The Standard Model
Parity, Charge Conjugation and Time Reversal

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Preface

Parity, charge conjugation and time-reversal are three discrete transformations which act on the fields in the Standard Model, parity and time-reversal being the improper Lorentz transformations with $\det \Lambda = -1$, and charge conjugation relating particles to their antiparticles. It was thought for a long time that physics was invariant under each of these transformations, and indeed, this is the case for both QED and QCD. However, the chiral nature of the electroweak interactions and the complex phase in the CKM matrix which mixes the quark families in the charged-weak current, mean that the electroweak sector is neither P (parity), C (charge conjugation) nor T (time-reversal) invariant.

1 Parity

The operation of parity inversion, P, is associated with the spatial map $x \rightarrow x_P = (x_0, -\mathbf{x})$.

1.1 Scalar Fields

If $\phi(x)$ is a classical scalar field, the operation of parity on ϕ is defined by the transformation

$$\phi(x) \longrightarrow \eta_P \phi(x_P) \ , \tag{1}$$

where η_P is the *intrinsic parity* of the field or particle. Since repeating the parity operation leaves x unchanged we would expect to have $P^2 = 1$. For a *classical* real field ϕ this means that $\eta_P = \pm 1$.

In the case of a quantum field theory P is represented by a unitary operator $\hat{P}$ acting on the Fock space of particle states. For a quantum scalar field $\phi(x)$

the parity transformation becomes

$$\hat{P}\phi(x)\hat{P}^{-1} = \eta_P \phi(x_P) \ . \tag{2}$$

Since a quantum state is arbitrary up to a phase factor, we need only require that $|\eta_P| = 1$. However, in practice one may always definite $\eta_P = \pm 1$. $\eta_P = -1$ is called *pseudo-scalar*.

1.2 Vector Field

For vector fields, the spatial components should reverse sign under parity, so one has

$$\hat{P}V^\mu(x)\hat{P}^{-1} = \eta_P V_\mu(x_P) \tag{3}$$

recalling that $V^0 = V_0$ and $V^i = -V_i$ for $i \in \{1, 2, 3\}$. If V^μ is hermitian, $\eta_P = \pm 1$ and $\eta_P = -1$ is called *pseudo-vector* or *axial-vector*.

1.3 Dirac Field

The Dirac quantum field transforms under parity as

$$\hat{P}\psi(x)\hat{P}^{-1} = \eta_P \psi^P(x) = \eta_P \gamma^0 \psi(x_P) \ , \qquad \hat{P}\overline{\psi}(x)\hat{P}^{-1} = \eta_P \overline{\psi}(x_P)\gamma^0 \ . \tag{4}$$

We have required that the transformed field also satisfies the Dirac equation, which is the equation of motion of the field. (It is not enough simply to invert the spatial coordinates $\mathbf{x}$ of the field $\psi(x)$.) To show this we have, letting $\mathbf{x} \to -\mathbf{x}$, since $\gamma^\mu \partial_\mu = \gamma^0 \partial_t + \gamma{\cdot}\nabla$,

$$\Big(i\gamma^0 \partial_t - i\gamma{\cdot}\nabla - m\Big)\,\psi(x_P) = 0 \ . \tag{5}$$

Now since $\gamma^0(\gamma^0, \gamma)\gamma^0 = (\gamma^0, -\gamma)$ and $(\gamma^0)^2 = I$, or $(\gamma^0)^{-1} = \gamma^0$, it is straightforward to see that, with the definition of ψ^P in (4),

$$(i\gamma^\mu \partial_\mu - m)\,\psi^P(x) = 0 \ , \tag{6}$$

as required.

Under the parity transformation the positive energy Dirac spinor of momentum p transforms as

$$u(p, \lambda)e^{-ip.x} \to \gamma^0 u(p, \lambda)e^{-ip.x_P} = u(p_P, \lambda)e^{-ip_P.x} \ , \tag{7}$$

using

$$\gamma^0 u(p, \lambda) = u(p_P, \lambda) \ , \tag{8}$$

That is, the spatial part of the momentum has been reflected but the spin state has been left unaltered which is just what is expected from a parity transformation.

