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Advanced Quantum Field Theory

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Contents

1. Recommended reading

The following textbooks may be useful, and in many places the notes borrow heavily from them:

- Srednicki, *Quantum Field Theory*. This is a very nice textbook, that deals with a lot of (sometimes quite advanced) material in a pedagogical and clear way. The book is divided into (often short) chapters and is generally accessible. However the approach taken is not always optimal, and the author often seems to make a point of choosing the opposite convention to that generally taken in the literature. In particular, the mostly positive metric is used throughout, which can lead to confusion if you are not used to it. It has the advantage of being available online at `http://web.physics.ucsb.edu/~mark/qft.html`

- Peskin and Schroeder, *Introduction to Quantum Field Theory*. A classic textbook that again covers a lot of material. Not always the best introduction to some of the material covered, but is always a useful resource.

- Zee, *Quantum Field Theory in a Nutshell.* Takes a slightly idiosyncratic approach in places, and does not cover everything. But those topics that are covered are explained very well, so this is a useful book.

- Schwartz, *Quantum Field Theory and the Standard Model.* A very good book, covering a lot of material in detail. However he is rather sloppy with raised and lowered indices, not keeping these consistently matching, as they should be.

- Ryder, *Quantum Field Theory.* The sections on path integrals and renormalization are in particular very useful.

There are of course many other textbooks available on the market, and essentially anything dealing with quantum field theory will cover at least some of the material contained in this course.

2. Introduction

In these lectures we aim to build up a description of the fundamental forces relevant to particle physics, namely the electromagnetic (EM), strong and (electro)weak forces. This is done using the tools of quantum field theory and gauge symmetry, through which we describe these forces by (quantum) gauge field theories:

- Electromagnetism - the abelian gauge theory of Quantum Electrodynamics (QED).

- Strong interaction - the non–abelian gauge theory of Quantum Chromodynamics (QCD).

- Electroweak interaction - a spontaneously broken non–abelian gauge theory (EW).

We will build up slowly to full QED, the theory of EM interactions between fermions (e.g. electrons), by beginning with the simpler toy case of *scalar QED*, where the fermionic fields are replaced by scalars, before demonstrating how spinor fields arise and applying these to formulate the full theory of QED, as it exists in Nature. In both cases we will demonstrate the important role that ($U(1)$) gauge symmetry plays. We will then generalise this so–called 'abelian' symmetry to the 'non–abelian' case and investigate the QCD gauge theory of the strong interaction.

In all of the above cases, we will apply the path integral formulation to derive the Feynman rules of the theory, and discuss the important role that gauge symmetry plays. We will also go beyond leading order, and show explicitly how these theories are renormalized, leading to the QED and QCD running couplings. This will require the introduction of various loop techniques in D–dimensions in order to deal with diverging integrals.

Finally, we will discuss the physics of spontaneous symmetry breaking, and we will show how this allows us to define a non-abelian gauge theory of the electroweak interaction, with massive W and Z gauge bosons (and fermions) and the famous Standard Model Higgs boson.

3. Notation

We have

$$x^\mu = (x^0, \mathbf{x}) \ , \tag{3.1}$$

and

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial x^0}, \nabla) \ , \tag{3.2}$$

while we use the mostly negative metric

$$g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1) \ . \tag{3.3}$$

We will often write the scalar product as

$$g_{\mu\nu} a^\mu b^\nu = (ab) \ . \tag{3.4}$$

4. Recap – Lagrangian formalism and Noether's theorem

A full explanation of these topics has been given in the QFT course. To set the scene, some brief and far from complete discussion is given below.

4.1 Lagrangians

In classical mechanics the behaviour of some system can be derived using the Lagrangian, given by

$$L = T - V \tag{4.1}$$

where T and V corresponds to the kinetic and potential energy of the system respectively. If we are interested in the evolution of the system between some times t_1 and t_2, say, then if we define the *action*

$$S = \int_{t_1}^{t_2} \mathrm{d}t \, L \ , \tag{4.2}$$

then the path taken by the system is given by that which corresponds to the minimum of this, known as the 'principle of least action'[1]. This can determined by solving for the case that first–order variation about the minimal path $\delta S = 0$, as must be the case about such an extremum.

Now, quantum mechanically we must sum at the amplitude level over all possible intermediate states (i.e. paths) that link the initial ($t = t_1$) and final ($t = t_2$) state of the system. This leads to a sum over paths with an amplitude

$$A \sim e^{iS/\hbar} \, . \tag{4.3}$$

To construct a relativistic field theory we apply this approach, with the Lagrangian now given as a function of the fields of interest, $\phi(x)$. Note that in the field theory approach, by 'path' we now mean a sum over the path in the space of field configurations.

The fundamental object in this course is the *Lagrangian density*, $\mathcal{L}$, in terms of which the action is given by

$$S = \int \mathrm{d}t \, L = \int \mathrm{d}^D x \, \mathcal{L} \, , \tag{4.4}$$

where $D = 4$ in the world we (seem to) live in. We will generally simply call this the 'Lagrangian', and it is typically taken to be a function of the fields ϕ and the derivatives $\partial_\mu \phi$. This defines the QFT we are working with: from it, we can amongst other things derive the equations of motion for the particles in theory, how they interact, and what symmetries we expect the theory to obey.

4.2 Euler–Lagrange equations

Consider a Lagrangian $\mathcal{L}(\phi_a, \, \partial_\mu \phi_a)$ for a set of fields ϕ_a (here a simply labels the field). Then if we make an infinitesimal change to the fields $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$, this will produce a change

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a(x)} \delta\phi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a(x))} \delta(\partial_\mu \phi_a(x)) \, , \tag{4.5}$$

where a is summed over. To derive the classical equations of motion we determine the

[1]See the Feynman lecture for a nice discussion.

variation

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right] ,$$
$$= \int \mathrm{d}^4x \left(\left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \right) . \tag{4.6}$$

The second contribution is an overall surface term that vanishes for any $\delta\phi_a$ that vanishes at spatial infinity and $t = t_{1,2}$. Thus we can drop this and we arrive at

$$\frac{\delta S}{\delta \phi_a} = \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) , \tag{4.7}$$

where we rewritten the expression as a functional derivative. This is defined by

$$\frac{\delta J(x_b)}{\delta J(x_a)} = \delta^4(x_a - x_b) , \tag{4.8}$$

which implies

$$\frac{\delta}{\delta J(x_a)} \int \mathrm{d}^4x J(x) G(x) = G(x_a) . \tag{4.9}$$

Applying the principle of least action, we demand that this variation (4.7) vanishes, and arrive at the classical equations of motion, or the *Euler–Lagrange* equations

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0 . \tag{4.10}$$

We have a set of equations, one for each field a.

4.3 Noether's theorem

A viable theory is constructed by demanding that the Lagrangian in (4.4) is invariant under the symmetries observed in Nature. A straightforward example is Lorentz invariance – if the Lagrangian obeys this manifestly then so too will the theory. Put another way, requiring that our theory obeys Lorentz invariance (something which we know from observation to be true to a high degree of precision) greatly restricts the sort of terms we could write down in $\mathcal{L}$, which must be a Lorentz scalar so that the action itself is (the measure d^Dx is invariant). When it comes to considering our interacting QFT, then as we will see the symmetries that the Lagrangian obeys tell us a great deal about which interactions can actually occur and the way the relationship between them. Symmetry is therefore a very tool in QFT. Most powerfully of all, by requiring that our Lagrangian is invariant under a class of continuous symmetries, known as gauge transformations, we are lead to a very specific and concrete *prediction* for the type of force carrying fields that must be present, as we will soon see.

The importance of symmetry in particle physics is perhaps made clearest by considering the role it plays in Noether's theorem. In particular, if the Lagrangian is invariant under a continuous symmetry there exists an object $j^\mu(x)$ that satisfies

$$\partial_\mu j^\mu(x) = 0 \,, \tag{4.11}$$

and is known as a conserved current. To see this, if we consider the infinitesimal transformation in the fields ϕ_a that we denote $\delta\phi_a$, then this is a symmetry if the Lagrangian changes by at most a total derivative

$$\delta\mathcal{L} = \partial_\mu F^\mu \,. \tag{4.12}$$

This can of course vanish (as will be the case in examples below). Now the variation in the Lagrangian is given by (4.5), which we can rewrite as

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) + \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\delta\phi_a \,, && (4.13)\\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) + \frac{\delta S}{\delta\phi_a}\delta\phi_a \,. && (4.14)
\end{aligned}$$

Now, if (4.12) and the equations of motion are satisfied this implies that

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu \,, \tag{4.15}$$

is indeed a conserved current, $\partial_\mu j^\mu = 0$.

Such a conserved current also implies that the charge Q given by

$$Q = \int \mathrm{d}^3\mathbf{x}\, j^0 \,. \tag{4.16}$$

is also conserved, i.e. does not change with time. To see this we note that

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}t} &= \int \mathrm{d}^3\mathbf{x}\, \partial_0 j^0 \,,\\
&= -\int \mathrm{d}^3\mathbf{x}\nabla\cdot\mathbf{j} = -\int_\infty \mathbf{j}\cdot\mathbf{S} = 0 \,, && (4.17)
\end{aligned}$$

where we have used (4.11), and we have used Gauss's theorem and assumed that $\mathbf{j} \to 0$

sufficiently quickly as $|\mathbf{x}| \to \infty$ in the last step.

In fact this charge is locally conserved. Writing (4.11) in the space and time components we have

$$\frac{\partial j^0(x)}{\partial t} + \nabla \cdot \mathbf{j}(x) = 0 \;, \tag{4.18}$$

which is a continuity equation, with $j^0(x)$ and $\mathbf{j}(x)$ corresponding to the charge and current density, respectively. Thus, the charge in some volume V is given by

$$Q_V = \int \mathrm{d}^3x \, j^0(x) \;, \tag{4.19}$$

and the rate of change of charge is given by

$$\frac{\partial Q_V}{\partial t} = -\int_V d^3x \, \nabla \cdot \mathbf{j} = -\int_A \mathbf{j} \cdot \mathbf{S} \;, \tag{4.20}$$

where A is the boundary of V. Thus, any charge leaving V must be accounted for by the current flowing through A.

Thus Noether's theorem tells us that if the Lagrangian is invariant under a continuous symmetry, then there exists a corresponding conservation law and conserved charge in the physics of the underlying QFT. For example, the invariance of the Lagrangian under spatial translations corresponds to momentum conservation, rotations to angular momentum conservation, time translations to energy conservation, and Lorentz symmetry to energy–momentum conservation. We will see some further examples of this in the context of QFT in what follows.

5. Scalar QED

5.1 The Klein–Gordon equation

We begin by consider the case of a single complex scalar field. The corresponding Lagrangian is:

$$\mathcal{L} = \partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x)\phi(x) \;. \tag{5.1}$$

Treating $\phi(x)$ and $\phi^*(x)$ as independent field variables, we have

$$\frac{\partial \mathcal{L}}{\partial \phi^*(x)} = -m^2\phi(x) \;, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*(x))} = \partial^\mu \phi(x) \;, \tag{5.2}$$

and the equations of motion (4.10) give

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0 \;, \tag{5.3}$$

and similarly for $\phi^*(x)$. This is the *Klein–Gordon* equation, a relativistic wave equation describing the free propagation of a scalar field.

5.2 Global $U(1)$ symmetry

The Lagrangian (5.1) is invariant under the transformation

$$\phi(x) \to e^{iq\alpha}\phi(x) \; , \tag{5.4}$$

where α is a real constant and q corresponds to the charge of the particle, which will be discussed more below. This corresponds to an abelian global $U(1)$ symmetry, that is a transformation that corresponds to simple multiplication by a spacetime independent complex phase. Now, we know from Section 4.3 that a symmetry of the Lagrangian implies the existence of a conserved current and charge. What does this $U(1)$ symmetry lead to?

To see what, we note that the $U(1)$ rephasing corresponds to the infinitesimal transformation

$$\delta\phi(x) = iq\alpha\phi(x) \; . \tag{5.5}$$

We then have

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\delta\phi(x) = iq\alpha\phi(x)\partial^\mu\phi^*(x) \; , \tag{5.6}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*(x))}\delta\phi^*(x) = -iq\alpha\phi^*(x)\partial^\mu\phi(x) \; , \tag{5.7}$$

No total derivative term is generated by the transformation, and therefore we have

$$j^\mu = -iq\left(\phi^*(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^*(x)\right) \; , \tag{5.8}$$

where we use the conventional normalization that the scale factor α is divided out: as any constant multiplied by the current still leads to a conserved current, we are completely free to do this.

To see how we can interpret the corresponding conserved charge we start with the free field expansion of the scalar fields

$$\begin{aligned} \phi(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left[a(\mathbf{k})e^{-ikx} + b^*(\mathbf{k})e^{ikx}\right] \; , \\ \phi^*(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left[b(\mathbf{k})e^{-ikx} + a^*(\mathbf{k})e^{ikx}\right] \; , \end{aligned} \tag{5.9}$$

where as we are considering the classical case for the time being we write $a^*(\mathbf{k})$ rather than $a^\dagger(\mathbf{k})$ and similarly for the b coefficient; at this point these expressions simply correspond to the most general Fourier decomposition of the field ϕ that is consistent with the Klein–Gordon equation (5.3). Note that the normalization convention is as discussed in Section 5.7. Using these and a little manipulation it is then straightforward to show that

$$\begin{aligned} Q &= -iq \int \mathrm{d}^3\mathbf{x} \left(\phi^*(x)\partial^0\phi(x) - \phi(x)\partial^0\phi^*(x)\right) \;, \\ &= q \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[a^*(\mathbf{k})a(\mathbf{k}) - b^*(\mathbf{k})b(\mathbf{k})\right] \;. \end{aligned} \tag{5.10}$$

When we quantise, these become the number operators $a^\dagger a$ and $b^\dagger b$ for the particle and antiparticle states, respectively. Thus this conserved charge corresponds to the number of particles minus antiparticles multiplied by their charge q. As this is time independent, any scattering process that changes the total Q will vanish. This also follows from the Feynman rules, where for any vertex corresponding to a Lagrangian with the global symmetry, the total charge Q must be conserved. Thus, if we had two fields $\phi_{1,2}(x)$ with different associated $U(1)$ charges $q_{1,2}$

$$\phi_{1,2}(x) \to e^{iq_{1,2}\alpha}\phi_{1,2}(x) \;, \tag{5.11}$$

then we can have no cross terms in the Lagrangian

$$\mathcal{L} = \partial_\mu\phi_1^*(x)\partial^\mu\phi_1(x) - m^2\phi_1^*(x)\phi_1(x) + \partial_\mu\phi_2^*(x)\partial^\mu\phi_2(x) - m^2\phi_2^*(x)\phi_2(x) \;, \tag{5.12}$$

so any interaction between the ϕ_1 and ϕ_2 fields must conserve the $q_{1,2}$ charges. For example, the term

$$\mathcal{L}_I = \lambda\phi_1^*\phi_1\phi_2\phi_2^* \to e^{i\alpha(-q_1+q_1+q_2-q_2)}\mathcal{L}_I = \mathcal{L}_I \;, \tag{5.13}$$

is invariant under the global $U(1)$ symmetries (5.11), while

$$\mathcal{L}_I' = \lambda\phi_1^*\phi_1\phi_1\phi_2^* \to e^{i\alpha(-q_1+q_1+q_1-q_2)}\mathcal{L}_I' \neq \mathcal{L}_I' \;, \tag{5.14}$$

is not. We have not discussed how to interpret such terms yet in terms of Feynman diagrams, but we shall see that they correspond as we might expect to 4–point interactions between the scalar fields. However, given the interpretation above about charge conservation, we expect this to be expressed in the corresponding interactions, i.e.

