The Standard Model
Introduction

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Preface

The Standard Model is a quantum field theory which incorporates all the currently known fundamental particles (leptons and quarks) and all forces (other than gravity), i.e. electromagnetic, weak and strong. It is a gauge theory where the forces are carried by the spin–1 vector bosons known as the photon (electromagnetic), $W\pm$ and Z (weak) and gluons (strong). It describes all the particle physics we currently know about up to scales of $\mathcal{O}(10\text{TeV})$, i.e. the scales of the largest colliders, LEP, HERA, the Tevatron and the LHC, and is, in principle, completely consistent until we reach scales of 10^{19}GeV, i.e. the Planck scale, at which point gravity at the very least must be added. However, it contains 19 free parameters (mainly masses) and there were indications, though no real proofs, that some extension/simplification would appear at the the LHC (Large Hadron Collider), which is now operating at a centre of mass energy near 14TeV. However, nothing has so far shown up.

In very general terms the Standard Model is just an extension of the simplest useful quantum gauge field theory QED, which contains only the electron and photon. The structure of the Standard Model has indeed been largely inspired by the particle content, where the particles are excitations of the quantum field about the vacuum, and are henceforth intrinsically linked to perturbation theory. However we can then obtain and test results using the full quantum field theory beyond perturbation theory, using for example lattice field theory. Nevertheless, the other sectors of the Standard Model are more complicated than QED, not just in terms of additional particle content and more parameters, but because the simplest extensions one can make are not suitable. In the case of the strong interaction the particles we actually observe in nature are not the fundamental degrees of freedom in the Lagrangian for the field theory, and we must explain this, and discover how we can in fact make calculations in such a theory. In the electroweak sector of the Standard Model the short range of the interactions makes it difficult to construct a theory which is renormalizable, i.e. where the divergences in the theory one automatically encounters beyond leading order are under control, and can be removed with a redefinition of the parameters in the theory, and which is unitary, i.e. $S^\dagger = S^{-1}$, where the S-matrix S evolves states from $t \to -\infty$ to $t \to \infty$, taking account of scattering. Lack of unitarity occurs if non-physical (sometimes negative norm) particle states actually contribute in scattering processes. Hence, both the electroweak and strong forces have complications which we will overcome (the former requiring the introduction of the recently discovered Higgs boson), and examine the consequences.

Hence, the structure of the course will be as follows:

1. Introduction. Conventions. Particles and fields. Gauge theories.

2. Problems encountered when describing short-range interactions. Spontaneous symmetry breaking. Massless particles and Goldstone's theorem. Massive vector bosons and Higg's model, both Abelian and Non-Abelian. Renormalization and unitarity.

3. Electroweak gauge theory. $SU(2)_L \times U(1)_Y$. Leptons and Quarks. Masses and mixing. CKM matrix.

4. Parity, charge conjugation and time-reversal. PCT. Violations of each in the Standard Model. Neutrino masses.

5. Weak interactions. Leptonic-processes. Semi and non-leptonic processes. CP violation in nature.

6. Strong interactions and Quantum ChromoDynamics (QCD). Structure of the theory. Running coupling, renormalization group, asymptotic freedom. Examples: $e^+e^- \to$ hadrons, deep inelastic scattering (DIS) - $e\,p \to e + X$ via a highly virtual photon.

7. Depending on time, limitations of the Standard Model.

1 Particles

The Standard Model contains:

$$\begin{array}{llll}
\text{fermions} & \begin{pmatrix} d \\ u \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix} \begin{pmatrix} b \\ t \end{pmatrix} & \text{quarks} \\
 & \begin{pmatrix} e \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} & \text{leptons} \\
\text{vector bosons} & \gamma, \quad W^\pm, Z \quad g & \\
\text{scalar boson} & \qquad\qquad H &
\end{array}$$

and the gauge group for the complete theory is $SU(3) \times SU(2)_L \times U(1)_Y$, where the $SU(2)_L \times U(1)_Y$ is broken by a choice of vacuum to $U(1)_Q$.