Using the properties of γ^0 it is easy to verify that under P bi-linears in fermion fields transform as

$$\begin{array}{lcll} \overline{\psi}(x)\psi(x) & \to & \overline{\psi}(x_P)\psi(x_P) & \text{scalar,} \\ \overline{\psi}(x)\gamma_5\psi(x) & \to & -\overline{\psi}(x_P)\gamma_5\psi(x_P) & \text{pseudoscalar,} \\ \overline{\psi}(x)\gamma^0\psi(x) & \to & \overline{\psi}(x_P)\gamma^0\psi(x_P) & \text{charge density,} \\ \overline{\psi}(x)\boldsymbol{\gamma}\psi(x) & \to & -\overline{\psi}(x_P)\boldsymbol{\gamma}\psi(x_P) & \text{current density.} \end{array} \tag{9}$$

2 Charge Conjugation

The operation of charge conjugation C exchanges particles and anti-particles.

2.1 Scalar Field

A scalar quantum field $\phi(x)$ has the decomposition in terms of creation and annihilation operators

$$\phi(x) = \sum_p \left[a(p)e^{-ip.x} + b(p)^\dagger e^{ip.x}\right] \ , \tag{10}$$

where $a(p)$ annihilates particles and $b(p)$ creates anti-particles of momentum p. Acting on the Fock space we require a unitary transformation $\hat{C}$ such that for a general single particle state $\hat{C}|p, \text{particle}\rangle = |p, \text{anti-particle}\rangle$. This is achieved by requiring $\hat{C}|0\rangle = |0\rangle$ and $\hat{C}a(p)\hat{C}^{-1} = b(p)$. Assuming also $\hat{C}b(p)\hat{C}^{-1} = a(p)$ then

$$\hat{C}\phi(x)\hat{C}^{-1} = \phi(x)^\dagger \ . \tag{11}$$

We have also

$$\hat{C}\phi(x)^\dagger\hat{C}^{-1} = \phi(x) \ . \tag{12}$$

2.2 Dirac Field

The charge conjugation operation on the Dirac field must again interchange particles and anti-particles. The transformation therefore involves the hermitian conjugation of the quantum field. However we define the field after charge conjugation so that it satisfies the Dirac equation.

In order to find the correct transformation property we use the following notational conventions:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \ , \quad \psi(x)^* = \begin{pmatrix} \psi_1(x)^\dagger \\ \psi_2(x)^\dagger \\ \psi_3(x)^\dagger \\ \psi_4(x)^\dagger \end{pmatrix} \ , \tag{13}$$

and

$$\psi(x)^\dagger = \left(\psi_1(x)^\dagger, \ \psi_2(x)^\dagger, \ \psi_3(x)^\dagger, \ \psi_4(x)^\dagger\right) \ . \tag{14}$$

and so $\overline{\psi}(x) = \psi(x)^\dagger\gamma^0$. $\psi_\alpha(x)^\dagger$ is the hermitian conjugate of a single component of the spinor.

Under charge conjugation we assume

$$\psi(x) \longrightarrow \psi^C(x) \ , \quad \psi^C(x) = C\overline{\psi}(x)^t \ = C(\gamma^t)^0\psi(x)^*, \tag{15}$$

with t denoting transpose. The matrix C is then chosen to ensure $\psi^C(x)$ satisfies the Dirac equation. Earlier we saw that the Dirac equation for $\bar{\psi}(x)$ was

$$\bar{\psi}(x)(-i\gamma\cdot\overleftarrow{\partial} - m) = 0. \tag{16}$$

So taking the transpose of this we immediately obtain

$$\Big(-i(\gamma^t)^\mu\partial_\mu - m\Big)\,\overline{\psi}(x)^t = 0 \ , \tag{17}$$

and so

$$\Big(-iC(\gamma^\mu)^t C^{-1}\partial_\mu - m\Big)\,\psi^C(x) = 0 \ . \tag{18}$$