$$\sum_i^{\text{in}} q_i - \sum_i^{\text{out}} q_i = 0 \,, \tag{5.15}$$

that is, the net charge in the initial state to should equal that in the final state. Bearing in mind the sign difference in how we treat the initial and final state in the above expression, we see that this is achieved if we for example interpret $\phi_1^{(*)}$ as an incoming (outgoing) particle or outgoing (incoming) antiparticle (or vice versa), and similarly for ϕ_2. Thus we can interpret (5.13) as corresponding to e.g.

$$1 + \overline{1} \to 2 + \overline{2} \,, \qquad \text{or} \qquad 1 + 2 \to 1 + 2 \,, \tag{5.16}$$

where we have used the bar to indicate the corresponding antiparticle. Both of these processes one can see indeed conserve the charges $q_{1,2}$. On the other hand, we can interpret (5.13) as corresponding to e.g.

$$1 + \overline{1} \to \overline{1} + 2 \,, \qquad \text{or} \qquad \overline{1} + \overline{2} \to 1 + \overline{1} \,, \tag{5.17}$$

both of which clearly do not. It turns out that this is indeed the correct interpretation of the corresponding interactions terms in the Lagrangian, though it takes a reasonable amount of work to show this is actually the case in the full QFT. We will see this sort of thing at work in the case fermion fields and their role in QED later on. For now though, it is worth emphasising that without doing all this work, just by requiring that the Lagrangian is invariant under (5.11) and the straightforward identification of the corresponding conserved charges $q_{1,2}$ in the above example leads us naturally to this interpretation and limits the sort of interactions that may occur in the theory.

Within the context of the Standard Model, exactly these type of global symmetries correspond to baryon, B, and lepton, L, number conservation. For example in QCD, as we will see later in any given term in the Lagrangian for each antiquark field $\overline{\psi}$ there is a corresponding quark field ψ, and hence the Lagrangian itself is invariant under the same type of global symmetry discussed above, with one such symmetry for each flavour of quark. This leads to the conservation of quark number, and hence baryon number. A similar result occurs for lepton in e.g. QED[2].

[2]In fact in the full Standard Model the situation here is not quite so simple, as these B and L symmetries are *anomalous*, that is they are present in the Lagrangian but are broken by quantum corrections. We will not discuss this further here, but it is discussed in many textbooks.

5.3 Local $U(1)$ gauge symmetry

The global symmetry described in the previous section corresponds to a conservation of the particle charge q: in QED we will associate this with the electric charge. As discussed above, in terms of Feynman rules, for any interaction vertex we associate a charge q with a particle and $-q$ with an anti–particle, and if $\sum q_i$ is not conserved the vertex is definitely forbidden. However we could just as well have associated a charge $-q$ with a particle and $+q$ with an anti–particle, so that (5.4) becomes

$$\phi(x) \to e^{-iq\alpha}\phi(x) \; . \tag{5.18}$$

This is a perfectly good symmetry of the Lagrangian, and leads to the same charge conservation requirement. Physically, whether we label a particle (anti–particle) as positively (negatively) charged and vice–versa is really a matter of convention, which leaves the physics untouched. This is only true because of the global $U(1)$ invariance of the Lagrangian, which tells us that we are free to relabel the particle/anti–particle charge $q \leftrightarrow -q$ and the physics will remain the same. In other words, there is a fundamental redundancy in our description.

However, this seems a somewhat unsatisfactory state of affairs, as we have performed this relabelling *everywhere* in the universe at the same time. This idea of performing a constant relabelling (or in the language of the Lagrangian, a constant rephasing of the fields) at all points in space at the same time seems to contradict the basic spirit of relativity; the relabelling of the charges on Andromeda should surely not affect physics on Earth. This leads us to consider a space–time dependent *local* gauge transformation

$$\phi(x) \to e^{iq\alpha(x)}\phi(x) \; , \tag{5.19}$$

where the scale parameter now depends on x. Under this transformation, we are free to rephase our fields in a different way at each point in space (and time). One can think of this as a suitable generalization of the above idea of particle/anti–particle charge $q \leftrightarrow -q$ relabelling, though it should be emphasised that this direct interpretation in terms of such a conserved charge is no longer present in the local case.

However immediately we have a problem. Due to this spacetime dependence the derivative term in the Lagrangian (5.1) becomes

$$\partial_\mu\phi(x) \to e^{iq\alpha(x)} \left(\partial_\mu\phi(x) + iq\phi(x)\partial_\mu\alpha(x)\right) \; , \tag{5.20}$$

which clearly doesn't lead to an invariant Lagrangian. To solve this, we need to consider a new Lagrangian which is invariant under such transformations. This will be achieved by introducing a derivative that transforms covariantly (in the same way as the fields $\phi(x)$) under the gauge transformation, with

$$D_\mu \phi(x) \to e^{iq\alpha(x)} D_\mu \phi(x) \ , \tag{5.21}$$

this covariant derivative needs an extra term which will cancel the second term in (5.20). As ∂_μ is a Lorentz vector, this requires the introduction of a new vector A_μ. If we define

$$D_\mu \equiv \partial_\mu - iqA_\mu \ . \tag{5.22}$$

and then demand that A_μ transforms according to

$$A_\mu \to A_\mu + \partial_\mu \alpha(x) \ , \tag{5.23}$$

then it readily follows that (5.21) holds:

$$\begin{aligned} D_\mu \phi(x) = (\partial_\mu - iqA_\mu)\phi(x) &\to (\partial_\mu - iq(A_\mu + \partial_\mu \alpha))e^{iq\alpha(x)}\phi(x) \ , \\ &= e^{iq\alpha(x)}(\partial_\mu - iqA_\mu - iq\partial_\mu\alpha(x) + iq\partial_\mu\alpha(x))\phi(x) = e^{iq\alpha(x)} D_\mu \phi(x) \ . \end{aligned} \tag{5.24}$$

From this we can immediately see that the new Lagrangian

$$\mathcal{L} = (D_\mu \phi(x))^* D^\mu \phi(x) - m^2 \phi^*(x)\phi(x) \ , \tag{5.25}$$

is invariant. Here, we simply replaced $\partial_\mu \to D_\mu$; this procedure, known as *minimal substitution*, is something we will use a lot. The equations of motion are

$$(D_\mu D^\mu + m^2)\phi = 0 \ . \tag{5.26}$$

Written another way, which will be useful later, it follows from (5.22) and (5.23) that

$$D_\mu \to e^{iq\alpha(x)} D_\mu e^{-iq\alpha(x)} \ , \tag{5.27}$$

from which we can again see that (5.25) is invariant.

Now, this updated Lagrangian will by construction still be invariant under the global symmetry (5.4) and hence there will be a conserved current associated with this. We find

$$j^\mu = -iq\left(\phi^*(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^*(x)\right) - 2q^2 A^\mu \phi(x)\phi^*(x) \ , \tag{5.28}$$

which allows us to write

$$\mathcal{L} = \partial_\mu \phi^*(x)\partial^\mu \phi(x) - m^2\phi^*(x)\phi(x) - j_\mu A^\mu + O(q^2) \ , \tag{5.29}$$

i.e. to $O(q)$ the vector field A_μ couples universally to the current j_μ. This all follows simply from gauge invariance. By demanding that our fields are invariant under the local gauge transformation (5.19) we are led to introduce a new vector field A_μ which couples to the current (5.28). This will induce terms $\sim A^\mu \phi^* \partial_\mu \phi$ in the Lagrangian, which as we will see correspond to interaction vertices between the scalar fields and the new bosonic field A_μ. This gauge field will therefore mediate an interaction (= a force) between the scalar fields at different points x, y in spacetime, such that the transformation properties of this field will correct for any arbitrary spacetime rephasing of the fields we wish to apply.

In QED we will associate A_μ with the photon field. Indeed, we recall that to describe the motion of a charged particle in an electromagnetic field in classical (and non–relativistic quantum) physics, the principle of minimal coupling tells us to make the replacement

$$p_\mu \to p_\mu + qA_\mu \; . \tag{5.30}$$

We have now derived this rule, which in momentum space is directly equivalent to (5.22), purely from gauge symmetry arguments, when we associate the gauge field A_μ with the EM 4–potential.

5.4 Photon kinetic term

Having introduced the photon field, we must now introduce a kinetic term for it. How do we do this in a Lorentz and gauge invariant way? Consider the commutator

$$F_{\mu\nu} = \frac{i}{q} \left[D_\mu, D_\nu \right] = \partial_\mu A_\nu - \partial_\nu A_\mu \; , \tag{5.31}$$

where in the last line the commutators of the Abelian fields A_μ and the derivatives ∂_μ vanish. Then very simply

$$F_{\mu\nu} \to \partial_\mu (A_\nu + \partial_\nu \alpha(x)) - \partial_\nu (A_\mu + \partial_\mu \alpha(x)) = F_{\mu\nu} \; , \tag{5.32}$$

is gauge invariant (exercise: show this from the commutator definition). To form a Lorentz and gauge invariant kinetic term we can take

$$\mathcal{L}_{\rm kin} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \; , \tag{5.33}$$

where the normalization will be discussed below. As we will show below this indeed corresponds to the kinetic term for a spin–1 massless field.

What if we wanted to include a mass term? This would look like

$$\mathcal{L}_{\rm kin} = \frac{1}{2} m_A^2 A_\mu A^\mu \;, \tag{5.34}$$

for some photon mass m_A, as can be verified by finding the equations of motion with this term included. However under a gauge transformation (5.23) we have

$$\frac{1}{2} m_A^2 A_\mu A^\mu \to \frac{1}{2} m_A^2 (A_\mu + \partial_\mu \alpha(x))(A^\mu + \partial^\mu \alpha(x)) \;, \tag{5.35}$$

which is clearly not invariant. Therefore any such explicit mass term is forbidden by gauge invariance. We have therefore arrived at the very important conclusions that no gauge–invariant mass term can be explicitly added for gauge fields such as the photon, and these must therefore be exactly massless. This gives a theoretical justification for the experimental fact that the photon is known to have at most a very small mass (astronomical observations limit it to be $m_\gamma \lesssim 10^{-26}$ eV, so pretty small). On the other hand, we know that the W and Z bosons, the force carriers of the weak nuclear force, *do* have mass. As we will see later, the concept of spontaneous symmetry breaking allows a way around this.

Aside: what else could we add?

In fact, there is a closely related objected known as the dual field strength tensor

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \;. \tag{5.36}$$

which we could also have introduced here. Here ϵ is the totally antisymmetric Levi–Civita tensor, with $\epsilon^{0123} = +1$. One can readily show that this is also gauge invariant. Now, it is straightforward to show that $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \propto F_{\mu\nu}F^{\mu\nu}$, and therefore such a term is redundant. On the other hand, we are in principle free to consider the contribution

$$\mathcal{L} \propto \tilde{F}_{\mu\nu} F^{\mu\nu} \;, \tag{5.37}$$

to the Lagrangian. However here one can show that this in fact enters as a total derivate contribution that will give a surface term in the action. This will not contribute in perturbation theory (it is in particular proportional to a factor of the momentum going into the vertex minus the momentum going out, which is zero), and in fact it gives no contribution at all in QED. However, for non–Abelian gauge theories, it does and this is the origin of the so–called *strong CP problem*; namely, such a term is allowed in QCD and would have physical effects but experimentally is found to be very small indeed.

Finally, we are in general free to add higher order terms

$$\mathcal{L} \propto (F_{\mu\nu}F^{\mu\nu})^2 + \cdots \ . \tag{5.38}$$

These would lead to non–linear interactions beyond those we have seen (so far) in electromagnetism. In fact, the latest limits on this have come from a measurement of so–called light–by–light scattering in lead ion collisions at the LHC, so it is still something that is being investigated today.

5.5 The Scalar QED Lagrangian

Combining the above terms, we have the scalar QED Lagrangian

$$\mathcal{L} = (D_\mu\phi(x))^* D^\mu\phi(x) - m^2\phi^*(x)\phi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{5.39}$$

which using (5.28) we can write as

$$\mathcal{L} = \partial_\mu\phi^*(x)\partial^\mu\phi(x) - m^2\phi^*(x)\phi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu - q^2 A_\mu A^\mu \phi^*(x)\phi(x) \ . \tag{5.40}$$

The final term corresponds to a pure $\gamma\gamma\phi\phi^*$ contact interaction, and is not present in spinor QED[3]. We can therefore ignore this for now, and we have for the 'QED' part of the Lagrangian

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \ . \tag{5.41}$$

The second term then corresponds to the usual EM interaction with matter. We have

$$\frac{\partial \mathcal{L}_{\rm QED}}{\partial A^\nu} = -j_\nu \ , \qquad \frac{\partial \mathcal{L}_{\rm QED}}{\partial(\partial^\mu A^\nu)} = -F_{\mu\nu} \ , \tag{5.42}$$

and therefore the Euler–Lagrange equations are

$$\partial_\mu F^{\mu\nu} = j^\nu \ , \tag{5.43}$$

which are nothing other than Maxwell's equations! How do we know this? Well, writing in terms of the notation of the usual electromagnetic 4–potential, $A^\mu = (A_0, \mathbf{A})$, we recall that

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \ , \qquad \mathbf{B} = \nabla \times \mathbf{A} \ . \tag{5.44}$$

[3] In fact from (5.28) we can see that a contact interaction term is also hidden in the $j_\mu A^\mu$ piece. For QED this will also be absent in the corresponding term, but as the analysis below makes no reference to the explicit form of j^μ, the same arguments will hold for QED proper.

This gives

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{5.45}$$

Writing the 4–current as $j^\mu = (\rho, \mathbf{j})$ in terms of the charge and current densities, if we consider e.g. the term

$$\partial^\mu F_{\mu 0} = j_0 , \tag{5.46}$$

then we get

$$\partial^\mu F_{\mu 0} = \frac{\partial}{\partial t} F_{00} - \nabla_i F_{i0} = \nabla \cdot \mathbf{E} = \rho , \tag{5.47}$$

as claimed. The other three equations can be similarly derived from other terms, and using

$$\partial^\mu \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\rho} \partial^\mu F^{\rho\sigma} = 0 . \tag{5.48}$$

Note that the normalization choice of $-1/4$ in (5.33) was required so that the Lagrangian matched the relativistic formulation of Maxwell's equations. This is however the *only* place where we had any freedom; the rest of the Lagrangian is obtained purely by the requirements of $U(1)$ gauge invariance.