There are 19 parameters in the Standard Model (assuming for the moment zero neutrino mass):

- 9 fermion masses
- 3 gauge couplings
- 4 quark mixings
- 2 Higgs potential terms (e.g. mass and self-coupling)
- 1 QCD Θ-vacuum term.

The last of these is known to be $\leq 10^{-9}$, and it is often assumed that there is some underlying explanation why it is zero in value.

All of the particles in the Standard Model are described by quantum field theory.

2 Scalar Field Theory

The Lagrangian for the complex scalar field is

$$\mathcal{L} = -\tfrac{1}{2}\phi(x)(\Box + m^2)\phi(x)^* - \mathcal{L}_{int}(|\phi(x)|), \tag{1}$$

where the part quadratic in the fields is often called $\mathcal{L}_f$, where f indicates free since this part represents a non-interacting theory. In terms of momentum modes the free scalar field has the expansion

$$\phi(x) = \sum_{\underset{\sim}{p}} [a(\underset{\sim}{p}) \exp(-ip \cdot x) + b^\dagger(\underset{\sim}{p}) \exp(ip \cdot x)], \tag{2}$$

where

$$\sum_{\underset{\sim}{p}} = \int \frac{d^3\underset{\sim}{p}}{(2\pi)^3\, 2E} \tag{3}$$

is the Lorentz invariance phase space for an on-shell particle, and the equation of motion for $\phi(x)$

$$(\Box + m^2)\phi(x) = 0 \tag{4}$$

enforces the on-shell condition $E^2 = \underline{p}^2 + m^2$. $a(\underline{p})$ is the annihilation operator for particles and $b^\dagger(\underline{p})$ is the creation operator for anti-particles. $a(\underline{p}) = b(\underline{p})$ for a real field.

The fields satisfy the quantization commutation condition $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$, where the conjugate momentum $\pi(\mathbf{x}) = \partial\mathcal{L}/\partial(\partial\phi/\partial t) = \partial\phi(\mathbf{x})/\partial t$. This results in the condition on the creation and annihilation operators of $[a_{\mathbf{q}}, a^\dagger_{\mathbf{p}}] = (2\pi)^3 2E(\mathbf{p})\delta^3(\mathbf{p} - \mathbf{q})$.

In perturbation theory the lines in Feynman diagrams represent the propagation of free particles between interactions and are related to the free part of the Lagrangian. i.e. if

$$\mathcal{L}_f = \phi(x)^* \mathcal{O}(\partial_\mu, m)\phi(x) \tag{5}$$

then the propagator (time-ordered two-point Green's function) is given by

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = -i\mathcal{O}^{-1}(x, y) \tag{6}$$

where

$$\mathcal{O}(\partial_\mu, m)\mathcal{O}^{-1}(x, y) = \delta^4(x - y) \tag{7}$$

In position space

$$-i\mathcal{O}^{-1}(x, y) \equiv G_F(x - y) = \int i \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}, \tag{8}$$

where ϵ is an infinitesimal constant. This is known as the Feynman propagator, and represents a free particle created at y and annihilated at x if $t > t'$ and created at x and annihilated at y if $t' > t$. It is easy to show that this Feynman Green's function is a Green's function of the Klein-Gordon operator:

$$(\partial_x^2 + m^2)G_F(x - y) = -i\delta^4(x - y). \tag{9}$$

In momentum space

$$\tilde{\mathcal{O}}(p^2, m^2)\tilde{\mathcal{O}}^{-1}(p^2, m^2) = i. \tag{10}$$

The propagator is normally expressed as the momentum space two-point function and in this case is

$$D(p^2) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{11}$$

where the $i\epsilon$ is a prescription for avoiding the pole and which leads to causal propagation. We generally also have interaction vertices, e.g. if $\mathcal{L}_{int} = -\lambda\frac{|\phi(x)|^4}{4!}$ this corresponds to a four-point vertex with factor $\frac{-i\lambda}{4!}$.