Assuming C satisfies

$$C(\gamma^\mu)^t C^{-1} = -\gamma^\mu \ , \tag{19}$$

then from (18)

$$(i\gamma^\mu\partial_\mu - m)\,\psi^C(x) = 0 \ , \tag{20}$$

as required. From (19) we can further straightforwardly obtain

$$C{\gamma_5}^t C^{-1} = \gamma_5 \ , \quad C(\gamma^\mu\gamma_5)^t C^{-1} = \gamma^\mu\gamma_5 \ . \tag{21}$$

In the Dirac representation we have an explicit form for C

$$C = i\gamma^0\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \ . \tag{22}$$

Charge conjugation on the quantum Dirac field is then given by

$$\begin{aligned} \hat{C}\psi(x)\hat{C}^{-1} &= C\overline{\psi}(x)^t \equiv \psi^C(x) \ , \\ \hat{C}\overline{\psi}(x)\hat{C}^{-1} &= -\psi(x)^t C^{-1} \ . \end{aligned} \tag{23}$$

A particularly important operator is the electric current $j_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$. We have

$$\begin{aligned} \hat{C}j_\mu(x)\hat{C}^{-1} &= \hat{C}\overline{\psi}_\alpha(x)\hat{C}^{-1}(\gamma_\mu)_{\alpha\beta}\hat{C}\psi_\beta(x)\hat{C}^{-1} \\ &= -(\psi(x)^t C^{-1})_\alpha(\gamma_\mu)_{\alpha\beta}(C\overline{\psi}(x)^t)_\beta \\ &= -\psi_{\alpha'}(x)(C^{-1}\gamma_\mu C)_{\alpha'\beta'}\overline{\psi}_{\beta'}(x) \ . \end{aligned} \tag{24}$$

But $C^{-1}\gamma_\mu C = -{\gamma_\mu}^t$ so therefore

$$\begin{aligned} \hat{C}j_\mu(x)\hat{C}^{-1} &= \psi_{\alpha'}(x)(\gamma_\mu)_{\beta'\alpha'}\overline{\psi}_{\beta'}(x) \\ &= -\overline{\psi}_{\beta'}(x)(\gamma_\mu)_{\beta'\alpha'}\psi_{\alpha'}(x), \end{aligned} \tag{25}$$

using the anticommutation property of the fermion fields. Hence

$$\hat{C}j_\mu(x)\hat{C}^{-1} = -j_\mu(x) \ . \tag{26}$$

Hence the current four-vector changes sign under charge conjugation, which is as we would expect since particles and antiparticles have opposite charges. For the axial current

$$\begin{aligned}\hat{C}j^A_\mu(x)\hat{C}^{-1} &= \hat{C}\overline{\psi}_\alpha(x)\hat{C}^{-1}(\gamma_\mu\gamma_5)_{\alpha\beta}\hat{C}\psi_\beta(x)\hat{C}^{-1} \\ &= -(\psi(x)^t C^{-1})_\alpha(\gamma_\mu\gamma_5)_{\alpha\beta}(C\overline{\psi}(x)^t)_\beta \\ &= -\psi_{\alpha'}(x)(C^{-1}\gamma_\mu\gamma_5 C)_{\alpha'\beta'}\overline{\psi}_{\beta'}(x)\ .\end{aligned} \tag{27}$$

But $C^{-1}\gamma_\mu\gamma_5 C = (\gamma_\mu\gamma_5)^t$, so therefore

$$\begin{aligned}\hat{C}j^A_\mu(x)\hat{C}^{-1} &= -\psi_{\alpha'}(x)(\gamma_\mu\gamma_5)_{\beta'\alpha'}\overline{\psi}_{\beta'}(x) \\ &= +\overline{\psi}_{\beta'}(x)(\gamma_\mu\gamma_5)_{\beta'\alpha'}\psi_{\alpha'}(x) \\ &= \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x)\ = j^A_\mu(x)\ .\end{aligned} \tag{28}$$

We note that the axial current is invariant under charge conjugation whereas the vector current changes sign. Theories where linear combinations of vector and axial currents appear, e.g. the electroweak sector of the standard model, where we have differentiated between left and right-handed currents, will not be invariant under charge conjugation.