Let's take a step back. We started by observing that the Lagrangian for a free complex scalar field which is invariant under a $U(1)$ global symmetry, i.e. a constant rephasing of the fields at every point in spacetime, must have charge conservation of the fields in their corresponding interactions. This allowed us to interpret such a symmetry physically as a relabelling of the particle and antiparticle charges, $q \leftrightarrow -q$, at every point in spacetime. As this leaves the physics unchanged, there is a fundamental redundancy in our description. We then make the reasonable requirement that this transformation should be local, i.e. spacetime dependent, which from relativistic considerations is a well motivated assumption. In this case the direct interpretation in terms of the particle/anti–particle charge $q \leftrightarrow -q$ relabelling is lost, but the idea of there being a basic redundancy in our description remains. To maintain the invariance of the Lagrangian in this case, we were forced to introduce a new gauge field A_μ. This transformed under this local symmetry in such a way as to maintain invariance of the Lagrangian. Having made no reference to electromagnetism or an interacting theory, this automatically introduced interactions between the scalar fields and the gauge boson of precisely the form we find in electromagnetism, while the simplest gauge invariant kinetic term we could introduce for the gauge field lead exactly to Maxwell's equations.

Thus, quite remarkably, the entirety of EM dynamics follows simply from the requirement of local gauge invariance. This is clearly a very powerful concept, and indeed it is used to construct the theory of all particle interactions that make up the Standard Model. Moreover, as we shall discuss later on, gauge invariance places further constraints that allow a sensible interacting quantum field theory to be constructed, which is not a trivial exercise.

5.6 Gauge choice

We have introduced a 4–vector potential A^μ to describe the EM field, which would naïvely correspond to 4 degrees of freedom. However, we know that the oscillation of the EM field that make up light waves are completely transverse to the direction of motion of the wave. This corresponds to two degrees of freedom, namely the two polarization states of the photon. The resolution to this apparent contradiction lies in the gauge symmetry of the Lagrangian. In particular, any fields which can be related via the gauge transformation (5.23) are not physically distinguishable. Thus there is clearly a redundancy in the system, which as we will see reduces the apparent 4 degrees of freedom to the 2 we expect.

To examine this further we note that the equations of motion (5.43) in the absence of matter correspond to

$$\partial^2 A_\nu - \partial_\nu(\partial_\mu A^\mu) = 0 \; . \tag{5.49}$$

Separating the time and spatial components this corresponds to

$$-\nabla^2 A_0 - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = 0 \; , \tag{5.50}$$

$$-\partial^2 \mathbf{A} - \nabla(\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A}) = 0 \; . \tag{5.51}$$

Now, under the gauge transformation (5.23), we have

$$\nabla \cdot \mathbf{A} \to \nabla \cdot \mathbf{A} - \nabla^2 \alpha \; , \tag{5.52}$$

and therefore we can choose the α such that

$$\nabla \cdot \mathbf{A} = 0 \; . \tag{5.53}$$

This is known as the Coulomb gauge. The first equation now becomes

$$\nabla^2 A_0 = 0 \; . \tag{5.54}$$

This has no time derivative, hinting that A_0 is not a physical, independent time varying variable. Indeed, we have not used the entirety of our gauge freedom, as for any α that satisfies $\nabla^2\alpha = 0$, the Coulomb gauge condition will still hold. We can then use the gauge transformation

$$A_0 \to A_0 + \frac{\partial\alpha}{\partial t} \ , \tag{5.55}$$

precisely to set $A_0 = 0$; note that the α obeying $\nabla^2\alpha = 0$ will still preserve (5.54). The remaining equation of motion then reads

$$\partial^2\mathbf{A} = 0 \ , \tag{5.56}$$

which describes the propagation of a field $\mathbf{A}$ with 3 degrees of freedom. However we also have the condition (5.53) which reduces this to 2, as required! We will show explicitly that this counting of degrees of freedom is correct below.

The Coulomb gauge is only one particular choice of gauge fixing. The basic idea of a gauge condition is that it places some constraint on the gauge field such that it restricts the choices of A_μ that correspond to the Lagrangian under consideration, i.e. it strips away all or some of the remaining gauge freedom. There are in fact an infinite number of ways of doing this, and while formally physical results must be independent of the gauge fixing, for given practical applications some choices are more convenient than others. As we have seen above, an advantage of the Coulomb gauge is that all gauge freedom can be stripped away, and we are left with two physical transverse polarization states (more of which below).

Some other common choices are:

- The Lorenz gauge, for which we require

  $$\partial_\mu A^\mu = 0 \ . \tag{5.57}$$

  We are free to do this by choosing a suitable α in (5.23). Note that this still leaves one remaining gauge degree of freedom, as we can still transform by any α satisfying $\partial^2\alpha = 0$.

  This has the advantage of being Lorentz invariant, greatly simplifying its application in actual calculations, which of course will almost always be cast in a Lorentz covariant fashion. We will see the benefit of this gauge in action later on when we compute loop diagrams.

- The R_ξ gauge chooses

  $$\partial^2\alpha = -\frac{1}{\xi}\partial_\mu A^\mu \ , \tag{5.58}$$

where ξ is an arbitrary real parameter. This is essentially a generalization of the Lorenz gauge, which defines a family of gauge fixing conditions, for different choices of ξ. We will see the benefit of this below, when defining the photon propagator.

- The condition

$$n^\mu A_\mu = 0 \tag{5.59}$$

where n is an arbitrary 4–vector corresponds to class of *axial* gauges. This again has the benefit of being Lorentz invariant. Moreover, under certain additional conditions it can be shown that in this case only two physical degrees of freedom propagate. In QCD, this has the advantage that the so–called ghost fields are not required (although it leads to a considerably more complicated gluon propagator). We we will discuss the role of these ghost fields further later on.

5.7 The Free Quantized Photon Field

Working in the Coulomb gauge, with $A_0 = 0$, in analogy to the scalar field theory case, we Fourier decompose the field operator $\mathbf{A}$ as

$$\mathbf{A}(x) = \sum_\lambda \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \left[\boldsymbol{\epsilon}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-ikx} + \boldsymbol{\epsilon}^*_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{ikx} \right] , \tag{5.60}$$

where $k_0 = |\mathbf{k}|$ is the photon energy, and the $\boldsymbol{\epsilon}$ are the photon polarization vectors (note that the same logic below would follow if one simply considered the expansion of the classical field). The $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ are the photon creation and annihilation operators after canonical quantisation. Note that the equation as it stands is simply a formal expression for $\mathbf{A}(x)$, which defines the $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$. Indeed we can readily invert the above equation to give expressions for these in terms of the fields $\mathbf{A}(x)$. In particular, the normalising factors are choices; here we have followed the Srednicki convention, but often this is multiplied by an overall factor of $\sqrt{2k_0}$.

The $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ obey the equal–time commutation relations

$$\begin{aligned}
[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k})] &= 0 \, , \\
[a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k})] &= 0 \, , \\
[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] &= (2\pi)^3 \, 2k_0 \, \delta^3(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'} \, ,
\end{aligned} \tag{5.61}$$

where in the last expression the factor of $2k_0$ would be missing if the $\sqrt{2k_0}$ normalization convention were taken above.

Note that in this Fourier expansion, the equations of motion (5.56) simply correspond to $k^2 = 0$, i.e. the on–shell condition for a massless photon, as we would expect. Thus, with this condition alone we would still have 3 degrees of freedom above, and have to expand in terms of 3 basis polarization vectors. However, with the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ we have in k space that $\mathbf{k} \cdot \mathbf{A} = 0$, and therefore we require

$$\mathbf{k} \cdot \boldsymbol{\epsilon}_\lambda(\mathbf{k}) = 0 \; . \tag{5.62}$$

Thus we indeed have only two degrees of freedom, corresponding to the independent polarization states that are transverse to the direction of motion $\mathbf{k}$. For the case that $\mathbf{k}$ lies along the z axis, then the two independent polarization vectors satisfying this can be chosen as

$$\boldsymbol{\epsilon}_+(\mathbf{k}) = -\frac{1}{\sqrt{2}}(1, i, 0) \; , \tag{5.63}$$

$$\boldsymbol{\epsilon}_-(\mathbf{k}) = \frac{1}{\sqrt{2}}(1, -i, 0) \; . \tag{5.64}$$

For other directions, we can simply rotate these. This is only one particular basis choice, but it turns out to be a physically intuitive one, as these in fact correspond to right and left–handed circular polarization states, respectively, i.e. with ± 1 angular momentum projected onto the direction of the photon. These satisfy the normalization and completeness relations

$$\boldsymbol{\epsilon}_\lambda(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'} \; , \tag{5.65}$$

$$\sum_{\lambda=\pm} \boldsymbol{\epsilon}^*_{i,\lambda}(\mathbf{k}) \boldsymbol{\epsilon}_{j,\lambda}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \; . \tag{5.66}$$

To keep things covariant, if we define

$$\epsilon^\mu_+(k) = (0, \boldsymbol{\epsilon}_+) \; , \tag{5.67}$$

$$\epsilon^\mu_-(k) = (0, \boldsymbol{\epsilon}_-) \; , \tag{5.68}$$

and introduce a vector n, which satisfies $n^2 = 0$, $n \cdot \epsilon_\lambda = 0$, but $n \cdot k \neq 0$, then we can write

$$\sum_{\lambda=\pm} \epsilon^*_{\mu,\lambda}(k) \epsilon_{\nu,\lambda}(k) = -g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{k \cdot n} \; . \tag{5.69}$$

To confirm this, for the case that $\mathbf{k}$ lies along the z axis, we can take $n^\mu = (1, 0, 0, -1)$. Then the first term on the RHS of (5.69) gives $\text{diag}(-1, 1, 1, 1)$ as usual, while the second gives diag$(1,0,0,-1)$, so that the combination is consistent with the explicit sum over the $\pm$ polarizations.

5.8 Lorentz invariance: Ward Identity

Let us consider a photon moving along the z axis with energy E, i.e.

$$k = (E, 0, 0, E) \ . \tag{5.70}$$

For demonstration purposes we take a slightly different choice of polarization vectors

$$\epsilon_1 = (0, 1, 0, 0) \ , \qquad \epsilon_2 = (0, 0, 1, 0) \ . \tag{5.71}$$

As above, these are *transverse* polarization vectors of the photon, that is they are orthogonal to the direction of the photon momentum. However if we consider the Lorentz invariant form of this statement, namely

$$\epsilon \cdot k = 0 \ . \tag{5.72}$$

then we quickly run into problems, as any vector satisfying $\epsilon \propto k$, that is with a longitudinal polarization mode, will also satisfy this requirement. This corresponds to a type of light wave that is in contradiction with our expectation from e.g. classical EM and observation, from which we know that the photon should be completely transverse. Why is this a problem? Because even if we start by defining completely transverse polarization vectors in some frame, as this in itself is not a Lorentz invariant requirement, if we change frame the vectors will not necessarily remain transverse. To see if this is indeed the case, we consider the action of some Lorentz transformation, which will in general take the form

$$\epsilon_1^{\mu'} = a_1(\Lambda)\epsilon_1^\mu + a_2(\Lambda)\epsilon_2^\mu + a_3(\Lambda)\frac{k^\mu}{E} \ , \tag{5.73}$$

and similarly for ϵ_2. Here, we are simply expanding our transformed vector in the basis of $\epsilon_{1,2}$ and k, as defined above in the original frame; the constraint (5.72) tells us that indeed these 3 vectors are enough to do that. The last term may then produce exactly such an unphysical longitudinal polarization that we would like to avoid. We are not necessarily in trouble yet, as such a transformation will also change k in general, and so perhaps the polarization vectors might still be transverse. Unfortunately this is not the case. To see this most clearly, we note that there exists a subgroup of Lorentz transformations, known as the 'little group', which leave the vector k unchanged. For the choice (5.70), one example is

$$\Lambda^\mu_\nu = \begin{pmatrix} \frac{3}{2} & 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \end{pmatrix} , \tag{5.74}$$

which satisfies $\Lambda^T g \Lambda = g$, and so is indeed a Lorentz transformation, while we also have $\Lambda^\nu_\mu k^\mu = k^\nu$. The action of this on ϵ_1 is

$$\Lambda^\mu_\nu \epsilon_1^\nu = (1, 1, 0, 1) = \epsilon_1^\mu + \frac{k^\mu}{E} . \tag{5.75}$$

This seems to be a disaster. The whole construction of the purely transverse polarization states of the photon is indeed not itself Lorentz invariant. The unphysical longitudinal polarization state that we wished to avoid is inevitably present in certain frames, and therefore cannot be omitted by hand. To be precise, if we consider the action of (5.75) on a general QED matrix element, then as we will discuss later this has the form

$$M = \epsilon_1^\mu M_\mu \to \left(\epsilon_1^\mu + \frac{k^\mu}{E}\right) M_{\mu'} , \tag{5.76}$$

where $M_\mu \to M'_\mu$ under the Lorentz transformation. If we want to maintain the Lorentz invariance of M, this would therefore apparently seem to require that in the primed frame the photon must couple to the QED process via a partly longitudinal polarization mode, in complete contradiction with what we know to be the case. In general, we can see that the requirement that our observable M only couple to physical transverse polarizations is not a Lorentz invariant statement. However, if we have

$$k_\mu M^\mu = 0 , \tag{5.77}$$

then the problem is resolved. In particular as k_μ in unchanged under our little group transformation, and the above expression is a Lorentz scalar we have

$$k_\mu M^{\mu'} = 0 , \tag{5.78}$$

and thus

$$M = \epsilon_1^\mu M_\mu \to \epsilon_1^\mu M'_\mu \tag{5.79}$$

so that only the physical polarization state contributes to the observable quantity M

in both frames[4], and M itself is Lorentz invariant.

Without (5.77) it would not be possible to satisfy this requirement. Remarkably, for theories with local gauge symmetry this is indeed true, and is known as a Ward identity. Even more remarkably, this follows directly from gauge symmetry.

