Higgs production via gluon fusion at next-to-leading order

Calculation notes
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1 Introduction

At hadron colliders, neutral Higgs bosons can be produced via gluon-gluon fusion, gg → H. The gluons do not couple directly to the Higgs boson, since they are massless. They do however interact with particles carrying colour charge (quarks and gluons), of which the quarks have a non-zero mass. The Higgs boson production via gluon fusion is therefore possible through a quark loop, with the Higgs boson coupling to the quarks via a Yukawa-type interaction. The strength of the Yukawa coupling is proportional to the quark mass. It turns out that the dominant contribution to the quark loop comes from the top quark, since it is by far the most massive quark. This is not immediately evident, because while the coupling of the Higgs boson to a heavier particle is stronger, the probability of producing a heavier particle in the loop is lower. All other flavours contribute little to the amplitude. (The top quark is about 40 times as heavy as the next heaviest known quark, the beauty quark.)

The goal here is to calculate and study the cross section of Higgs boson production via gluon fusion first at leading, then at next-to-leading order.

2 The process gg → H at leading order

Before doing the next-to-leading order calculation, we calculate the cross section of the process at leading order.

2.1 Writing the amplitude

At leading order, there are two Feynman diagrams that contribute to the process. They are shown in Figure 2.1. While the diagrams are topologically distinct, they contribute the same amount to the total amplitude by symmetry.

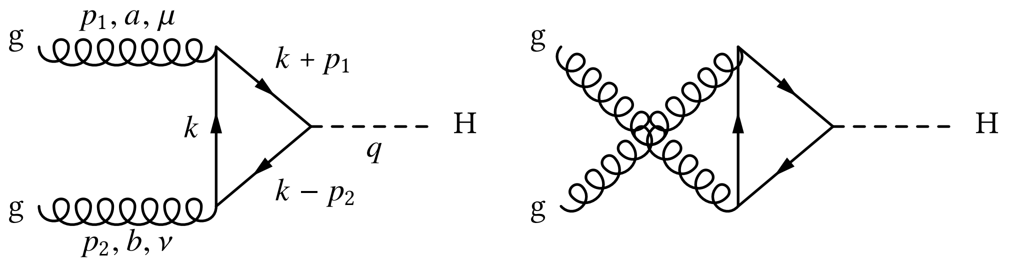

Figure 2.1: The lowest (non-trivial) order diagrams for Higgs boson production via gluon fusion. Their corresponding amplitudes are equal. The labels characterising the particles in the diagram to the right are the same as to the left.

Feynman rules

External gluon:	$\varepsilon_\mu(p)$
External scalar:	1
Fermion loop:	-1
Fermion propagator:	$\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$
Quark-gluon vertex:	$ig\gamma^\mu t^a$
Yukawa vertex:	$-i\lambda$

Amplitude

We make the approximation that only the contribution due to the top quark matters. We denote the mass of the top quark m.

$$
\begin{aligned}
i\mathcal{M} = -2\int \frac{d^4k}{(2\pi)^4}\, 1(-i\lambda)\frac{i(\not{k} + \not{p}_1 + m)}{(k + p_1)^2 - m^2 + i\varepsilon}(ig\gamma^\mu t^a) \\
\times \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon}(ig\gamma^\nu t^b)\frac{i(\not{k} - \not{p}_2 + m)}{(k - p_2)^2 - m^2 + i\varepsilon}\varepsilon_\mu(p_1)\varepsilon_\nu(p_2)
\end{aligned}
\tag{2.1}
$$

Because all the fermion lines in the diagrams form closed loops, the amplitude has no free Dirac indices. This means that we can take the trace of the Dirac matrices, which simplifies the expression considerably, since the trace of any odd number of Dirac matrices vanishes. Writing the trace and reordering the commuting factors gives

$$
\begin{aligned}
i\mathcal{M} = -2\lambda g^2\, \varepsilon_\mu(p_1)\varepsilon_\nu(p_2)t^a t^b \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\Big[\frac{(\not{k} + \not{p}_1 + m)}{(k + p_1)^2 - m^2 + i\varepsilon}\gamma^\mu \\
\times \frac{(\not{k} + m)}{k^2 - m^2 + i\varepsilon}\gamma^\nu \frac{(\not{k} - \not{p}_2 + m)}{(k - p_2)^2 - m^2 + i\varepsilon}\Big].
\end{aligned}
\tag{2.2}
$$