3 Fermionic Fields - Spinors

Fermions are represented by 4-component spinors $\psi_\alpha(x)$ satisfying the Dirac equation

$$(i\gamma \cdot \partial - m)\psi(x) = 0. \tag{12}$$

The γ-matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4 \tag{13}$$

where I_4 is the 4-dimensional identity matrix in spinor space, and is usually left implicit, and g is the metric tensor with convention $g_{00} = 1, g_{ii} = -1, i = 1, 2, 3$.

If $\underline{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$ then we usually specify

$$(\gamma^0)^\dagger = \gamma^0, \qquad \underline{\gamma}^\dagger = -\underline{\gamma}. \tag{14}$$

The Dirac Hamiltonian is

$$H_D = -i\underline{\alpha} \cdot \underline{\nabla} + \beta m, \tag{15}$$

where $\beta = \gamma^0$ and $\underline{\alpha} = \gamma^0 \underline{\gamma}$, so the above conditions guarantee the explicit hermiticity of the Hamiltonian. An equivalent way of expressing the condition is

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu \tag{16}$$

since $(\gamma^0)^2 = 1$. The representations of the γ-matrices are given as 4×4 matrices, though any representation satisfying (13) is equivalent. Note that γ^μ are not the components of a 4-vector.

The Lagrangian for fermion fields which results in the Dirac equation is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma \cdot \partial - m)\psi(x) \tag{17}$$

where $\bar{\psi}(x) = \psi(x)^\dagger \gamma^0$. Treating $\psi(x)$ and $\bar{\psi}(x)$ as the two independent fields one obtains the equations of motion

$$(i\gamma \cdot \partial - m)\psi(x) = 0 \tag{18}$$

$$\bar{\psi}(x)(-i\gamma \cdot \overleftarrow{\partial} - m) = 0. \tag{19}$$

In a purely fermionic theory no interaction Lagrangian is allowed. This is because the simplest interaction is a four-fermion operator, and this has dimension six, and hence is accompanied by a negative dimension (-2) coupling constant. Such interactions lead to a lack of renormalizability in a quantum field theory. This problem will be discussed in more detail later.

The solutions of the equations of motion for the field operators have the Fourier expansion

$$\psi_\alpha(\underline{x}, t) = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{1}{2E_{\underline{k}}} \sum_{s=1,2} [b(s,\underline{k}) u_\alpha(s,\underline{k}) e^{-ik\cdot x} + d^\dagger(s,\underline{k}) v_\alpha(s,\underline{k}) e^{ik\cdot x}] \tag{20}$$

$$\bar{\psi}_\alpha(\underline{x}, t) = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{1}{2E_{\underline{k}}} \sum_{s=1,2} [b^\dagger(s,\underline{k}) \bar{u}_\alpha(s,\underline{k}) e^{ik\cdot x} + d(s,\underline{k}) \bar{v}_\alpha(s,\underline{k}) e^{-ik\cdot x}] \tag{21}$$

where $b(s,\underline{\mathrm{k}})$ annihilates particles of momentum $\underline{\mathrm{k}}$ and spin s and $d^\dagger(s,\underline{\mathrm{k}})$ creates anti-particles of momentum $\underline{\mathrm{k}}$ and spin s. The fermionic fields and operators satisfy anticommutation relations instead of commutation relations.

The spinors $u_\alpha(s,\underline{p})$ and $v_\alpha(s,\underline{p})$ have four components each. The label $\alpha \in \{1,2,3,4\}$ is a spinor index that will usually be suppressed. They satisfy the equations

$$(\gamma \cdot p - m)u(s,\underline{p}) = 0 \qquad\qquad (\gamma \cdot p + m)v(s,\underline{p}) = 0, \tag{22}$$

which again ensures that $E^2 = \underline{p}^2 + m^2$. They correspond to the positive and negative energy solutions in relativistic quantum mechanics.

