frac{T\sin^2\theta}{2m}}.$$

[4]