2.2 Calculating the loop integral

Denominator of the integrand

The integrand in Equation (2.2) can be simplified using the method of Feynman parameters[1] to combine the three factors in the denominator. Specifically, we use the identity

$$\frac{1}{ABC} = \int_0^1 dx\, dy\, dz\, \frac{2!}{[xA + yB + yC]^3} \tag{2.3}$$

to write

$$\frac{1}{((k+p_1)^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)((k-p_2)^2 - m^2 + i\varepsilon)} = \int_0^1 dx\, dy\, dz\, \frac{2}{D^3}, \tag{2.4}$$

where

$$\begin{aligned} D &= x(k+p_1)^2 + yk^2 + z(k-p_2)^2 - (x+y+z)m^2 + (x+y+z)i\varepsilon \\ &= k^2 + 2k\cdot(xp_1 - zp_2) - m^2 + i\varepsilon. \end{aligned} \tag{2.5}$$

In simplifying the expression for D the identity $x+y+z = 1$ and the fact that gluons are massless ($p_1^2 = p_2^2 = 0$) have been used. Since all values of the momentum k are integrated over, we can shift it to $\ell = k + xp_1 - zp_2$. This completes the square in D, which simply becomes

$$\begin{aligned} D &= \ell^2 + 2xz\, p_1\cdot p_2 - m^2 + i\varepsilon \\ &= \ell^2 + xzm_H^2 - m^2 + i\varepsilon \\ &= \ell^2 - \Delta + i\varepsilon, \end{aligned} \tag{2.6}$$

where m_H is the mass of the Higgs boson and $\Delta \equiv m^2 - xzm_H^2$ can be thought of as an effective mass of the fermion loop, if the loop is treated like an effective fermion propagator.

Before moving on, we can already make a simplification. Since ℓ does not contain the Feynman parameter y, the integrand I will not depend on y either. Therefore we can get rid of y right away by making the replacement

$$\int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) I(x,z) = \int_0^1 dx \int_0^{1-x} dz\, I(x,z) \tag{2.7}$$

Numerator of the integrand

Having dealt with the denominator, we now turn to the numerator of the integrand in Equation (2.2). We need to express it in terms of the new variable of integration, ℓ. From the fact that D only depends on the magnitude of ℓ it follows that[2]

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0; \tag{2.8}$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{4} g^{\mu\nu}\ell^2}{D^3}. \tag{2.9}$$

[1] See for instance Peskin & Schroeder, p. 189ff.

[2] Peskin & Schroeder, p. 191

The second integral is divergent in four dimensions. Therefore we need to modify it to make it well-defined. This is called *regularising*. We use dimensional regularisation, where the integral is calculated in d energy-momentum space dimensions (corresponding to d space-time dimensions). After the calculation, we will take the limit $d \to 4$ to recover our physical space-time. In d dimensions, the analog of Equation (2.9) is

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{d} g^{\mu\nu} \ell^2}{D^3}. \tag{2.10}$$

Based on this, we see that we can bring the integral into the form

$$A \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{D^3} + B \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{D^3}, \tag{2.11}$$

where A and B are factors that do not depend on ℓ and can therefore be taken out of the integrals. Note that the integrands are now Dirac scalars, depending only on ℓ^2, while the entire Dirac structure is contained in A and B.

Making use of the linearity property of the trace and only keeping terms containing an even number of Dirac matrices, we find

$$\begin{aligned} \text{Numerator} = m \, \text{Tr}\big[&\not{p}_1 \gamma^\mu \not{k} \gamma^\nu - \not{p}_1 \gamma^\mu \gamma^\nu \not{p}_2 + \not{p}_1 \gamma^\mu \gamma^\nu \not{k} + \not{k} \gamma^\mu \not{k} \gamma^\nu \\ &- \not{k} \gamma^\mu \gamma^\nu \not{p}_2 + \not{k} \gamma^\mu \gamma^\nu \not{k} - \gamma^\mu \not{k} \gamma^\nu \not{p}_2 + \gamma^\mu \not{k} \gamma^\nu \not{k} + m^2 \gamma^\mu \gamma^\nu\big]. \end{aligned} \tag{2.12}$$