The conventional choice of the γ-matrices is the *Dirac representation*:

$$\underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{23}$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the 2×2 identity matrix and σ_i are the Pauli matrices. Using this convention the explicit expressions for $u_\alpha(s,\underline{p})$ and $v_\alpha(s,\underline{p})$ are

$$\sqrt{E+m} \begin{pmatrix} \chi_r \\ \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_r \end{pmatrix}. \tag{24}$$

and

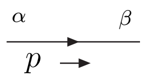
$$\sqrt{E+m} \begin{pmatrix} \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_r \\ \chi_r \end{pmatrix}. \tag{25}$$

The argument r takes the values $1, 2$, with

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{26}$$

For the simple case $\underline{p} = 0$ we may interpret χ_1 as the spin-up state and χ_2 as the spin-down state. Thus for $\underline{p} = 0$ the 4-component wave function has a very simple interpretation: the first two components describe electrons with spin-up and spin-down, while the second two components describe positrons with spin-up and spin-down. Thus we understand on physical grounds why the wave function had to have four components.

The fermion propagator is represented as below

fermion line 

$\dfrac{i(\gamma.p + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$

And we also have

- Outgoing electron $\bar{u}_\alpha(s,p)$
- Incoming electron $u_\alpha(s,p)$
- Outgoing positron $v_\alpha(s,p)$
- Incoming positron $\bar{v}_\alpha(s,p)$

4 Chirality

We can examine a particular additional property of the Dirac field known as the chiral structure. In order to do this it is necessary to define

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{27}$$

where the last equality is valid in the Dirac representation. This new matrix satisfies

$$\gamma^{5\dagger} = \gamma^5, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0, \qquad (\gamma^5)^2 = 1, \tag{28}$$

where all the above are independent of representation. This matrix allows us to introduce the concept of chiral symmetry, which is related to the massless limit of fermion fields.

Let us consider the field corresponding to a massless particle. Such a field satisfies the massless Dirac equation:

$$\gamma.\partial\psi(x) = 0\ . \tag{29}$$

This has the important property that, because $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$, $\gamma_5\psi(x)$ also satisfies the same equation,

$$\gamma.\partial\gamma_5\psi(x) = 0\ . \tag{30}$$

It follows that the fields

$$\psi_L(x) = \tfrac{1}{2}(1-\gamma_5)\psi(x) \equiv P_L\psi(x) \quad \text{and} \quad \psi_R(x) = \tfrac{1}{2}(1+\gamma_5)\psi(x) \equiv P_R\psi(x) \tag{31}$$

do so as well. Since ${\gamma_5}^2 = 1$, P_L and P_R define a set of projection operators

$$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_LP_R = P_RP_L = 0, \qquad P_L + P_R = 1. \tag{32}$$

The fields in (31) are eigenvectors of γ_5 with eigenvalues ∓ 1 and describe particles of definite *helicity* or handedness. To show this we use the result that the angular momentum operator acting on solutions of the Dirac equation $\psi(x)$ is $\mathbf{J} = -i\mathbf{x} \times \nabla + \mathbf{S}$ where the spin $S_i = i\frac{1}{4}\epsilon_{ijk}\gamma^j\gamma^k$. Using $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it is possible to see that $\gamma_5\mathbf{S} = \frac{1}{2}\gamma^0\underline{\gamma} = \mathbf{S}\gamma_5$. We have therefore for a Dirac spinor of momentum p

$$(\gamma.p)u(p,\lambda) = 0 \quad \Rightarrow (\gamma^0|\mathbf{p}| - \underline{\gamma}\cdot\mathbf{p})u(p,\lambda) = 0 \Rightarrow (1 - \gamma^0\underline{\gamma}\cdot\hat{\mathbf{p}})u(p,\lambda) = 0, \tag{33}$$

where we have used $\hat{\mathbf{p}} = \mathbf{p}/E$ ($|\mathbf{p}| = E$ for massless particles). Multiplying through by γ_5 we obtain

$$\gamma_5 u(p,\lambda) = 2\mathbf{S}.\hat{\mathbf{p}}u(p,\lambda) \tag{34}$$

The component of the spin of a particle along the direction of motion, measured by $\mathbf{S}.\hat{\mathbf{p}}$, is defined to be the helicity. Hence for the projections in eq.(31)

$$(1 + 2\mathbf{S}.\hat{\mathbf{p}})u_L(p,\lambda) = 0\ , \qquad (1 - 2\mathbf{S}.\hat{\mathbf{p}})u_R(p,\lambda) = 0\ , \tag{35}$$

so that these describe left-handed and right-handed particles, with negative and positive helicity $\mp\frac{1}{2}$, respectively.