5.9 Ward Identity and gauge symmetry

Consider (5.60) written in a covariant form:

$$A^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[\epsilon^\mu_\lambda(k) a_\lambda(k) e^{-ikx} + \epsilon^{\mu *}_\lambda(k) a^\dagger_\lambda(k) e^{ikx}\right] , \tag{5.80}$$

where given $k^2 = 0$, a dependence on $\mathbf{k}$ can just as well be written as a dependence on k. Now we know that our Lagrangian must be invariant under the local gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) . \tag{5.81}$$

How does this affect the above field expansion? Well, one can Fourier transform the $\alpha(x)$ in the same way, and then in momentum space the $\partial_\mu \alpha(x) \to k^\mu \tilde{\alpha}(k)$. Then, we require invariance under the transformation

$$\epsilon^\mu_\lambda(k) \to \epsilon^\mu_\lambda(k) + k^\mu f(k, \lambda) , \tag{5.82}$$

where $f(k, \lambda)$ is some arbitrary function, related to the momentum space decomposition of $\alpha(x)$. Given our general form for a matrix element involving the absorption of a photon

$$M = \epsilon^\mu(k) M_\mu , \tag{5.83}$$

and similarly for emission, the only way this can be a symmetry for arbitrary $f(k, \lambda)$ is if indeed we have

$$k_\mu M^\mu = 0 . \tag{5.84}$$

Thus, by constructing a gauge theory we find that we have automatically achieved the non–trivial result of a sensible Lorentz invariant theory where photons have two physical polarization states. This is one example (but not the only one) of the power of gauge symmetry and a reason why these are so ubiquitous in particle physics, where we are of course interested in constructing Lorentz invariant theories of massless spin–1 gauge bosons, most obviously in the case of QED.

[4]For more general Lorentz transformations beyond the little group we have for simplicity considered here, this will also guarantee that polarization states $\epsilon'_\mu \sim k'_\mu$ do not contribute.

Note the above demonstration is perhaps a little heuristic, and we have not actually proved in terms of e.g. arbitrary Feynman diagrams that (5.77) does indeed hold in general, but we clearly expect it to given the discussion above. A more complete account within the path integral context will be given in Section 9.

6. The Photon Propagator

6.1 Path Integrals: Recap

Recall that in the path integral formulation of quantum mechanics we must sum at the amplitude level over all possible intermediate states (i.e. paths) that link the initial ($t = t_1$) and final ($t = t_2$) state of the system . This leads to a sum over paths with an amplitude

$$A \sim e^{iS/\hbar} , \tag{6.1}$$

To construct a relativistic field theory we apply this approach, with the Lagrangian now given as a function of the fields of interest, $\phi(x)$. Note that in the field theory approach, by 'path' we now mean a sum over the path in the space of field configurations. The quantum mechanical sum over field configurations is then determined by the path integral

$$Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int \mathrm{d}^4x \, \mathcal{L}} , \tag{6.2}$$

keeping $\hbar$ explicit here. In the classical $\hbar \to 0$ limit, the path is dominated by the minimum of the exponent and can be calculated using the method of steepest descent. This leads back to the usual classical Euler–Lagrange equations. Away from this limit new quantum phenomena enter, i.e. virtual particles, loop corrections and so on, due to this sum over paths.

Now the above expression corresponds to the quantum mechanical transition amplitude from vacuum to vacuum, which while in principle interesting (it gives the energy of the vacuum, which is a non–trivial thing), is not what we want here. Rather we want to know what happens when we disturb the vacuum – when particles are created/annihilated and interact with each other. To do this we need to act on the vacuum in some way, to disturb it. This is achieved by introducing a source term $J(x)$, giving the generating functional

$$Z[J] = \int \mathcal{D}\phi \, e^{iS(\phi) + i\int \mathrm{d}^4x \, J(x)\phi(x)} , \tag{6.3}$$

where $S(\phi) = \int \mathrm{d}^4x \mathcal{L}$. In some situations this corresponds to a physical quantity of the system, e.g. in condensed matter systems it can correspond to the applied magnetic field. Here it is more of a calculation aid. This in particular provides a nice way to calculate the vacuum expectation of time–ordered fields products, via

$$\langle 0|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle = \frac{1}{Z[0]}\frac{\delta^N Z[J]}{i\delta J(x_1)\cdots i\delta J(x_n)}\bigg|_{J=0} , \tag{6.4}$$

where the $\hat{\phi}$ indicates that on the left hand side we deal with field operators, rather than the field value ϕ. This result will be particularly useful when deriving the Feynman rules of the theory. While the fact that each derivative with respect to $iJ(x_a)$ brings down a $\phi(x_a)$ from the exponent in $Z[J]$ is clear, the derivation of this result, and the fact that is really time–ordered products which are generated, requires more thought. We will not go through this here, and will simply state the result; further details can be found in the first QFT course, as well as Srednicki chapters 6-8 and Schwartz chapter 14.

Finally, it will sometimes be useful to introduce a slightly more concise notation, with

$$\delta_a \equiv \frac{1}{i}\frac{\delta}{\delta J(x_a)} . \tag{6.5}$$

and thus

$$\langle 0|\mathrm{T}\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle = \frac{1}{Z[0]}\delta_1\cdots\delta_n Z[J]|_{J=0} . \tag{6.6}$$

6.2 Klein–Gordon propagator

Consider the Lagrangian for a real free scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \stackrel{!}{=} -\frac{1}{2}(\phi\partial^2\phi + m^2\phi^2) , \tag{6.7}$$

where in the last step we have integrated by parts and dropped a total derivative, which will only contribute as a surface term. Now we introduce the Fourier conjugate $\tilde{\phi}$ via

$$\tilde{\phi}(p) = \int d^4x\, e^{-ipx}\phi(x) , \qquad \phi(x) = \int \frac{d^4p}{(2\pi)^4}e^{ipx}\tilde{\phi}(p) . \tag{6.8}$$

Then, the exponent of (6.3) in Fourier space becomes

$$iS + i\int \mathrm{d}^4x J(x)\phi(x) = \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\left[\tilde{\phi}(p)(p^2-m^2)\tilde{\phi}(-p) + \tilde{J}(p)\tilde{\phi}(-p) + \tilde{J}(-p)\tilde{\phi}(p)\right] . \tag{6.9}$$

If we shift the fields by

$$\tilde{\chi}(p) = \tilde{\phi}(p) + \frac{\tilde{J}(p)}{p^2 - m^2} , \tag{6.10}$$

then (6.9) becomes

$$\frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\left[\tilde{\chi}(p)(p^2-m^2)\tilde{\chi}(-p) - \frac{\tilde{J}(p)\tilde{J}(-p)}{p^2-m^2}\right] = iS_0 - \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\frac{\tilde{J}(p)\tilde{J}(-p)}{p^2-m^2} . \tag{6.11}$$

Going back to coordinate space we thus have

$$Z_0[J] = Z[0]\exp\left[\frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right] . \tag{6.12}$$

where we have defined

$$\Delta(x-y) = -\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon} . \tag{6.13}$$

Thus, by introducing a simple shift in the field variables, the result associated with performing the path integral over ϕ becomes completely contained within $Z[0]$, with

$$Z[0] = \int \mathcal{D}\phi\, e^{iS_0[\phi]} . \tag{6.14}$$

For the vacuum expectation of time–ordered fields products, as in (6.6), this will simply cancel and so we have completely avoided the potential complication of actually having to write down a well–defined formulation for and calculation of this object (although this can be done).

In (6.13) we have introduced a $i\epsilon$ term in the numerator. This picks out the correct contour to give the time–ordered product of fields and define the Feynman propagator in the usual way, see e.g. Section 2.4 of *Peskin* and the QFT course. From the path integral point of view, this corresponds to a shift $m^2 \to m^2 - i\epsilon$, where ϵ is an infinitesimal positive parameter to make the path integral well–defined in Minkowski space.

Above, by introducing a source term and changing variables in this way, we have completely solved the non–interacting theory. The final object (6.13) which is brought down by the double functional differentiation in (6.4), gives the two point correlation function for a particle to propagate from x_1 to x_2 in the free (i.e. non–interacting theory). In particular, from (6.12) we simply have

$$\langle 0|\mathrm{T}\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \frac{1}{i}\Delta(x-y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon} \ , \tag{6.15}$$

and the momentum–space propagator is given by

$$\tilde{\Delta}(p) = -\frac{1}{p^2-m^2+i\epsilon} \ , \tag{6.16}$$

6.3 The Photon Propagator

To calculate the photon propagator we try to follow the same procedure as before, but starting with the free QED Lagrangian (5.41). Thus

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \overset{!}{=} \frac{1}{2}A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu - J_\mu A^\mu \ , \tag{6.17}$$

where as in the scalar case, we have rearranged up to a total derivative to make the manipulation below easier. We capitalise the EM current, as we will treat this as a source for the purposes of this derivation. We have as before

$$\tilde{A}_\mu(p) = \int d^4x\, e^{-ipx}A_\mu(x) \ , \qquad A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}e^{ipx}\tilde{A}_\mu(p) \ , \tag{6.18}$$

giving

$$iS = -\frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\tilde{A}_\mu(p)\left[p^2P^{\mu\nu}\right]\tilde{A}_\nu(-p) + \tilde{J}_\mu(p)\tilde{A}^\mu(-p) + \tilde{J}_\mu(-p)\tilde{A}^\mu(p) \ , \tag{6.19}$$

where we have defined the projection matrix

$$P^{\mu\nu}(p) = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \ , \tag{6.20}$$

which satisfies $P^{\mu\nu}(p)P_\nu{}^\lambda(p) = P^{\mu\lambda}(p)$, as a projection should. Now to calculate the propagator we will want to make the field definition

$$\tilde{\chi}_\mu = \tilde{A}_\mu + \frac{1}{p^2}\tilde{J}^\sigma P^{-1}_{\mu\sigma} \ , \tag{6.21}$$

where P^{-1} is the inverse of the projection matrix. However we have that

$$P^{\mu\nu}p_\nu = 0 \ , \tag{6.22}$$

and therefore this matrix has a zero eigenvalue, and is not invertible.

What is causing this issue? If we imagine decomposing the field $\tilde{A}_\mu$ in a basis of linearly independent 4–vectors, then the component that is $\propto p_\mu$ gives no contribution at all to (6.19), due to (6.22) and the fact that $\partial_\mu J^\mu = 0$ for a free photon, i.e. $p_\mu \tilde{J}^\mu(p) = 0$. Therefore it makes no sense at all to integrate over it. Instead we are interested in the subspace of $\tilde{A}_\mu$ that lies orthogonal to p. We therefore define our path integral $\mathcal{D}A$ to only span those field components that satisfy $p_\mu \tilde{A}^\mu = 0$, of which there are three. This is equivalent to imposing the Lorenz gauge $\partial_\mu A^\mu = 0$.

In this subspace $P_{\mu\nu}$ is simply equivalent to $g_{\mu\nu}$, which is its own inverse, i.e. $P^{-1}_{\mu\nu} = P_{\mu\nu}$. Thus we can take

$$\tilde{\chi}_\mu = \tilde{A}_\mu + \frac{1}{p^2 + i\epsilon} \tilde{J}^\sigma P_{\mu\sigma} \ , \tag{6.23}$$

where we have introduced the usual $i\epsilon$ to render the path integral convergent. We then arrive at

$$iS = iS_0 + \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \tilde{J}_\mu(p) \frac{P^{\mu\nu}}{p^2 + i\epsilon} \tilde{J}_\nu(-p) \ , \tag{6.24}$$

and in position space

$$Z[J] = Z[0] \exp\left[\frac{i}{2} \int d^4x d^4y J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(y)\right] , \tag{6.25}$$

with

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{P^{\mu\nu}}{p^2 + i\epsilon} \ . \tag{6.26}$$

Thus the momentum space propagator is given by

$$\tilde{\Delta}^{\mu\nu}(p) = \frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} \ . \tag{6.27}$$

As described above, the condition $k_\mu \tilde{A}^\mu = 0$ corresponds to $\partial_\mu A^\mu = 0$, and therefore this is the photon propagator in the Lorenz gauge.

Aside: what about the scalar propagator?

Why does the gauge redundancy of our Lagrangian not cause issues with the scalar propagator? One way to motivate this is to consider our gauge transformations in the free field limit, i.e. with $q \to 0$. In such a case we have

$$\mathcal{L} = \partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x)\phi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;, \tag{6.28}$$

and our gauge transformation becomes

$$\phi \to \phi \;, \qquad A_\mu \to A_\mu + \partial_\mu \alpha(x) \;, \tag{6.29}$$

The terms involving ϕ are clearly invariant (!), while as we have shown above the field strength is also invariant and thus so is the kinetic term

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu \;, \tag{6.30}$$

which is guaranteed as

$$(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) \partial_\nu \alpha = 0 \;. \tag{6.31}$$

It is precisely this zero eigenvalue, when taken to momentum space, which leads to the non–invertible projection matrix for the photon above. On the other hand in the free field limit the scalar field, and more generally the matter field, decouples from the gauge symmetry transformation entirely. We therefore experience no issues in this case in defining the corresponding propagator, and this remains true when we turn interactions back on, $q \neq 0$.

6.4 Faddeev–Popov gauge fixing

We saw in the previous section that the redundancy in the gauge description did not allow us to define the photon propagator until we identified essentially by eye and removed by hand the contribution from the redundant $\tilde{A}_\mu \propto k_\mu$ component. However, we would like to develop a more general and systematic approach to this, where we restrict the path integral so that it is does not run over redundant field configurations that are related by gauge transformations. This gauge fixing is implemented by the Faddeev–Popov approach.

Assume for a general path integral

$$Z = \int \mathcal{D}A \, e^{iS[A]} \;, \tag{6.32}$$

that under the gauge transformation, which we denote by $A \to A_g$, the action and measure remain invariant, i.e. $S[A] = S[A_g]$ and $\mathcal{D}A = \mathcal{D}A_g$. These transformations of course form a group, as the action and measure remain invariant under the combined transformation $(A_g)_{g'} = A_{gg'}$. We would like to rewrite the path integral schematically as

$$Z = \int \mathrm{d}g Z' \ , \tag{6.33}$$

where the redundant gauge degrees of freedom have been integrated out and Z' is independent of g. As a simple example of how this works, consider the regular integral

$$I = \int \mathrm{d}x\mathrm{d}y \, g(x^2 + y^2) \ . \tag{6.34}$$

for some function g. Now as this depends only on x^2+y^2 we can simplify the integration by changing to polar coordinates, with

$$I = \int \mathrm{d}\theta \int r\mathrm{d}r \, g(r^2) = 2\pi \int r\mathrm{d}r \, g(r^2) \ . \tag{6.35}$$

Another way of phrasing this is that the system has an overall rotational ($SO(2)$) symmetry, and so the redundant angular group degree of freedom can be integrated out, with 2π being the volume of this finite group of rotations in 2 dimensions. The integral over r corresponds the unique degree of freedom of the problem, with all redundancy factored out. This is the analogue of the physical degree(s) of freedom in the gauge theory case. Ignoring this overall normalization we therefore have

$$I \propto \int r\mathrm{d}r \, g(r^2) \ . \tag{6.36}$$

i.e. we can drop the θ integral entirely. This is essentially what was done in the previous section when we dropped the $\tilde{A}_\mu \propto k_\mu$ component of the action.