2.3 Vector Field

We can work out the charge conjugation properties of the photon by imposing that the electromagnetic interaction $j^\mu(x)A_\mu(x)$ is invariant under charge conjugation. This implies that

$$\hat{C}A_\mu(x)\hat{C}^{-1} = -A_\mu(x). \tag{29}$$

For a general quantum vector field $V_\mu(x)$ this becomes

$$\hat{C}V_\mu(x)\hat{C}^{-1} = \eta_C V_\mu(x)^\dagger\ . \tag{30}$$

An N photon state therefore has charge conjugation $(-1)^N$ and a π^0 meson (which has charge conjugation $+1$) can decay to two photons but not three, assuming charge conjugation is an exact symmetry of electromagnetic and strong interactions.

3 Time Reversal

This interchanges initial and final states with identical positions but opposite velocities and hence momenta. $\hat{T}$ must be defined as an anti-linear transformation, i.e. it complex conjugates scalar products. To see why we consider the action of time-reversal on a matrix element for initial and final states $|i\rangle$ and $|f\rangle$, i.e. $\langle f|S^\dagger|i\rangle$. Using the notation $|i_T\rangle = \hat{T}|i\rangle$ etc. we have

$$\langle f|S^\dagger|i\rangle = \langle f_T|{S_T}^\dagger|i_T\rangle^*, \tag{31}$$

that is

$$\langle f|S^\dagger|i\rangle = \langle i_T|S_T|f_T\rangle \ . \tag{32}$$

Under $\hat{T}$ symmetry of the Hamiltonian one can show that since $S = \exp(-i\int_{t_i}^{t_f} H_{\rm int}\, dt)$ and $(H_{\rm int})^\dagger = H_{\rm int}$, then $S_T = S^\dagger$ therefore

$$\langle f|S^\dagger|i\rangle = \langle i_T|S^\dagger|f_T\rangle \ . \tag{33}$$

In turn this implies that the probabilities, rates or cross-sections are equal for two processes related by time-reversal, and we obtain this desired result by demanding that the time-reversal transformation is **anti-linear**.

For time reversal of quantum scalar fields

$$\hat{T}\phi(x)\hat{T}^{-1} = \phi(x_T). \tag{34}$$

For the Dirac field itself, we have a complicated transformation, but under T are

$$\hat{T}\overline{\psi}(x)\psi(x)\hat{T}^{-1} = \overline{\psi}(x_T)\psi(x_T) \ , \tag{35}$$

and for the electric current

$$\hat{T}j^0(x)\hat{T}^{-1} = j^0(x_T) \qquad \text{and} \qquad \hat{T}\mathbf{j}(x)\hat{T}^{-1} = -\mathbf{j}(x_T) \ . \tag{36}$$

Time-reversal therefore leaves the charge density unchanged but reverses the flow of the current, as one might expect.

For vector fields one has

$$\hat{T}V^\mu(x)\hat{T}^{-1} = \eta_T V_\mu(x_T) \ . \tag{37}$$

This maintains the invariance of the interaction term with the current.

4 CP Violation

Many quantum field theories are invariant under C, P and T separately, e.g. QED, QCD. However, it is straightforward to check that C and P are violated for chiral interactions:

$$\begin{aligned}
\mathcal{L}_{int}(x) &= \overline{\psi}(x)\gamma^\mu(1-\gamma^5)\psi(x)A_\mu(x) && (38)\\
&\equiv \overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) - \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)A_\mu(x) && (39)\\
&= j_V^\mu(x)A_\mu(x) - j_A^\mu(x)A_\mu(x), && (40)
\end{aligned}$$

where $j_V^\mu(x)$ is the vector current and $j_A^\mu(x)$ is the axial current. Under C

$$\begin{aligned}
A_\mu(x) &\to -A_\mu(x) && (41)\\
j_V^\mu(x) &\to -j_V^\mu(x) && (42)\\
j_A^\mu(x) &\to j_A^\mu(x), && (43)
\end{aligned}$$