We can also write massive fields in terms of their chiral components,

$$\psi(x) = \psi_R(x) + \psi_L(x) \ , \tag{36}$$

where

$$\psi_R(x) = \tfrac{1}{2}\left(1+\gamma_5\right)\psi(x) \quad \text{and} \quad \psi_L(x) = \tfrac{1}{2}\left(1-\gamma_5\right)\psi(x) \ . \tag{37}$$

The conjugate fields are

$$\overline{\psi_R}(x) = \psi_R(x)^\dagger\gamma^0 = \psi(x)^\dagger\tfrac{1}{2}\left(1+\gamma_5\right)\gamma^0 = \bar{\psi}(x)\tfrac{1}{2}\left(1-\gamma_5\right) \ , \tag{38}$$

and similarly

$$\overline{\psi_L}(x) = \bar{\psi}(x)\tfrac{1}{2}\left(1+\gamma_5\right) \ . \tag{39}$$

It follows that

$$\overline{\psi_R}(x)\psi_R(x) = \overline{\psi_L(x)}\psi_L(x) = 0 \ , \tag{40}$$

and

$$\overline{\psi_R}(x)\gamma^\alpha\psi_L(x) = \overline{\psi_L(x)}\gamma^\alpha\psi_R(x) = 0 \ . \tag{41}$$

The Lagrangian $\mathcal{L} = \bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x)$ can then be rewritten as

$$\mathcal{L}(x) = \overline{\psi_R}(x)i\gamma.\partial\psi_R(x) + \overline{\psi_L}(x)i\gamma.\partial\psi_L(x) - m\left[\overline{\psi_R}(x)\psi_L(x) + \overline{\psi_L}(x)\psi_R(x)\right] \ . \tag{42}$$

The 'kinetic' part of the Lagrangian is therefore a sum of two terms involving the right and left chiral components separately, while the mass term couples right to left and left to right. Hence massless particle have definite helicity and chiral symmetry, while mass breaks this symmetry. This has a clear physical interpretation since if the massive particle has its spin along its direction of motion in one frame one can always boost to another frame where it is in the opposite direction relative to its motion. This is not possible for a massless particle.

In the Standard Model chiral symmetry is spontaneously broken. Therefore although (most and probably all) particles have mass, vestiges of chirality remain in the complete theory, e.g. only left handed particles couple to the charged weak bosons.

5 Gauge Fields - Abelian

I will start with the assumption that one can construct a quantum field theory involving vector fields just by maintaining Lorentz covariance. Hence, we consider the single real vector field $A_\mu(x)$ for some general Lorentz-invariant Lagrangian. The quantum field theory can be constructed by definining the conjugate momentum field $\Pi_\nu = \frac{\partial\mathcal{L}}{\partial(\partial A^\nu/\partial t)}$ and imposing the commutation relations

$$[A_\mu(\underline{x},t), \Pi_\nu(\underline{y},t)] = -ig_{\mu\nu}\delta^3(\underline{x}-\underline{y}). \tag{43}$$

The Fourier expansion of the quantum field may be written

$$A_\mu(\underline{x}, t) = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{1}{2E_{\underline{k}}} \sum_{\lambda=0,3} [\epsilon_\mu(\lambda, \underline{k}) a(\lambda, \underline{k}) e^{-ik\cdot x} + \epsilon^*_\mu(\lambda, \underline{k}) a^\dagger(\lambda, \underline{k}) e^{ik\cdot x}] \quad (44)$$

where the $a(\lambda, \underline{k}), a^\dagger(\lambda, \underline{k})$ have their usual interpretation as creation and annihilation operators and $\epsilon_\mu(\lambda, \underline{k})$ are polarization vectors which are generally two transverse, i.e. $\underline{\epsilon} \cdot \underline{k} = 0$, one longitudinal and one timelike.