Now in the simple example above it was straightforward to change variables and eliminate the redundant angular degree of freedom explicitly, however this may not always be the case. What we want instead is a more general method that we can apply when we do not know what the suitable variables are to do this. To achieve this, we note that as (6.34) is independent of θ, we are free to fix its value by inserting a factor of $\delta(\theta - \phi)$, for some arbitrary ϕ, giving

$$I = \int \mathrm{d}\tilde{\theta} \int \mathrm{d}x\mathrm{d}y \, g(x, y)\delta(\theta - \phi) \ , \tag{6.37}$$

where the integration over the dummy variable $\tilde{\theta}$, which gives the group factor of 2π as above, is inserted to keep the normalization consistent. In fact we can be even more general than this, fixing instead some function of θ (and the other coordinates, in this case r), $f(r,\theta)$. In this case we need to take care of the usual Jacobian factor, given via

$$1 = \Delta(r) \int \mathrm{d}\theta \delta(f(r,\theta)) \;, \tag{6.38}$$

i.e. with $\Delta(r) = \sum_i |f'(r,\theta_i)|$ summing over the solutions to the delta function condition as usual. So:

$$I = \int \mathrm{d}\tilde{\theta} \int \mathrm{d}x\mathrm{d}y \, \Delta(r) \, g(x,y)\delta(f(r,\theta)) \;, \tag{6.39}$$

where $r^2 = x^2 + y^2$ as usual. Sticking with (6.37) for now, we note that as this is independent of ϕ, we are free to integrate over it, or indeed integrate over any arbitrary function of it:

$$\begin{aligned} I &= \frac{1}{\int \mathrm{d}\phi' h(\phi')} \left[\int \mathrm{d}\phi h(\phi) \int \mathrm{d}\tilde{\theta} \int \mathrm{d}x\mathrm{d}y \, g(x,y)\delta(\theta-\phi) \right] \;. & (6.40) \\ &= \frac{1}{\int \mathrm{d}\phi' h(\phi')} \left[\int \mathrm{d}\tilde{\theta} \int \mathrm{d}x\mathrm{d}y \, g(x,y) h(\theta) \right] \;. & (6.41) \end{aligned}$$

This may seem like a slightly strange manipulation, but the point here is that the integrand of the x, y integral is now explicitly dependent on θ, and hence no longer has the redundancy of the function $g(x,y)$ alone, i.e. precisely what we are aiming at for our modified path integral. Now, in this simple example, which is amenable to a simple change of variables to a set of coordinates that are independent of the redundant variable θ, there is of course no reason at all to go through these manipulations. The point is that in more general situations, where the symmetry of the problem is not quite so simple, this is the approach we must take.

In this more general case, we instead take the form as in (6.39), where $f(r,\theta)$ is some function, that as it depends on θ, will fix the redundancy in our system, as required. One can also again integrate over an additional function h, as above, but we will not consider this explicitly here within the example above as it does not add much to the discussion, though we will do this below. For Faddeev–Popov gauge fixing we apply (6.39), suitably generalized to the path integral case, as we shall now discuss. The rotational symmetry becomes a gauge symmetry, and the analogous arguments to those above are now expressed in terms of the gauge invariance of various objects, but the logic is essentially the same.

Returning to the path integral, we write

$$1 = \Delta(A) \int \mathcal{D}g \delta(f(A_g)) \;, \tag{6.42}$$

which defines the function $\Delta(A)$, known as the Faddeev–Popov determinant. Now we have

$$[\Delta(A_{g'})]^{-1} = \int \mathcal{D}g\delta(f(A_{gg'})) = \int \mathcal{D}g''\delta(f(A_{g''})) = [\Delta(A)]^{-1} , \tag{6.43}$$

where we have defined $g'' = gg'$ and used the fact that the measure $\mathcal{D}g$ is invariant under gauge transformations. Thus the Faddeev–Popov determinant is gauge invariant. Now we insert (6.42) into (6.32) to give

$$Z = \int \mathcal{D}Ae^{iS[A]}\Delta(A)\int \mathcal{D}g\delta(f(A_g)) , \tag{6.44}$$

$$= \int \mathcal{D}g\int \mathcal{D}Ae^{iS[A]}\Delta(A)\delta(f(A)) , \tag{6.45}$$

where in the second line we have performed the transformation $A \to A_{g^{-1}}$ and used the fact that S, Δ and $\mathcal{D}A$ are all invariant. Now, the factor $\int \mathcal{D}g$ exactly corresponds to the 2π group volume above, although in gauge theories there is a separate group at every point in spacetime, and this factor is infinite. However as we are not interested in overall factors for observable quantities we can throw this away. Thus we can take

$$Z = \int \mathcal{D}A\, e^{iS[A]}\Delta(A)\delta(f(A)) , \tag{6.46}$$

Now, we consider the specific example of the EM Lagrangian, with its corresponding $U(1)$ gauge invariance. We have

$$A^\mu_\alpha = A^\mu + \partial^\mu\alpha , \tag{6.47}$$

where we relabel $g \equiv \alpha$. For our gauge fixing function we take

$$f(A) = \partial_\mu A^\mu - \sigma(x) , \tag{6.48}$$

where $\sigma(x)$ is some arbitrary function; of course then $f(A)$ is also strictly speaking a function of $f(A,\sigma)$, but for brevity this argument is generally dropped, as it is in the Faddeev–Popov determinant itself. We then have

$$[\Delta(A)]^{-1} = \int \mathcal{D}\alpha\,\delta(f(A_\alpha)) = \int \mathcal{D}\alpha\,\delta(\partial_\mu A^\mu + \partial^2\alpha - \sigma(x)) , \tag{6.49}$$

$$\stackrel{!}{=} \int \mathcal{D}\alpha\,\delta(\partial^2\alpha) , \tag{6.50}$$

where in the last line we have used the fact that this will appear in (6.46) accompanied by a $\delta(f(A))$ and therefore this term can be set to zero. However this is clearly independent of A, and therefore will only contribute as an overall factor to the path integral. This will always be divided out in any observable we care to contribute, and therefore we can simply throw it away. Now, using the argument that led to (6.39) we are free to integrate over any arbitrary function of σ. We will consider in particular

$$Z_\sigma = \int \mathcal{D}\sigma \exp\left(-\frac{i}{2\xi}\int d^4x\sigma(x)^2\right) . \tag{6.51}$$

The reason for this particular form is simply that it allows the gauge boson propagator to be derived in a straightforward way; other choices are in principle possible, and formally equivalent, but would in practice lead to results that would not be amenable to the analytic manipulations we know how to apply. Thus we have

$$Z = \int \mathcal{D}\sigma \exp\left(-\frac{i}{2\xi}\int d^4x\sigma(x)^2\right)\int \mathcal{D}A\, e^{iS[A]}\delta(\partial_\mu A^\mu - \sigma) , \tag{6.52}$$

$$= \int \mathcal{D}A \, \exp\left(iS[A] - \frac{i}{2\xi}\int d^4x(\partial_\mu A^\mu)^2\right) , \tag{6.53}$$

and we can see that indeed the path integral is no longer invariant under (6.47). This is equivalent to adding a gauge–fixing contribution to the Lagrangian

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\xi}\partial_\mu A^\mu \partial_\nu A^\nu \to \frac{1}{2\xi}A_\mu \partial^\mu \partial^\nu A_\nu , \tag{6.54}$$

where in the second step we have integrated by parts and set the surface term to zero. Thus, the QED Lagrangian becomes

$$\mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{g.f.}} = \frac{1}{2}A_\mu(\partial^2 g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu)A_\nu - J_\mu A^\mu \tag{6.55}$$

with the projection matrix

$$P_\xi^{\mu\nu}(p) = g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{p^\mu p^\nu}{p^2} , \tag{6.56}$$

where the ξ subscript is taken to distinguish this from (6.20). This obeys $P_\xi^{\mu\nu}p_\mu = 1/\xi p^\nu$, and so it is invertible. Indeed we have

$$P^{-1}_{\xi,\mu\nu} = g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2} , \tag{6.57}$$

which gives the propagator

$$\tilde{\Delta}^{\mu\nu}(p) = \frac{g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} \; . \tag{6.58}$$

Two specific and often used cases are the $\xi = 0$ and 1 choices, known as the Lorenz and Feynman gauges. Note sometimes these are called the Landau and 't Hooft–Feynman gauges, respectively. Note also, that if we include the gauge fixing term (6.54) in the Lagrangian and calculate the corresponding equations of motion for A, then we find that indeed for $\xi = 0$ these are satisfied if $\partial_\mu A^\mu = 0$, consistent with the above assignment.

7. Scalar QED: Feynman Rules

7.1 LSZ Reduction

We saw in Section 6.1 how the vacuum expectation of time–ordered field products could be calculated within the path integral formulation, via the generating functional $Z[J]$. However, the cross sections that are directly measured in experiments such as the LHC are given in terms of scattering amplitudes between appropriate initial and final states. These correspond to suitably defined momentum eigenstates at $t \to \pm\infty$, which are generally taken to be localised wavepackets, although the derivation does not depend on the details of this construction. The connection to these is achieved via the so–called *LSZ reduction formula*, see the QFT notes and e.g. Srednicki chapter 5 and Schwartz chapter 6 for more details.

The basic idea is to define initial and final states at $t \to \pm\infty$ via creation operators acting on the vacuum $|0\rangle$

$$|i\rangle \sim a_1^\dagger(t\to-\infty)a_2^\dagger(t\to-\infty)|0\rangle \; , \qquad |f\rangle \sim a_3^\dagger(t\to+\infty)a_4^\dagger(t\to+\infty)|0\rangle \; . \tag{7.1}$$

We then make use of the identities

$$\begin{aligned} a_i^\dagger(-\infty) - a_i^\dagger(+\infty) &= +i\int \mathrm{d}^4x e^{-ik_i x}(\partial^2 + m^2)\phi(x) \; , \\ a_i(+\infty) - a_i(-\infty) &= +i\int \mathrm{d}^4x e^{ik_i x}(\partial^2 + m^2)\phi(x) \; , \end{aligned} \tag{7.2}$$

where k_i is the wavepacket momentum, which we can assume to be infinitely narrow. These results follow after some manipulation of the Fourier decomposition of the fields. The second operators, $a_i^\dagger(+\infty)$ and $a_i(-\infty)$ annihilate the vacuum state, and therefore can be dropped, so that we can write

$$
\begin{aligned}
a^\dagger(\mathbf{k}_i)_{\rm in} &\to +i\int \mathrm{d}^4x e^{-ik_i x}(\partial^2+m^2)\phi(x)\;,\\
a(\mathbf{k}_i)_{\rm out} &\to +i\int \mathrm{d}^4x e^{ik_i x}(\partial^2+m^2)\phi(x)\;,
\end{aligned}
\tag{7.3}
$$

which define the operators that create the incoming and outgoing states for a particle of momentum k_i appearing in

$$
\langle f|i\rangle = \langle 0|\, a(\mathbf{k}_3)_{\rm out}a(\mathbf{k}_4)_{\rm out}a^\dagger(\mathbf{k}_1)_{\rm in}a^\dagger(\mathbf{k}_2)_{\rm in}\,|0\rangle \tag{7.4}
$$

We therefore find, for the $2\to 2$ scattering of scalar particles, that

$$
\begin{aligned}
\langle f|i\rangle = i^4 &\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4 e^{i(k_3x_3+k_4x_4-k_1x_1-k_2x_2)}(\partial_1^2+m^2)(\partial_2^2+m^2)\cdot\\
&(\partial_3^2+m^2)(\partial_4^2+m^2)\,\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle\;,
\end{aligned}
\tag{7.5}
$$

thus providing a direct relation between the scattering probability amplitude of the physical initial and final states and the expectation value of time–ordered field products, as claimed; this is the LSZ reduction formula for scalar fields. How does this change for photons? Well, for the case of a scalar field the first step in deriving (7.2) is to invert the decomposition

$$
\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left[a(\mathbf{k})e^{-ikx}+a^\dagger(\mathbf{k})e^{ikx}\right]\;, \tag{7.6}
$$

to give

$$
a(\mathbf{k}) = i\int \mathrm{d}^3x\, e^{ikx}\overleftrightarrow{\partial}_0\phi(x)\;, \tag{7.7}
$$

see the QFT course for more details (recalling the factor of $2k_0$ differs due to our normalization convention). For the photon case we instead have

$$
A^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left[\epsilon^\mu_\lambda(k)a_\lambda(k)e^{-ikx}+\epsilon^{\mu *}_\lambda(k)a^\dagger_\lambda(k)e^{ikx}\right]\;, \tag{7.8}
$$

which clearly has the same form, with the exception of the photon polarization vectors, which are now present due to the vector nature of the photon field. However, we can straightforwardly use the orthogonality condition (5.65) to invert this in the same way, giving

$$a_\lambda(\mathbf{k}) = -i\epsilon_\lambda^{\mu *} \int \mathrm{d}^3x \, e^{ikx} \overleftrightarrow{\partial}_0 A_\mu(x) \; . \tag{7.9}$$

The rest of the derivation of the LSZ formula then follows through in the same way as for the scalar field, but in this case (7.3) simply becomes

$$\begin{aligned} a_\lambda^\dagger(\mathbf{k})_{\rm in} &\to -i\epsilon_\lambda^{\mu} \int \mathrm{d}^4x e^{-ikx} (\partial^2) A_\mu(x) \; , \\ a_\lambda(\mathbf{k})_{\rm out} &\to -i\epsilon_\lambda^{\mu *} \int \mathrm{d}^4x e^{ikx} (\partial^2) A_\mu(x) \; . \end{aligned} \tag{7.10}$$

The overall minus sign can be dropped, and the result of this is that when it comes to the Feynman rules for QED, we associate a polarization vector $\epsilon_\lambda^{\mu(*)}$ with an incoming (outgoing) photon.

Finally, we note that it is convenient to define a scattering amplitude $\mathcal{M}$ via

$$\langle f|i\rangle \equiv i(2\pi)^4 \delta^4(k_{\rm in} - k_{\rm out}) \mathcal{M} \; , \tag{7.11}$$

that is, with a factor of $i(2\pi)^4$ multiplied by an overall momentum conserving delta function, which appear universally in calculations, factored out.