We insert $\not{k} = \not{\ell} - x\not{p}_1 + z\not{p}_2$ into this expression, simplify it using the identity $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$, and discard terms proportional to an odd power of ℓ, which vanish by virtue of Equation (2.9). We can also discard terms proportional to $p_i^\alpha p_i^\beta$ (with $i = 1, 2$). The reason is the following. Since the matrix element is a Dirac scalar, all Dirac indices will be contracted. For terms of the form just mentioned, there are only two ways in which this can be done. They will either be proportional to p_i^2, which is zero because the gluons are massless, or to $p_i \cdot \varepsilon(p_i)$, which is again zero due to the fact that the external gluons can only have polarizations transverse to their momentum. All in all, we find

$$\begin{aligned} \text{Numerator} \to m \, \text{Tr}\Big[& z\not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu - \not{p}_1 \gamma^\mu \gamma^\nu \not{p}_2 - z\not{p}_1 \gamma^\mu \gamma^\nu \not{p}_2 \\ &+ (\not{\ell} - x\not{p}_1 + z\not{p}_2)\gamma^\mu(\not{\ell} - x\not{p}_1 + z\not{p}_2)\gamma^\nu + x\not{p}_1 \gamma^\mu \gamma^\nu \not{p}_2 \\ &+ (\not{\ell} - x\not{p}_1 + z\not{p}_2)\gamma^\mu \gamma^\nu (\not{\ell} - x\not{p}_1 + z\not{p}_2) + x\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2 \\ &+ \gamma^\mu (\not{\ell} - x\not{p}_1 + z\not{p}_2)\gamma^\nu (\not{\ell} - x\not{p}_1 + z\not{p}_2) + m^2 \gamma^\mu \gamma^\nu\Big] \\ \to m \, \text{Tr}\Big[& \ell_\sigma \ell_\rho (\gamma^\sigma \gamma^\mu \gamma^\rho \gamma^\nu + \gamma^\sigma \gamma^\mu \gamma^\nu \gamma^\rho + \gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\rho) \\ &- p_{1,\alpha} p_{2,\beta} \Big\{ -z\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu - x\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \\ &+ xz(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu + \gamma^\beta \gamma^\mu \gamma^\alpha \gamma^\nu + \gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\beta \\ &+ \gamma^\beta \gamma^\mu \gamma^\nu \gamma^\alpha + \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta + \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\alpha) \\ &+ (1 - z - x)\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\beta \Big\} + m^2 \gamma^\mu \gamma^\nu \Big]. \end{aligned} \tag{2.13}$$

Now we can calculate the traces the usual way, leading to the numerator

$$4m\left(4\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu} + (4xz - 1)p_1^\nu p_2^\mu. + (1/2 - xz)m_H^2 g^{\mu\nu} + m^2 g^{\mu\nu}\right). \tag{2.14}$$

The entire amplitude then becomes

$$
\begin{aligned}
i\mathcal{M} = -8m\lambda g^2 \varepsilon_{1,\mu}\varepsilon_{2,\nu} t^a t^b \int_0^1 dx \int_0^{1-x} dz \bigg(g^{\mu\nu} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta+i\varepsilon)^3} \frac{4-d}{d} \\
- \Big((4xz-1)p_1^\nu p_2^\mu + (1/2-xz)m_H^2 g^{\mu\nu} - m^2 g^{\mu\nu} \Big) \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta+i\varepsilon)^3} \bigg),
\end{aligned}
\tag{2.15}
$$

where $\varepsilon_i \equiv \varepsilon(p_i)$. The values of the integrals can be found in tables. The second integral is

$$
\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta+i\varepsilon)^3} = \frac{-i}{32\pi^2\Delta}, \tag{2.16}
$$

which is finite. The first integral might lead to terms diverging in the limit $d \to 4$. It gives

$$
\int \frac{d^d\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \tag{2.17}
$$

We expand the gamma function and the factor containing Δ^{-1} near $d = 4$, using the definition $\varepsilon \equiv 4 - d$:

$$
\Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon), \tag{2.18}
$$

where $\gamma \approx 0.58$ is the Euler-Mascheroni constant, and

$$
\left(\frac{1}{\Delta}\right)^{\varepsilon} = \left(\frac{1}{\Delta}\right)^{0} + \left(\frac{1}{\Delta}\right)^{0} \log\left(\frac{1}{\Delta}\right)\varepsilon + \mathcal{O}(\varepsilon^2) = 1 - \varepsilon\log\Delta + \mathcal{O}(\varepsilon^2). \tag{2.19}
$$

Using this, we find

$$
\begin{aligned}
\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^3} \frac{4-d}{d} &= \frac{i}{(4\pi)^{d/2}} \frac{d}{4} \left(\frac{2}{\varepsilon} - \gamma\right) (1 - \varepsilon\log\Delta) \frac{\varepsilon}{d} + \mathcal{O}(\varepsilon) \\
&= \frac{i}{2(4\pi)^2} + \mathcal{O}(\varepsilon),
\end{aligned}
\tag{2.20}
$$

which has the finite value $i/32\pi^2$ as $\varepsilon \to 0$. In the end, there were no divergences arising in the loop calculation. We simplify the term that we are left with.