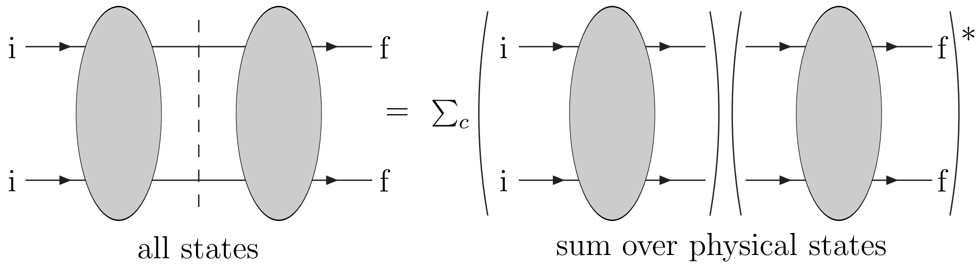


Figure 1: Diagrammatic representation of unitarity.

The commutation relation for the fields leads to the commutator for the creation and annihilation operators.

$$[a(\lambda, \underline{k}'),), a^\dagger(\lambda', \underline{k})] = -g_{\lambda\lambda'} 2E_{\underline{k}} (2\pi)^3 \delta^3(\underline{k} - \underline{k}'), \quad (45)$$

i.e. the timelike vector bosons have a negative norm. This unavoidably leads to problems with unitarity. The S-matrix for particle scattering is defined by

$$|out\rangle = S^\dagger |in\rangle. \quad (46)$$

Therefore in order for probability to be conserved we must have $\langle in|in\rangle = \langle out|out\rangle$, which means

$$S^\dagger S = SS^\dagger = 1, \qquad S^\dagger = S^{-1}, \quad (47)$$

which is known as unitarity of the S-matrix. For incoming state labelled by a and final state labelled by b, we can express the S-matrix in the form

$$S_{ab} = \delta_{ab}\delta^4(p_a - p_b) + i(2\pi)^4\delta^4(p_a - p_b)T_{ab}, \quad (48)$$

where p_a and p_b represent the sum of the four momenta for all incoming and outgoing particles respectively, δ_{ab} represents the identical outgoing state to incoming state (hence no scattering), and T_{ab} is known as the scattering amplitude. Unitarity demands that

$$\mathrm{Im}\, T_{ab} = \tfrac{1}{2}(2\pi)^4 \sum_c T_{ac} T^*_{bc} \delta^4(p_a - p_c), \quad (49)$$

where c is the sum over all physical states. The imaginary part of a diagram can be shown to be equivalent to taking a line which separates (cuts) the initial state from the final state by crossing intermediate propagators, replacing all cut propagators with the on-shell condition for these particles $(\Theta(p_0)\delta^4(p^2 - m^2))$ taking the complex conjugate of the right-hand side and summing over all cuts. The imaginary part of the amplitude is then obtained by summing over all diagrams. Hence, (49) is represented as below:

Even if one restricts negative norm states from the external states, in an interacting quantum field theory they will appear in the diagrams represented in the left-hand-side of Figure 1 and spoil the equivalence of both sides and violate unitarity.

In order to have a physically viable quantum field theory of vector bosons we must restrict the degrees of freedom so that the negative norm states do not contribute. In fact we have a well-known vector particle, the photon, which we know has only two degrees of freedom, i.e. the two transverse polarizations $\underline{\epsilon} \cdot \underline{p} = 0$, rather than the four apparent degrees of freedom in a vector field. In order to describe this theory we impose the (Abelian) gauge symmetry on the theory, i.e. define a Lagrangian which is invariant under the transformation of the gauge field

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \chi(x). \tag{50}$$

This is achieved by defining the Lagrangian

$$\mathcal{L}_g = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{51}$$

where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the factor of $\frac{1}{4}$ is a normalization chosen to obtain Maxwell's equations when coupling to a current.