and hence

$$\mathcal{L}_{int}(x) \to j_V^\mu(x)A_\mu(x) + j_A^\mu(x)A_\mu(x). \tag{44}$$

However, under P

$$\begin{aligned} A_\mu(x) &\to A^\mu(x) && (45) \\ j_V^\mu(x) &\to j_{\mu V}(x) && (46) \\ j_A^\mu(x) &\to -j_{\mu A}(x), && (47) \end{aligned}$$

and therefore

$$\mathcal{L}_{int}(x) \to j_V^\mu(x)A_\mu(x) - j_A^\mu(x)A_\mu(x) \equiv \mathcal{L}_{int}(x_P). \tag{48}$$

Hence, $\mathcal{L}_{int}$ is violated by both C and P but is invariant under C and P combined.

Most of the terms in the Standard Model Lagrangian are invariant under C and P or under CP, but consider the quark coupling to the W

$$\mathcal{L}_{qW} = \sum_i \sum_j -\frac{g}{2\sqrt{2}}(\bar{u}_i\gamma^\mu(1-\gamma^5)V_{ij}d_jW_\mu + \bar{d}_j\gamma^\mu(1-\gamma^5)V_{ij}^*u_iW_\mu^\dagger). \tag{49}$$

Under C and P

$$\begin{aligned} \bar{u}_i\gamma^\mu(1-\gamma^5)d_jW_\mu &\to \bar{d}_j\gamma^\mu(1-\gamma^5)u_iW_\mu^\dagger && (50) \\ \bar{d}_j\gamma^\mu(1-\gamma^5)u_iW_\mu^\dagger &\to \bar{u}_i\gamma^\mu(1-\gamma^5)d_jW_\mu && (51) \end{aligned}$$

which leads to

$$\mathcal{L}_{qW} \to \sum_i \sum_j -\frac{g}{2\sqrt{2}}(\bar{u}_i\gamma^\mu(1-\gamma^5)V_{ij}^*d_jW_\mu + \bar{d}_j\gamma^\mu(1-\gamma^5)V_{ij}u_iW_\mu^\dagger). \tag{52}$$

This would be invariant if the mixing matrix were real, i.e $V_{ij} = V_{ij}^*$. However, this is not true for the CKM matrix for three fermion families which has a complex phase. Hence the quark coupling to the W particles violates CP in the Standard Model.

However, T leaves this interaction term invariant except that it interchanges V_{ij} and V_{ij}^* (from the fact that it complex conjugates c-numbers). Hence, the above term is invariant under the combined CPT transformation as is every other term in the Standard Model Lagrangian (all the rest being invariant under CP and T separately). It is a general theorem that any Lorentz invariant Lagrangian $\mathcal{L}(x)$ formed from products of quantum fields at the point x is invariant under CPT.

5 Neutrino Masses

We now know that neutrinos do have masses, albeit very small ones, or more precisely there is a very small difference between the neutrino masses, which from comparison with other lepton families we assume implies that the masses are of a similar size. If the neutrino mass eigenstates are labelled by 1,2,3, then

$$\Delta m_{12} \sim 10^{-5}\mathrm{eV} \qquad\qquad \Delta m_{23} \sim 10^{-3}\mathrm{eV}. \tag{53}$$

In order to obtain this neutrino mass we must introduce a right-handed neutrino field ν_R into the Standard Model. We might think that this is simply done in an analogous fashion to up-type quarks, i.e. the mass term is

$$\mathcal{L}_{lept,\phi} = -\sqrt{2}[\bar{L}_N f^{-}_{NM} \phi R_{M-} + \bar{L}_N f^{0}_{NM} \phi^c R_{M0} + \mathrm{h.c.}], \tag{54}$$

where $R_{M0} = (\nu_M)_R$.