$$
\begin{aligned}
& i\frac{\varepsilon_1\cdot\varepsilon_2}{32\pi^2} + i\frac{(4xz-1)(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1) + \left((1/2-xz)m_H^2 - m^2\right)\varepsilon_1\cdot\varepsilon_2}{32\pi^2\left(m^2 - xzm_H^2\right)} \\
&= i\frac{(4xz-1)(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1) + \left((1/2-2xz)m_H^2\right)\varepsilon_1\cdot\varepsilon_2}{32\pi^2\left(m^2 - xzm_H^2\right)} \\
&= \frac{i}{32\pi^2}\left(\frac{\varepsilon_1\cdot\varepsilon_2}{2}m_H^2 - (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)\right)\frac{1-4xz}{m^2 - xzm_H^2} \\
&= \frac{i}{32\pi^2}\frac{\tau}{m^2}\left(\frac{\varepsilon_1\cdot\varepsilon_2}{2}m_H^2 - (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)\right)\frac{1-4xz}{\tau - 4xz},
\end{aligned}
\tag{2.21}
$$

where $\tau \equiv 4m^2/m_H^2$. With the momentum integral carried out, the amplitude becomes

$$i\mathcal{M} = \frac{-i\tau\lambda g^2 t^a t^b}{4\pi^2 m^2}\left(\frac{\varepsilon_1\cdot\varepsilon_2}{2}m_H^2 - (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)\right)\int_0^1 dx \int_0^{1-x} dz \frac{1-4xz}{\tau - 4xz}. \quad (2.22)$$

The integration over the Feynman parameter z can be carried out by parts,

$$\begin{aligned}\int_0^{1-x} dz\frac{1-4xz}{\tau-4xz} &= \left[(4xz-1)\frac{\log|\tau-4xz|}{4x}\right]_0^{1-x} - \int_0^{1-x}\log|\tau-4xz| \\ &= \frac{\tau-1}{4x}\log\left[1-\frac{4x(1-x)}{\tau}\right] + x - 1.\end{aligned}$$

In the integration the fact that τ is positive and greater than $4xz$ (because $m > m_H$ and $x, z \leq 1$) has been used, so absolute values do not need to be taken explicitly.

The last thing that remains to be done is the integration over the Feynman parameter x. The integral has three terms, of which two are very easy and one very difficult to integrate. The easy terms give $\int_0^1 dx(x-1) = -\frac{1}{2}$. The difficult integral can be solved using dilogarithms. The result can also be found in the literature.It is

$$\frac{\tau-1}{4}\int_0^1 \frac{dx}{x}\log\left[1-\frac{4x(1-x)}{\tau}\right] = \frac{\tau-1}{2}\arcsin^2\sqrt{\frac{1}{\tau}}, \quad (2.23)$$

which holds for $\tau > 1$ (as is the case here).

2.3 Result for the amplitude

Colour factors

The amplitude is proportional to $t^a t^b$, where a and b are indices of the adjoint representation (describing gluons) of the SU(3) colour symmetry group. Explicitly writing the indices of the fundamental representation (describing quarks), this factor is $t^a_{ij} t^b_{ji}$ due to conservation of colour charge. Repeated indices are summed over. It can be shown (though I'm too lazy to do it) that this corresponds to taking the trace of the matrices t in the fundamental representation space,

$$t^a t^b \to \mathrm{Tr}\left[t^a t^b\right] = C(N)\delta_{ab} = \frac{\delta_{ab}}{2}. \quad (2.24)$$

The values of the coupling constants

The Yukawa coupling constant is $\lambda = -im/v$, where $v \approx 246$ GeV is the vacuum expectation value of the Higgs field. The strong coupling constant is $\alpha_s = g^2/4\pi$.

Putting everything together

The final result for the amplitude is

$$i\mathcal{M} = \frac{\alpha_s\delta_{ab}}{4\pi v m}\left(\frac{\varepsilon_1\cdot\varepsilon_2}{2}m_H^2 - (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)\right)\left(\tau - \tau(\tau-1)\arcsin^2\sqrt{1/\tau}\right). \quad (2.25)$$

2.4 Unpolarised cross section

We wish to calculate the absolute value squared of the amplitude, averaging over the different possible spin polarisation and colour states of the incoming gluons.