We can couple the gauge field to fermionic fields to produce an interacting gauge theory by demanding that the theory be invariant under additional transformations

$$\psi(x) \to \exp(i\chi(x))\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)\exp(-i\chi(x)), \tag{52}$$

and by defining the covariant derivative

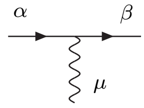
$$D_\mu \equiv \partial_\mu + ieA_\mu. \tag{53}$$

The fermionic part of the Lagrangian is then

$$\mathcal{L}_f = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi, \tag{54}$$

and this gives $\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu A_\mu \psi$ resulting in the below interaction vertex:

Vertex



$-ie\gamma^\mu_{\alpha\beta}$

However, the full quantization also requires an explicit fixing of the gauge e.g. $\partial^\mu A_\mu = 0$ is the conventional choice. This is required for two reasons. Firstly, using the standard gauge field Lagrangian the conjugate momentum $\Pi^0(x) = (\partial\mathcal{L}/\partial((\partial A^0(x)/\partial t)))$ does not exist due to the antisymmetry of the field strength, thus making the canonical commutation procedure impossible. Secondly, there is no inverse to the quadratic part of the Lagrangian, making the definition of a propagator impossible. This can be seen as follows. In momentum space the free part of the Lagrangian is

$$\mathcal{L}_{free} = -\tfrac{1}{2}A^\mu(-p)\Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)p^2 A^\nu(p), \tag{55}$$

where $g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} = P^T_{\mu\nu}$ and $\frac{p_\mu p_\nu}{p^2} = P^L_{\mu\nu}$ define generalized projection operators

$$P^T_{\mu\lambda}P^{T\lambda\nu} = P^{T\nu}_\mu = \delta^\nu_\mu - \frac{p_\mu p^\nu}{p^2}, \qquad P^L_{\mu\lambda}P^{L\lambda\nu} = P^{L\nu}_\mu = \tfrac{p_\mu p^\nu}{p^2},$$
$$P^T_{\mu\lambda}P^{L\lambda\nu} = P^L_{\mu\lambda}P^{T\lambda\nu} = 0, \qquad P^T_{\mu\nu} + P^L_{\mu\nu} = g_{\mu\nu}.$$

(The latter identity illustrating the problem of a timelike negative norm state.) Hence, the free part of the Lagrangian has momentum dependence which is $\propto P^T_{\mu\nu}$ and it is not difficult to see that there is no inverse of this tensor. This can be rectified by adding the gauge condition to the Lagrangian as a constraint, i.e. adding a gauge-fixing term $\lambda(\partial_\mu A^\mu)^2$, where λ is a Lagrange multiplier, and for convenience we often write $-1/(2\xi)(\partial_\mu A^\mu)^2$, instead. Similarly the canonical quantization problem is solved by adding this term to the Lagrangian and additionally demanding that $\langle\psi|(\partial_\mu A^\mu)|\psi\rangle = 0$, where $|\psi\rangle$ is a physical state.

Introducing the gauge-fixing term the free part of the Lagrangian becomes

$$\mathcal{L}_{free} = -\tfrac{1}{2}A^\mu(-p)\Big[\Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)p^2 - \frac{1}{\xi}\frac{p_\mu p_\nu}{p^2}p^2\Big]A^\nu(p). \tag{56}$$

This then leads to the propagator

$$D_{\mu\nu}(p) = \Big(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\Big)\frac{-i}{p^2 + i\epsilon}, \tag{57}$$

which corresponds to massless particles.

Note, that even after fixing the gauge via $\partial_\mu A^\mu = 0$ and therefore removing 1 degree of freedom we can still make a gauge transformation $A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\chi(x)$ as long as $\Box\chi(x) = 0$. Hence we still have one unphysical degree of freedom, and the physical degrees of freedom indeed number $4-2$, i.e the two transverse polarizations.