7.2 Path Integrals and interactions - some key identities

Consider some (say) real scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_I \equiv \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) \; , \tag{7.12}$$

where we keep for now the ϕ argument of the Lagrangian explicit, and $\mathcal{L}_I$ contains some interaction terms, for example

$$\mathcal{L}_I(\phi) = -\frac{g}{6}\phi^3 - \frac{\lambda}{4!}\phi^4 \; , \tag{7.13}$$

for a cubic and quartic interaction, where g and λ are the couplings associated with these. Now, in general we cannot solve the interacting theory in the way we did for the free theory case. However, we can still make use of the same overall approach, by noting that we can formally write

$$Z[J] = \int \mathcal{D}\phi \, e^{i\int \mathrm{d}^4x (\mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) + J\phi)} = \int \mathcal{D}\phi \, e^{i\int \mathrm{d}^4x \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} e^{i\int \mathrm{d}^4x (\mathcal{L}_0(\phi) + J\phi)} \; , \tag{7.14}$$

which follows as the functional derivative $\frac{1}{i}\frac{\delta}{\delta J(x)}$ with respect to the source term acting on the exponential corresponds as usual to $\phi(x)$. We can then perform the same shift in field variables as for the free theory case, to give

$$Z[J] = \int \mathcal{D}\phi\, e^{i\int \mathrm{d}^4x \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} e^{iS_0[\phi]} e^{\frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)}\ , \tag{7.15}$$

$$= e^{i\int \mathrm{d}^4x \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0[J]\ , \tag{7.16}$$

where $Z_0[J]$ is the generating functional for the free theory, as in (6.12). We can see again that the $Z_0[0]$ term due to the path integral associated with the source–free action S_0 in the non–interacting theory factorises entirely from the rest of the expression, and so will cancel when considering the vacuum expectation of time–ordered field products. For the scalar QED case, we have two sources J^*, J, for the two independent fields ϕ, ϕ^*, respectively, and a source J^μ for the photon field. That is, we have

$$Z[J^*, J, J^\mu] = \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}A \exp\left[iS[\phi, \phi^*, A^\mu] + \int \mathrm{d}^4x\, \phi(x) J^*(x) \right.$$
$$\left. + \phi^*(x)J(x) + J^\mu(x)A_\mu(x)\right]\ . \tag{7.17}$$

Then, with $\mathcal{L}_I[\phi, \phi^*, A^\mu]$, the above result generalises to

$$Z[J^*, J, J^\mu] = Z_0[0,0,0] \exp\left[i\int \mathrm{d}^4x \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J^*(x)}, \frac{1}{i}\frac{\delta}{\delta J(x)}, \frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\right]$$
$$\cdot \exp\left[i\int d^4x d^4y J^*(x)\Delta(x-y)J(y)\right] \exp\left[\frac{i}{2}\int d^4x d^4y J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right]\ , \tag{7.18}$$

where the absence of the factor of $1/2$ in the complex scalar case (in contrast to the real scalar before) comes from the fact that there are two independent fields entering the propagator; this will be discussed further when we consider fermionic fields, where the same effect occurs. In what follows, until we discuss scalar QED again in Section 7.5, we will for simplicity consider the real scalar case, as in (7.15).

The action of the exponential terms in (7.16) and (7.18) on Z_0 cannot be solved for exactly in the general case, but what we can do is expand the exponential term by term. If the coupling g associated with the interaction is small, then the first few terms will give a good approximation to the full result; in other words, we apply perturbation theory. This allows us to derive a set of Feynman rules that can be applied to calculate observables to (in principle) arbitrary accuracy within the perturbative approach. Here, higher order terms in the expansion in g are generated by adding more interaction vertices to the associated Feynman diagrams.

The vacuum expectation value of time–ordered products of fields can then be calculated in the usual way, i.e. via

$$\langle 0|\,\mathrm{T}\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\,|0\rangle = \frac{1}{Z[0]}\delta_1\cdots\delta_n Z[J]|_{J=0}\;, \tag{7.19}$$

with $Z[J]$ calculated at a given perturbative order by (7.16), while the scalar QED result follows in direct analogy. Note that in what follows we will drop the $\hat{\phi}$ on the LHS for simplicity, but it is always understood that these are to be interpreted as field operators, as should be clear from the context. Now, it is useful to define

$$iW[J] = \log Z[J]\;, \tag{7.20}$$

which generates all connected diagrams, that is where every external vertex connects to every other vertex somehow. This omits both vacuum bubble diagrams and those without such contributions but where all external lines are nonetheless not connected to each other. The former correspond to vacuum–to–vacuum transitions which occur independently of the scattering process (thinking physically, the 'sea' of particle/antiparticles that are always popping in and out of existence in the vacuum), and are cancelled by the factor of $Z[0]$ in the denominator of (6.4). The latter corresponds to separate scatters which should be treated independently.

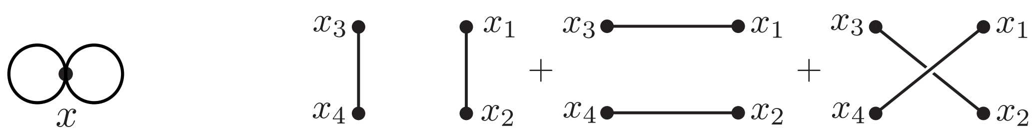

Figure 1: Example Feynman rules for (left) vacuum bubble and (right) disconnected contributions to path integral.

What do these contributions look like? If we consider for example the contribution from the quartic interaction in (7.13) of the form

$$Z[J] \sim -i\frac{\lambda}{4!}\int \mathrm{d}^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4 Z_0[J] = i\frac{\lambda}{8}\int \mathrm{d}^4x \Delta(0)^2 Z_0[J]\;,$$

that is, where the derivatives act (twice) on the sources associated with the same Δ, and we have picked up a symmetry factor of 3. This corresponds to a simple case of precisely the vacuum bubble diagram described above. The Feynman diagram for this contribution is shown in Fig. 1 (left), and corresponds to two closed loops at

$\Delta(x-x)=\Delta(0)$, that is disconnected from all external points. We can see that this will contribute to $Z[0]$ in the denominator of (7.19), and it can indeed be shown that these contributions will cancel the corresponding such terms in the numerator. The second class of disconnected diagrams can be seen by considering the contribution to the 4–point correlation function

$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = -\Delta(x_1-x_2)\Delta(x_3-x_4) = \langle 0|\mathrm{T}\phi(x_1)\phi(x_2)|0\rangle\,\langle 0|\mathrm{T}\phi(x_3)\phi(x_4)|0\rangle \ ,$$

which is non–zero in the non–interacting ($g=0$) theory, and comes from the term where the $\delta_1\delta_2$ and $\delta_3\delta_4$ derivatives act on the same term from the exponent of $Z_0[J]$. This clearly corresponds to two independent free propagators, which are entirely disconnected, and is certainly not what we are interested in when we calculate the 4–point function in the interacting theory. The Feynman diagrams for this contribution, as well as those for other permutations of the external indices, are shown in Fig. 1 (right).

To see how the connected generating functional avoids both of these contributions, consider

$$i\delta_a W[J]|_{J=0} = \frac{1}{Z[0]}\delta_a Z[J]|_{J=0} = \langle\phi(x_a)\rangle \ , \tag{7.21}$$

$$\begin{aligned} i\delta_a\delta_b W[J]|_{J=0} &= \frac{1}{Z[0]}\delta_a\delta_b Z[J]|_{J=0} - \frac{1}{Z[0]^2}\delta_a Z[J]\delta_b Z[J]|_{J=0} \\ &= \langle\phi(x_a)\phi(x_b)\rangle - \langle\phi(x_a)\rangle\langle\phi(x_b)\rangle \ . \end{aligned} \tag{7.22}$$

In the second case, we can see that the denominator factor of $Z[0]$ is automatically present, cancelling the vacuum bubble diagrams, while the second term subtracts off the disconnected contribution. This generalises to all orders, see Srednicki chapter 9, 10 and Peskin & Schroeder chapter 4 for more details. Thus we have

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle_{\mathrm{c}} = i\delta_1\cdots\delta_n W[J]|_{J=0} \ , \tag{7.23}$$

where the 'c' indicates that these are connected diagrams.

Finally, if we want to calculate the scalar propagator to any given order we use

$$\langle 0|T\phi^*(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip(x-y)} S(p) \ . \tag{7.24}$$

At leading order, from (6.15) we have simply

$$S(p) = \frac{i}{p^2-m^2+i\epsilon} \ , \tag{7.25}$$

as we would expect, but this expression provides us with a more general definition of the propagator, that goes beyond leading order. This can be calculated perturbatively by expanding the $\mathcal{L}_I$ exponent in (7.18). To give one slightly more complicated example, in scalar QED we can write

$$\langle 0|TA^\mu(x)\phi(y)\phi^*(z)|0\rangle_c = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4p'}{(2\pi)^4} e^{ip(z-x)}e^{-ip'(y-x)}S(p')ie\Gamma_v(p,p')S(p)D^{\mu\nu}(p-p') \;, \tag{7.26}$$

where D is the photon propagator and Γ is the vertex with one photon and two scalars.

A few simple examples follow below.

7.3 An example: ϕ^3 theory, cubic vertex

As a toy example, we will consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\phi^3 \;, \tag{7.27}$$

for a real scalar field ϕ with a cubic interaction term. Let us take the observable

$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)|0\rangle = \delta_1\delta_2\delta_3 iW[J]|_{J=0} \;, \tag{7.28}$$

that is, the connected 3–point function. First we expand the $\mathcal{L}_I$ exponent in (7.16), with

$$\exp\left[i\int \mathrm{d}^4x\mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] = 1 - i\frac{g}{6}\int \mathrm{d}^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3 + \cdots \;, \tag{7.29}$$

where we omit higher order terms in g, as we will only consider the lowest–order non–zero contribution here. Up to $O(g)$ we therefore have

$$\begin{aligned} Z[J] =& Z[0]\left\{1 - i\frac{g}{6}\int \mathrm{d}^4y_a\left(\frac{1}{i}\frac{\delta}{\delta J(y_a)}\right)^3\right\}\exp\left[\frac{i}{2}\int d^4xd^4yJ(x)\Delta(x-y)J(y)\right] \;, \\ &= Z[0]\Bigg\{1 - i\frac{g}{6}\left(\frac{1}{i}\right)^3\left(\frac{i}{2}\right)^3 2^3\int \mathrm{d}^4y_a\int\prod_{i=1,2,3}\mathrm{d}^4y_i\, J(y_i)\Delta(y_i-y_a) \\ &- i\frac{g}{6}\left(\frac{1}{i}\right)^3\left(\frac{i}{2}\right)^2 2^2\cdot 3\,\Delta(0)\int \mathrm{d}^4y_a\int \mathrm{d}^4y_4\, J(y_4)\Delta(y_4-y_a)\Bigg\}\exp\left[\cdots\right] \;, \end{aligned} \tag{7.30}$$

where the 3 and 2^3 are symmetry factors; in e.g. the latter case this comes from acting on $Z[J]$, where there are two sources for each derivative to act on. Now the $\sim \Delta(0)$

term is precisely the sort of disconnected vacuum bubble contribution discussed above that will not contribute here. As we will discuss below, this happens automatically to $O(g)$, but in reality one can simply trust that this will happen and drop this term. Doing this, we get

$$iW[J] = \log\left[1 - i\frac{g}{6}\left(\frac{1}{i}\right)^3\left(\frac{i}{2}\right)^3 2^3\int \mathrm{d}^4y_a \int \prod_{i=1,2,3}\mathrm{d}^4y_i\, J(y_i)\Delta(y_i - y_a)\right] + \\ + \log Z[0] + \frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\ . \tag{7.31}$$

Now, the second and third terms will give a vanishing contribution when acted on by the three δ derivatives in (7.28) and setting $J = 0$. So we only need consider the action of these on the first term. We have e.g.

$$i\delta_3 W[J] = -i\frac{g}{6}\delta_3\int \mathrm{d}^4y_a \int \prod_{i=1,2,3}\mathrm{d}^4y_i\, J(y_i)\Delta(y_i - y_a)\Big[\cdots\Big]^{-1}\ , \tag{7.32}$$

where the '$\cdots$' denotes the argument of the first logarithm in (7.31). One then readily identifies that the only non–zero contribution upon setting $J = 0$ comes from acting on the first integral with all three derivative, giving

$$i\delta_1\delta_2\delta_3 W[J]|_{J=0} = -i\frac{g}{6}\delta_1\delta_2\delta_3\int \mathrm{d}^4y_a \int \prod_{i=1,2,3}\mathrm{d}^4y_i\, J(y_i)\Delta(y_i - y_a) \tag{7.33}$$

$$= -i\frac{g}{6}\left(\frac{1}{i}\right)^3 3!\int \mathrm{d}^4y_a \prod_{i=1,2,3}\Delta(x_i - y_a)\ . \tag{7.34}$$

Now to get the scattering amplitude we use the LSZ formula, to give

$$\langle f|i\rangle = i^3 g\int \mathrm{d}^4y_a \prod_{i=1,2,3}\mathrm{d}^4x_i e^{i(k_1x_1+k_2x_2-k_3x_3)}\prod_{i=1,2,3}(\partial_i^2 + m^2)\Delta(x_i - y_a)$$

$$= -ig\int \mathrm{d}^4y_a e^{iy_a(k_1+k_2-k_3)}\ , \tag{7.35}$$

$$= (2\pi)^4\delta^4(k_1 + k_2 - k_3)\cdot(-ig)\ , \tag{7.36}$$

where we have used that

$$(\partial_i^2 + m^2)\Delta(x_i - x) = \delta^4(x_i - x)\ , \tag{7.37}$$

i.e. the propagator is a Green's function of the KG equation (exercise: check this for yourself). Absorbing the momentum conserving delta function and factors of 2π into the definition of the scattering amplitude as in (7.11) we have

$$i\mathcal{M} = -ig \;, \tag{7.38}$$

and so to calculate $i\mathcal{M}$ we have the simple rule that we simply assign a factor $-ig$ for every 3–point scalar vertex and conserve momentum at the vertex.

Finally, a quick comment about the $\sim \Delta(0)$ term in (7.30). If we did keep this term, then at (7.32) we would have a non–zero contribution $\sim \Delta(0)\int \mathrm{d}^4 y_a \Delta(x_3 - y_a)$, however this leaves no additional source terms for the $\delta_{1,2}$ to act on, beyond those in the '$\cdots$' of (7.32). However, these carry additional factors of g and therefore are zero at this order, as promised.

7.4 A second example: ϕ^3 theory, $2 \to 2$ scattering

We now consider the leading amplitude for $2 \to 2$ scattering, which occurs at $O(g^2)$ and thus involves the second order term in the expansion (7.29). Working this through is left as an exercise in the problem classes. We find:

$$\langle f|i\rangle = -ig^2(2\pi)^4\delta(k_1+k_2-k_3-k_4)\left[\frac{1}{(k_1+k_2)^2-m^2}+\frac{1}{(k_1-k_3)^2-m^2}\right. \tag{7.39}$$

$$\left. +\frac{1}{(k_1-k_4)^2-m^2}\right] . \tag{7.40}$$

We therefore have

$$i\mathcal{M} = -ig^2\left[\frac{1}{(k_1+k_2)^2-m^2}+\frac{1}{(k_1-k_3)^2-m^2}+\frac{1}{(k_1-k_4)^2-m^2}\right] . \tag{7.41}$$

We see that these three terms can be assigned to the Feynman diagrams shown in Fig. 2, with a factor of $-ig$ as before assigned to each vertex, and a scalar propagator given by

$$\frac{i}{k^2-m^2-i\epsilon} \;, \tag{7.42}$$

which is consistent, up on overall factor of $-i$, with (6.16).