However, for neutrinos there is an added complication. In the second term we have the hypercharge assignments $\bar{L}_N : Y = \frac{1}{2}$, $\phi^c : Y = -\frac{1}{2}$, and therefore $R_{M0} : Y = 0$. This is consistent since R_{M0} is also a weak singlet and we want Q for R_{M0} to be zero.

This means that the right-handed neutrino carries no quantum numbers at all and is automatically invariant under any gauge transformations. (It also means it is real.) This invariance allows it to appear in a completely new mass term which is forbidden for all other fermion fields in the Standard Model.

We have defined the charge conjugate field by $\psi^c(x) = C\gamma^{0t}\psi^*(x)$ (and $(\overline{\psi}(x))^c = -\psi(x)^t C^{-1}$), and this means we can have a Majorana mass term

$$-\tfrac{1}{2} M_m (\overline{\psi}_R)^c \psi_R + \mathrm{h.c.}\ , \tag{55}$$

where we make the definition

$$(\overline{\psi}_R)^c = (\overline{\psi}^c)\tfrac{1}{2}(1+\gamma^5). \tag{56}$$

This type of term would not usually be allowed since $(\overline{\psi}_R)^c$ and ψ_R carry the same quantum numbers, rather than opposite, so the term would not be gauge invariant. However, here there is no gauge transformation for the field, and hence no problem.

For a single neutrino the complete mass term in the Lagrangian is thus

$$\mathcal{L}_m = -M_D \overline{\psi}_L \psi_R - \tfrac{1}{2} M_m (\overline{\psi}_R)^c \psi_R + \mathrm{h.c.} \tag{57}$$

It is possible to show that the Lagrangian can be written such that we obtain a mass matrix $\mathcal{M} = \begin{pmatrix} 0 & M_D \\ M_D & M_m \end{pmatrix}$. If M_D is real and positive and $|M_m| \gg M_D$, diagonalizing the matrix results in masses which are approximately $|M_m|$ and $M_D^2/|M_m|$, with corresponding eigenstates approximately $(\psi_R)^c$ and ψ_L respectively. Therefore the heavy neutrino is more or less neutral, and decouples from the rest of the physics, and the light neutrino is very light and is the one which takes part in the weak interactions. This scenario is known as the "seesaw mechanism" and often assumes that M_D is similar to the masses of the charged leptons, whereas $M_m \sim 10^{15}$GeV (justified by many beyond the Standard Model theories). This is then a way of justifying the very small masses of the neutrinos we observe.

When we consider the full three families of neutrinos we have to diagonalize both the 2-d matrices incorporating the mixing between Dirac and Majorana mass terms and the 3-d matrices which exist in family space, i.e. the elements M_D above actually come from the f^0_{NM} in eq.(54) and there will be an equivalent 3×3 matrix for M_m. This diagonalization in family space will lead to unitary matrices equivalent to the U and V matrices defined in section 3.1 for the quark sector, and because these will not be identical for the charged leptons and neutrinos, expressing the charged weak current for leptons

$$\begin{aligned} J^\mu &= 2\bar{L}_N \gamma^\mu \sigma_+ L_N \\ &= 2 \begin{pmatrix} \bar{\nu}\tau & \bar{\nu}\mu & \bar{\nu}e \end{pmatrix}_L \gamma^\mu \begin{pmatrix} \tau' \\ \mu' \\ e' \end{pmatrix}_L , \end{aligned} \tag{58}$$

in terms of lepton mass eigenstates will, as for the quarks, lead to a mixing matrix

$$J^\mu = \begin{pmatrix} \bar{\nu}_3 & \bar{\nu}_2 & \bar{\nu}_1 \end{pmatrix} \gamma^\mu (1-\gamma_5) V_{lept} \begin{pmatrix} \tau \\ \mu \\ e \end{pmatrix} . \tag{59}$$

The parameters of this matrix are determined by neutrino oscillation experiments, and the mixing angles are quite large. As for the quarks the matrix leads to CP-violation. Note that this time the mixing is associated with the neutrinos by convention, i.e. charged leptons are simultaneously mass and weak eigenstates, whereas neutrinos are not, while in the quark sector it was the down type quarks which were defined to mix.