The sum over physical spin polarisation states of a gluon is

$$\sum_{\text{physical polarisations}} \varepsilon_\mu(k)\varepsilon_\nu^*(k) = -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{n \cdot k} + n^2 \frac{k_\mu k_\nu}{(n \cdot k)^2}, \tag{2.26}$$

where n is a four-vector that is linearly independent of the gluon's momentum k but otherwise arbitrary. We take n to be equal to p_1 for the gluon with momentum p_2 and vice versa. This is a smart choice, as it will lead to cancellation and vanishing of terms earlier on in the calculation. The spin polarisation sums in our case are

$$\begin{aligned}
&\sum \left(\frac{m_H^2}{2}\varepsilon_{1,\mu}\varepsilon_2^\mu - p_{1,\nu}\varepsilon_2^\nu p_{2,\xi}\varepsilon_1^\xi\right)\left(\frac{m_H^2}{2}\varepsilon_{1,\rho}\varepsilon_2^\rho - p_{1,\sigma}\varepsilon_2^\sigma p_{2,\tau}\varepsilon_1^\tau\right)^* \\
&= \frac{m_H^4}{4}\sum \varepsilon_{1,\mu}\varepsilon_{1,\rho}^* \sum \varepsilon_2^\mu \varepsilon_2^{*\rho} - m_H^2 p_{1,\nu}p_{2,\xi}\sum \varepsilon_1^\xi \varepsilon_{1,\rho}^* \sum \varepsilon_2^\nu \varepsilon_2^{*\rho} \\
&\quad + p_{1,\nu}p_{2,\xi}p_{1,\sigma}p_{2,\tau}\sum \varepsilon_1^\xi \varepsilon_1^{*\tau}\sum \varepsilon_2^\nu \varepsilon_2^{*\sigma} \\
&= \frac{m_H^4}{2}
\end{aligned} \tag{2.27}$$

(A remark: this result is the same as we would have obtained if the term proportional to $(p_1 \cdot \varepsilon_2)(p_2 \cdot \varepsilon_1)$ had not been present in the amplitude, Equation (2.25). Their corresponding terms in the polarisation sum vanish separately. Apparently, when averaging over all possible polarisations, the average projections of polarisations are zero on average.) After evaluating the polarisation sums, we divide the amplitude-squared by the square of the number of possible gluon polarisations, 2^2 to obtain the averaging factor, $m_H^4/8$.

Averaging over the possible colour states in the amplitude-squared gives a factor of

$$\frac{\delta_{ab}\delta_{ab}}{2^2 \times 8^2} = \frac{8}{4 \times 64} = \frac{1}{32}. \tag{2.28}$$

Thus we obtain the unpolarised, colour-averaged amplitude-squared

$$\overline{|\mathcal{M}|^2} = \frac{\alpha_s^2 m_H^2}{(32\pi)^2 v^2}\tau\left(1 - (\tau - 1)\arcsin^2\sqrt{1/\tau}\right)^2 \tag{2.29}$$

Kinematics of the decay: phase space integral

In order to arrive at the cross section $\sigma(\mathrm{gg} \to \mathrm{H})$ based on the amplitude, the kinematics of the decay need to be considered, summing all allowed kinematical configurations of the final state. This corresponds to calculating the phase space integral in

$$\sigma(\mathrm{gg} \to \mathrm{H}) = \frac{1}{\mathcal{F}}\int \frac{d^3q}{(2\pi)^3 2E_H}(2\pi)^4\delta^4\left(q - p_1 - p_2\right)\overline{|\mathcal{M}|^2}, \tag{2.30}$$

where q and E_H are the momentum and energy of the Higgs boson, and $\mathcal{F} = 2s$ is a flux factor. Since the unpolarised amplitude squared is independent of external momenta, the phase space integration simply gives a numerical factor,

$$\int \frac{d^3q}{(2\pi)^3 2E_H}(2\pi)^4\delta^4(q - p_1 - p_2) = 2\pi\delta(s - m_H^2), \tag{2.31}$$

where the Mandelstam variable $s = (p_1 + p_2)^2 = 2p_1 \cdot p_2$ appears.

The cross section is

$$\sigma(\mathrm{gg} \to \mathrm{H}) = \frac{\pi}{s}\delta(s - m_H^2)\overline{|\mathcal{M}|^2} \tag{2.32}$$

What happens if the top mass is considered to be infinite?
We study the cross section in the limit of $m \to \infty$. This corresponds to the limit $\tau \to \infty$. Using the limits

$$\lim_{\tau\to\infty} \tau \arcsin^2 \sqrt{1/\tau} = 1; \quad \lim_{\tau\to\infty} \arcsin^2 \sqrt{1/\tau} = 0, \tag{2.33}$$

we find

$$\lim_{m\to\infty} \overline{|\mathcal{M}|^2} \tag{2.34}$$

3 Going to next-to-leading order