7.5 Feynman rules for scalar QED

The scalar QED Lagrangian is

$$\mathcal{L} = (D_\mu\phi(x))^* D^\mu\phi(x) - m^2\phi^*(x)\phi(x) - \frac{\lambda}{4}(\phi(x)\phi^*(x))^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \;, \tag{7.43}$$

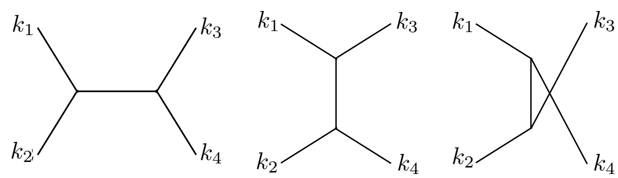

Figure 2: Leading order diagrams contributing to $2 \to 2$ scattering via 3–point interactions of real scalar fields.

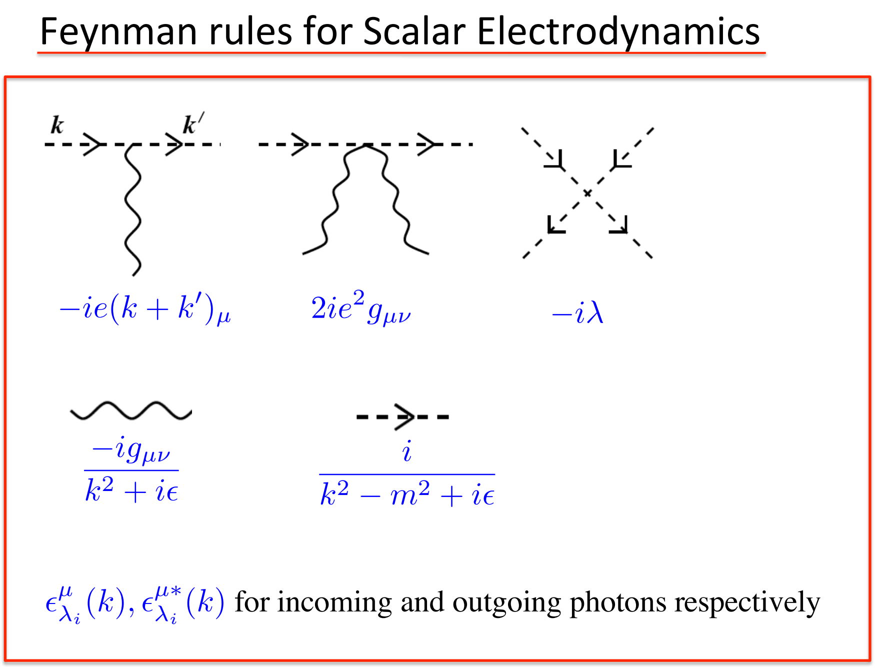

Figure 3: Feynman rules for scalar QED in the Feynman gauge.

where we have added a quartic scalar interaction term with coupling λ to the earlier expression (5.39). We are in principle free to add any gauge invariant contributions we would like, and as a cubic term would not be gauge invariant, this is the lowest dimension operator we can add. This provides us with a suitably general theory with which to investigate the application of Feynman rules and higher–order corrections below. Taking $q = -e$, corresponding to the case of the scalar electron having negative charge, the interaction part of the Lagrangian is now given by

$$\mathcal{L}_I = -ieA_\mu \left(\phi^*(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^*(x)\right) - \frac{\lambda}{4}(\phi(x)\phi^*(x))^2 + e^2 A_\mu A^\mu \phi^*(x)\phi(x)\,. \tag{7.44}$$

As described in earlier sections, the $\phi(x)$ and $\phi^*(x)$ carry opposite $U(1)$ charge, which we associate with electric charge. We can then associate the $\phi(x)$ ($\phi^*(x)$) with a fictional scalar e^- (e^+) particle. This theory, while not identical to true QED has many similar properties, and has the benefit of avoiding the complexity which comes from treating fermions, which we will turn to later.

The Feynman rules can be derived by following the same procedure outlined in the previous sections. However, without resorting to this these can be essentially read off from the Lagrangian, guided by the results above. For the 3–point vertex, the ∂_μ derivatives in the first term will generate a k_μ in the momentum space Feynman rules, with the relative sign given by noting that

$$\langle k'|(\partial_\mu \phi^*(x))\phi(x)|k\rangle = ik'_\mu e^{i(k'-k)x} \,, \tag{7.45}$$

$$\langle k'|\phi^*(x)\partial_\mu \phi(x)|k\rangle = -ik_\mu e^{i(k'-k)x} \,, \tag{7.46}$$

where $|k\rangle$ ($\langle k'|$) are incoming (outgoing) scalar electron states. These follow from the quantum version of the field decomposition (5.9) and the commutation rules for the particle creation operators, which are given by straightforward analogy with (5.61).

Another point to note is the use of an arrow when indicating the scalar particle lines. The rules for these are given below, with perhaps the most important thing being to remember the sign of the momenta k, k' with respect to the arrow direction for the $\gamma\phi\phi^*$ vertex. We will discuss the reason for this more in Section 12, but for now recall from Section 5.2 that we interpret $\phi_1^{(*)}$ as an incoming (outgoing) particle or outgoing (incoming) antiparticle. Given this, and the fact that the Lagrangian itself (in particular the interaction part) always features ϕ and ϕ^* together in equal number, the rules given below for the assignment of the arrow directions ensure that the direction of these arrows always flows continuously along the scalar line, with one arrow pointing towards a vertex and one away from it along the line. The reason for the sign of the momentum assignment with respect to the direction of the arrow is not immediately obvious, and follows from the form of the field decompositions of ϕ and ϕ^* as applied within the context of the LSZ reduction formula. We we will show this concretely in the case of fermion in Section 12, but for now simply state the result.

The Feynman rules for scalar QED are:

- For every incoming (outgoing) scalar particle draw a dashed line with an arrow pointing towards (away from) the vertex.

- For every outgoing (incoming) scalar antiparticle draw a dashed line with an arrow pointing towards (away from) the vertex.

- For every incoming (outgoing) photon, associate a polarization vector $\epsilon^\mu_{\lambda_i}$ ($\epsilon^{\mu*}_{\lambda_i}$).

- The allowed vertices are shown in Fig. 3, along with the corresponding Feynman rules. For the 3–point vertex, the momenta k and k' point in the direction of the arrows. Note the direction of the arrows for the vertices (e.g. for the $\gamma\phi\phi^*$ one arrow points towards the vertex and one away from it); these must be preserved.

- For every internal scalar line, draw a dashed line with an arrow that is consistent with the rules above, and associate a factor

$$\frac{i}{k^2 - m^2} , \tag{7.47}$$

where k is the momentum carried by the line.

- For every internal photon, associate a factor

$$-i\frac{g^{\mu\nu} - (1-\xi)\,\frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} . \tag{7.48}$$

where p is the momentum carried by the photon, for general gauge parameter ξ.

- For every loop, we must integrate over the unconstrained internal momentum l with measure $\int \frac{\mathrm{d}^4 l}{(2\pi)^4}$, where the factor of $(2\pi)^4$ enters due to the standard conversion to momentum space, as in e.g. (6.13). We will discuss this more in the following section.

7.6 Calculating cross sections and decay rates: brief summary

The squared scattering amplitude $|\mathcal{M}|^2$ corresponds to the scattering probability for a given process. This can then be related further to a suitably defined object, known as a *cross section*, that is particularly amenable to experimental interpretation. We will only quote a few key results here, and will not go into the details, which can be found in e.g. Schwartz chapter 5 and Srednicki chapter 11. For $2 \to 2$ scattering we have

$$\mathrm{d}\sigma = \frac{1}{2E_1 2E_2}\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\mathbf{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)|\mathcal{M}|^2 . \tag{7.49}$$

Assuming the particles all have the same mass m, in the centre–of–mass frame we have $E_1 = E_2 = \sqrt{s}/2$, and $|\mathbf{v}_{1,2}| = |\mathbf{p}|/E$, with opposite signs. Using this, as well as

$$\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i} = \frac{\mathrm{d}^4 p_i}{(2\pi)^3}\delta(p_i^2 - m_i^2) , \tag{7.50}$$

(aside: this is why this measure is Lorentz invariant), we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \; , \tag{7.51}$$

where $\mathrm{d}\Omega = \mathrm{d}\phi \mathrm{d}\cos\theta$ is the decay solid angle of particle 3 in this frame. This expression in fact also holds for $m_1 \neq m_2$, provided the masses $m_1 = m_3$ and $m_2 = m_4$.

A further process we are often interested in measuring experimentally, is the decay rate of a particle into a number of other particles. Considering the two–body decay of a particle with momentum P to particles with momenta $p_{1,2}$, the decay width is in general given by

$$\Gamma = \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(P - p_1 - p_2) |\mathcal{M}|^2 \; . \tag{7.52}$$

This is most simply evaluated in the rest frame of the decaying particle, in which case we find

$$\Gamma = \frac{1}{32\pi^2} \frac{|\mathbf{p}|}{M^2} \int \mathrm{d}\Omega \, |\mathcal{M}|^2 \; , \tag{7.53}$$

where $|\mathbf{p}| = |\mathbf{p}_1| = |\mathbf{p}_2|$ is the magnitude of the momentum of the final state particle in this frame, which is given by

$$|\mathbf{p}| = \frac{\left[(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)\right]^{1/2}}{2M} \; . \tag{7.54}$$

7.7 First example: Møller scattering

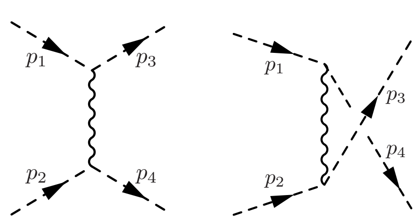

Figure 4: t, u–channel diagrams for Moller scattering.

Møller scattering is the 2 body $e^-(p_1)e^-(p_2) \to e^-(p_3)e^-(p_4)$ (or e^+) process. The amplitude for t–channel exchange (the reason for this name will become clear below) shown in Fig. 4 (left) is given by

$$i\mathcal{M}_t = (-ie)(p_1^\mu + p_3^\mu)\left[-i\frac{g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2}\right](-ie)(p_2^\nu + p_4^\nu) \; , \tag{7.55}$$

where k is the momentum in the photon propagator. Now we have

$$k_\mu(p_1^\mu + p_3^\mu) = (p_{1\mu} - p_{3\mu})(p_1^\mu + p_3^\mu) = p_1^2 - p_3^2 = m^2 - m^2 = 0; \tag{7.56}$$

and thus the ξ dependence vanishes, as it must; a physical amplitude cannot depend on this unphysical parameter. In fact we need to include the u–channel diagram to get the final physical result, so it is not guaranteed that it would vanish for this individual diagram, but here it does. Defining the Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2m^2 + 2(p_1 p_2) = 2m^2 + 2(p_3 p_4) \; , \tag{7.57}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = 2m^2 - 2(p_1 p_3) = 2m^2 - 2(p_2 p_4) \; , \tag{7.58}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = 2m^2 - 2(p_1 p_4) = 2m^2 - 2(p_2 p_3) \; , \tag{7.59}$$

(exercise: show that $s + t + u = \sum m_i^2$), we have

$$\mathcal{M}_t + \mathcal{M}_u = 4\pi\alpha\left(\frac{s-u}{t} + \frac{s-t}{u}\right) \; , \tag{7.60}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. As the second u–channel diagram shown in Fig. 4 (right) comes from interchanging p_3 with p_4, this is simply given by swapping $t \leftrightarrow u$ in $\mathcal{M}_t$. The name t or u–channel is then simply because the propagator carries momentum corresponding to the t or u Mandelstam.

Returning to the Møller scattering case, we then have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(\frac{s-u}{t} + \frac{s-t}{u}\right)^2 \; , \tag{7.61}$$

7.8 Second example: $e^+e^- \to \gamma\gamma$ and the Ward Identity

We saw in Section 5.8 that an amplitude $\epsilon_\mu \mathcal{M}^\mu$ for the emission (or absorption) of an on–shell physical photon state must vanish if we replace $\epsilon^\mu \to k^\mu$ for the result to be sensible from the point of view of Lorentz invariance. As we argued there, this identity follows from gauge invariance. In particular, demanding invariance of observable quantities under the transformation $A_\mu \to A_\mu + \partial_\mu \alpha$ precisely corresponds to a symmetry under $\epsilon_\mu \to \epsilon_\mu + k_\mu$; in momentum space, the $\partial_\mu \alpha \propto k_\mu$.

We now verify that this does indeed hold, considering the $e^-(p_1)e^+(p_2) \to \gamma(p_3)\gamma(p_4)$ process as an example. Writing $\mathcal{M} = \mathcal{M}^{\mu\nu}\epsilon^*_{3\mu}\epsilon^*_{4\nu}$, the t–channel amplitude shown in Fig. 5 (left) is given by

$$\mathcal{M}_t^{\mu\nu} = (-ie)^2 \frac{(2p_1 - p_3)^\mu (p_4 - 2p_2)^\nu}{(p_1 - p_3)^2 - m^2} , \tag{7.62}$$

where we have only assumed that the electrons are on–shell. Now consider replacing $\epsilon^*_{3\mu} \to p_3$ and contracting:

$$p_{3\mu}\mathcal{M}_t^{\mu\nu} = e^2(p_4 - 2p_2)^\nu , \tag{7.63}$$

For the u–channel case shown in Fig. 5 (centre) we simply replace p_2 with p_1, giving

$$p_{3\mu}(\mathcal{M}_t^{\mu\nu} + \mathcal{M}_u^{\mu\nu}) = 2e^2(p_4 - p_2 - p_1)^\nu \neq 0 , \tag{7.64}$$

and we appear to be in trouble. However we have omitted the contact interaction shown in Fig. 5 (right), which gives

$$\mathcal{M}_{\text{contact}} = 2e^2 g^{\mu\nu}\epsilon^*_{3\mu}\epsilon^*_{4\nu} , \tag{7.65}$$

adding this in we get

$$p_{3\mu}(\mathcal{M}_t^{\mu\nu} + \mathcal{M}_u^{\mu\nu} + \mathcal{M}_{\text{contact}}^{\mu\nu}) = 2e^2(p_4 + p_3 - p_2 - p_1)^\nu = 0 , \tag{7.66}$$

which vanishes from momentum conservation. Thus the Ward identity does indeed hold in this more general case, but only when we include all contributing diagrams required by the Lagrangian (7.43) and (7.44). This is not surprising: it is only when including all terms to a given order, including that corresponding to the contact interaction, that the gauge invariance of the Lagrangian holds. Thus the Ward Identity, which itself follows from gauge invariance, will not necessarily hold in explicit calculations unless all Feynman diagrams are included. Put another way, it is only the contribution from the sum of contributing Feynman diagrams to a given process that is itself a physical observable, and not the contributions from individual diagrams, which are themselves interrelated by the gauge symmetry of the Lagrangian.