6 Non-Abelian Gauge Theories

One can generalize the above Abelian quantum field theory to more complicated symmetry transformations. In general we can define the field transformation

$$\psi_i(x) \to \exp(i\underline{T}\cdot\underline{\theta}(x))_{ij}\psi_j(x) \equiv U_{ij}\psi_j \tag{58}$$

where T_a are the generators in a particular n-dimensional representation of a given gauge group. This gauge group has number of generators, N.

The generators have the property

$$[T_a, T_b] = if_{abc}T_c, \tag{59}$$

i.e. they form a closed algebra. f_{abc} are known as the structure constants. The generators are also orthogonal and normalized to

$$\mathrm{tr}\{T_a T_b\} = T(R)\delta_{ab}, \tag{60}$$

where $T(R)$ is representation dependent, but $T(R) = \frac{1}{2}$ for the fundamental representation, i.e, the represenatation with the lowest dimension. The f_{abc} are completely antisymmetric under an interchange of indices. As well as the fundamental representation another commonly encountered representation is the adjoint representation defined by

$$(T_a)_{ij} = -if_{aij}. \tag{61}$$

This clearly has dimension equal to the number of generators N.

The gauge fields in this theory transform according to

$$(\underline{T} \cdot \underline{A}_\mu) \to U(\underline{T} \cdot \underline{A}_\mu)U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \tag{62}$$

where g is the gauge coupling and the covariant derivative is

$$D_\mu^{ij} = \partial_\mu \delta^{ij} + ig(\underline{T} \cdot \underline{A}_\mu)^{ij} \tag{63}$$

so that $D_\mu \psi(x) \to U(x)D_\mu \psi(x)$.

The fermionic part of the Lagrangian is defined by

$$\mathcal{L}_f = \bar{\psi}_i(i\gamma^\mu D_\mu^{ij} - M\delta^{ij})\psi_j \tag{64}$$

and is clearly gauge invariant. We define the field strength for the non-Abelian field as

$$F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - gf_{abc}A_{b\mu}A_{c\nu}, \tag{65}$$

which is derived from the equivalent definition

$$T_a F_{a\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]. \tag{66}$$

From the transformation rule of the covariant derivative it is easy to see that $T_a F_{a\mu\nu} \to UT_a F_{a\mu\nu}U^{-1}$. Hence the gauge part of the Lagrangian

$$\mathcal{L}_{gauge} = -\frac{1}{4}\mathrm{tr}(F_a^{\mu\nu}F_{a\mu\nu}) \tag{67}$$

is invariant under gauge transformations. There are no mass terms in the Lagrangian, and the gauge fields are massless, as in QED. In fact the part of the

Lagrangian which is quadratic in gauge fields alone is the same for each of the N fields as it was for the photon field, and polarization is transverse.

In order to quantize it is again necessary to fix the gauge for each field. This can be done in exactly the same way e.g. adding $-1/(2\xi)(\partial_\mu A_a^\mu)^2$ to the Lagrangian. However, the gauge fixing procedure is more complicated due to the self-interaction of the gauge bosons. In order to cancel the unphysical modes it is now necessary to introduce ghost fields into the Lagrangian via, e.g. the term

$$\mathcal{L}_{gh} = \partial_\mu C_a^\dagger (D_{ab}^\mu C_b). \tag{68}$$

The ghosts C_a have dimensions of scalar fields, but satisfy anticommutation relations like fermions. The full Lagrangian, including gauge fixing and ghost satisfies a generalized symmetry (BRS symmetry).

Due to the gauge field self-interaction there are now additional vertices in the Feynman rules, i.e.

$$g f_{abc}(\partial_\mu A_{a\nu}) A_b^\mu A_c^\nu \qquad -\tfrac{1}{4} g^2 f_{abc} f_{ade} A_b^\mu A_c^\nu A_{d\mu} A_{e\nu}$$

There will also be ghost-gauge field interactions, but these will depend on the gauge-fixing. (Physical gauge-fixing can decouple ghosts, but at the expense of other complications, e.g. lack of manifest Lorentz covariance.)