7.9 Photon Polarization Sum

To calculate the cross section corresponding to the process in the previous section, we could in principle substitute explicit expressions for the photon polarizations, giving four so–called helicity amplitudes, i.e. one for each combination of photon helicities $(++, +-, -+, --)$, which we could then square in the usual way. However, generally

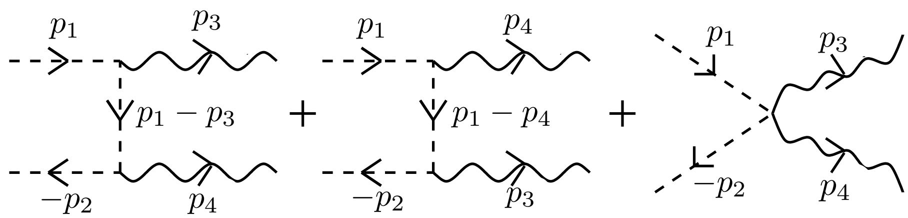

Figure 5: t, u–channel and contact contributions to $e^+e^- \to \gamma\gamma$.

speaking the photon polarization is not measured, and we are therefore interested in the unpolarized cross section, given by summing these four squared amplitudes incoherently. From the previous section, we can then see that we are interested in evaluating the sum

$$\sum_{\lambda=\pm} \epsilon^*_{\mu,\lambda}(k)\epsilon_{\nu,\lambda}(k)\ . \tag{7.67}$$

In Section 5.7 we derived an expression for precisely this, with

$$\sum_{\pm\lambda} \epsilon^*_{\mu,\lambda}(\mathbf{k})\epsilon_{\nu,\lambda}(\mathbf{k}) = -g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{k\cdot n}\ , \tag{7.68}$$

where as before n satisfies $n^2 = 0$, $n \cdot \epsilon_\lambda = 0$, but $n \cdot k \neq 0$. Thus we have:

$$\sum_{\pm\lambda} \epsilon^*_{\mu,\lambda}(\mathbf{k})\epsilon_{\nu,\lambda}(\mathbf{k})\mathcal{M}^\mu\mathcal{M}^{\nu*} = \left(-g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{k\cdot n}\right)\mathcal{M}^\mu\mathcal{M}^{\nu*}\ . \tag{7.69}$$

However, we know from the Ward identity that for

$$\mathcal{M} = \epsilon^\mu_\lambda(k)\mathcal{M}_\mu \Rightarrow k^\mu\mathcal{M}_\mu = 0\ . \tag{7.70}$$

and thus the second term vanishes in (7.69) vanishes! We can therefore make the simple replacement:

$$\sum_{\pm\lambda} \epsilon^*_{\mu,\lambda}(k)\epsilon_{\nu,\lambda}(k) \to -g_{\mu\nu}\ , \tag{7.71}$$

where the '$\to$' indicates that this is not an actual equality, but that those terms which contribute in addition to this give a vanishing contribution when contracted with a physical amplitude.

8. Radiative corrections

8.1 Renormalization

While the Lagrangian for scalar QED (7.43), given by

$$\mathcal{L} = (D_\mu\phi(x))^* D^\mu\phi(x) - m^2\phi^*(x)\phi(x) - \frac{\lambda}{4}(\phi(x)\phi^*(x))^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \; , \tag{8.1}$$

is sufficient for calculating all leading order processes, it requires modification when we go beyond this and start to consider (quantum) loop corrections. These loops are generically divergent in the *ultraviolet* (UV) $k \to \infty$ limit, where k is the momentum flowing through the loop. Divergences aside, it is a bit of worry that our calculation is apparently sensitive to this high k region, as this will include a regime that is way beyond the energy scale we currently have access to, even at the LHC, and where various new physical phenomena (supersymmetry, string theory...) may well be present. In such regimes, we could hardly expect our simple Lagrangian above to apply.

However, it turns out that in certain cases we can systematically re–express our theory so that it relates purely observable quantities to other observable quantities in a way that these decouple from such UV physics. This process is known as *renormalization*. Here we will present an overview, without going into all of the details (of which there are many); Schwartz part III and Zee part III also give nice discussions.

The basic point comes from the realisation that the parameters of the above Lagrangian, e.g. the couplings e, λ and mass m, do not actually correspond directly to what is measured experimentally. Our Feynman rules tells us to associate a coupling e to the leading order $\phi\phi^*\gamma$ vertex, however if we were to go away and measure the electron charge by e.g. scattering electrons off each other, then this observable will not just be given in terms of the 'bare' $\phi\phi^*\gamma$ vertex, i.e. the one we write down at leading order, but rather to the vertex including an entire set (infinite in number) of higher–order loop corrections. We should therefore express the 'bare' parameters of the Lagrangian (which correspond to nothing more than those parameters we use when implementing the Feynman rules, and are themselves *not* observable) in terms of the physical, or 'renormalised' parameters, which correspond to those that we actually measure.

The calculation in terms of these renormalized parameters will be finite, and our renormalized Lagrangian can be used to express other (by definition, finite) observables in terms of these. The crucial point is that for a *renormalizable* theory, this only involves a suitable adjustment of a finite number of inputs. It is in general not trivial to prove that this is the case for a given theory, however one important point to make is that the proof of renormalizability of the Standard Model relies on the underlying gauge symmetry of the theory. Thus, once again gauge symmetry is playing an important role here.

8.2 Motivation: quartic scalar interaction

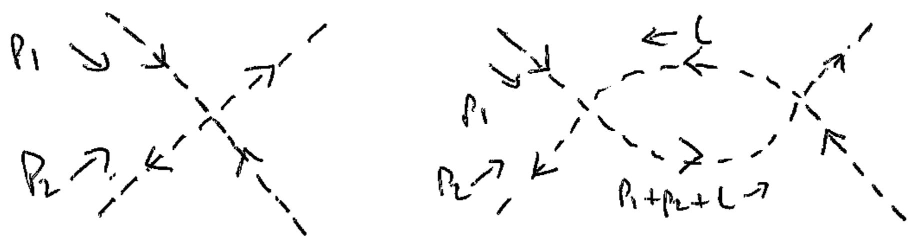

Figure 6: Leading–order and one–loop contributions to $\phi\phi^* \to \phi\phi^*$ scattering.

To clarify some of the general discussion above, let us consider for simplicity the $\sim \lambda$ term in (8.1), forgetting about the QED part of the Lagrangian for now; this is of course perfectly reasonable as the pure scalar part of this Lagrangian leads to a perfectly well defined theory. This will allow us to introduce most of the basic concepts in a relatively simplified scenario.

Where/Why renormalization comes in

The leading order diagram is shown in Fig. 6 (left), and we have simply

$$i\mathcal{M}^{(0)} = -i\lambda \,, \tag{8.2}$$

where the '0' superscript indicates that this is lowest order in λ contribution. If we wish to be more precise, we can then include the $O(\lambda^2)$ contribution to this, for which a representative diagram is shown in Fig. 6 (right). In this case we have

$$i\mathcal{M}^{(1)} = -i\lambda + \lambda^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{(l^2-m^2)((q+l)^2-m^2)} \,, \tag{8.3}$$

where $q = p_1 + p_2$. In general we must also include those diagrams due to permutations of the external legs, however these simply corresponds to taking other values of q ($= p_1 - p_3$ and so on) and will not affect the discussion which follows, so for simplicity we drop them. Concentrating on the second term, focussing on the $l \to \infty$ limit of the integral we have[5]

$$i\mathcal{M}^{1-\text{loop}} \to \lambda^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^4} \sim \lambda^2 \int \frac{\mathrm{d}l}{l} \, , \tag{8.4}$$

which is infinite[6]! This is precisely an example of the UV divergences mentioned above, and these are completely ubiquitous in such loop diagrams, where we are free to integrate up to arbitrarily large internal momenta. Now, following the arguments above we might say that it is rather presumptuous to assume that our theory extends up to infinite values of the field momenta, and so we might argue for introducing some cutoff Λ at the energy scale where we might expect physics beyond this particular model (be it quantum gravity, or supersymmetry or anything else) to enter and the calculation to break down anyway. In this case we have

$$i\mathcal{M}^{1-\text{loop}} \to \lambda^2 \int^{\Lambda} \frac{\mathrm{d}l}{l} \sim \lambda^2 \log \Lambda \, . \tag{8.5}$$

The result is then finite, but still potentially rather large and certainly highly sensitive to precisely how/when we decide to cutoff our ignorance. More precisely, cutting off the integral for $l > \Lambda$ in this way we have

$$i\mathcal{M}^{(1)}(q^2) = -i\lambda + i\lambda^2 \left[C \log\left(\frac{\Lambda^2}{q^2}\right) + D \right] \, , \tag{8.6}$$

where C and D are finite numerical factors which depend on the details of the integration, and the factor of i in the second term comes from the Wick rotation that is needed to evaluate the integral. Now, while this result is as before strongly sensitive to Λ, as well as being infinite if we remove this cutoff, we find that if we consider instead the amplitude at two different scales q_1 and q_2, then we simply have

$$i\mathcal{M}^{(1)}(q_2^2) = i\mathcal{M}^{(1)}(q_1^2) + i\lambda^2 C \log\left(\frac{q_1^2}{q_2^2}\right) \, , \tag{8.7}$$

and hence this relation is finite and perfectly well defined! This is in a line the essence of renormalization: while the naive application of our original Lagrangian to give (8.6) leads to unstable and potentially infinite results, this is because we are asking the wrong question of our theory. If we instead focus on predicting observable quantities (in this case the scattering amplitude at scale q_2) in terms of observable quantities (in this case the scattering amplitude at a scale q_1) then the theory gives completely sensible

[5]Strictly speaking this comes after Wick rotation, with $l = |l_E|$, see the QFT notes, but we omit this notation for simplicity here.

[6]Note in the infrared $l \to 0$ limit we must keep the mass, m, dependence, and the result is finite.

results[7].

So what was wrong with our original calculation? The problem becomes apparent when we think about the nature of the parameter λ in the original Lagrangian (8.1), and what numerical value we should actually give it. This characterises the strength of the quartic interaction and hence the value we take for λ in our theory precisely comes from a measurement of $\mathcal{M}$ above. However as we have seen this amplitude is not just proportional to λ, but also has a λ^2 correction, and depending on which order in perturbation theory we calculate at, many higher order in λ corrections besides that, see Fig. 7.

Thus there is no unique or completely direct way to extract λ from the scattering process, however what we can do is associate a physical, or 'renormalized' coupling by definition with the measured one at some input scale, which we label by q_1, by writing

$$-i\lambda_R \equiv i\mathcal{M}(q_1^2) = -i\lambda_0 + i\lambda_0^2 \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] + O(\lambda^3) \ , \tag{8.8}$$

where we write the result to the first order we are calculating at here, but note that this result can be defined at arbitrary order. Here and in what follows we have introduced a '0' subscript to the so–called 'bare' λ parameter which appears directly in the Lagrangian (8.1), but which we do not directly associate with the measured value. This fixing of λ_R is known as a *renormalization condition*; more precisely we should include a q_1 argument for λ_R, but we drop this for simplicity. Physically, in the above case one can interpret this procedure as including all contributions from loop momenta $q_1^2 < l^2 < \Lambda^2$ in the *definition* of the coupling λ_R.

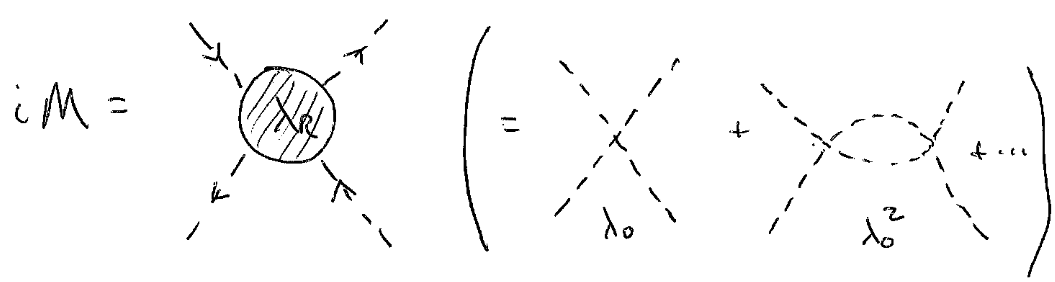

Figure 7: Schematic representation of the renormalized quartic coupling, λ_R.

[7] More precisely the experimental quantity of interest would be the $\phi\phi^* \to \phi\phi^*$ scattering cross section, but this is directly related to $|\mathcal{M}|^2$, and so we can continue to refer to the amplitude $\mathcal{M}$, or as we shall see the coupling λ, as our observable without loss of generality.

We note that to lowest order we have

$$\lambda_0 = \lambda_R \,, \tag{8.9}$$

consistently with the fact that in the absence of loop diagrams we were free to associate the bare parameter of the Lagrangian with the measured one. To first order we have

$$\lambda_0 = \lambda_R + \lambda_R^2 \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] + O(\lambda_R^3) \,. \tag{8.10}$$

Now, substituting this into (8.6) evaluated at scale q_2 we find

$$\begin{aligned} i\mathcal{M}^{(1)}(q_2^2) &= -i\lambda_R - i\lambda_R^2 \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] + i\lambda_R^2 \left[C \log\left(\frac{\Lambda^2}{q_2^2}\right) + D \right] + O(\lambda_R^3) \,, \\ &= -i\lambda_R + i\lambda_R^2 C \log\left(\frac{q_1^2}{q_2^2}\right) + O(\lambda_R^3) \,, \end{aligned} \tag{8.11}$$

which we can see is directly analogous to (8.7). Thus by noticing that the original bare parameter λ_0 of the Lagrangian has no direct physical interpretation, introducing a renormalized coupling that does (i.e. has a precisely defined relationship (8.8) to the measured scattering process), and then re–expressing our predictions in terms of this we arrive at a perfectly finite result which corresponds to a uniquely predicted and well–defined relationship between physical observables. The renormalization process becomes more complex in the full case discussed in the following sections, due to the fact that one must consider multiple renormalized inputs (namely the masses, other couplings present in the theory and the particle propagators) with in general more than one loop diagram contributing in each case, while the introduction of dimensional regularization renders some of the physical interpretation necessarily a little less direct. However, behind all of this the basic idea is exactly the same as in the above example. Before moving on to the general case of scalar QED, this example can also be used to introduce some further useful tools for performing renormalization in perturbation theory.

Renormalization constants

It is conventional to write

$$\lambda_0 \equiv Z_\lambda \lambda_R \tag{8.12}$$

which is simply a definition of Z_λ, known as a *renormalization constant*; in the more general case it is often more tractable to work with this. However the content is the same as above, and indeed we can very straightforwardly substitute this expression into

(8.10) to give

$$Z_\lambda = 1 + \lambda_R \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] + O(\lambda_R^2) \,. \tag{8.13}$$

However from a practical point of view what we tend to do is work directly with (8.12) from the beginning. In other words when evaluated at q_1 (8.6) becomes

$$i\mathcal{M}^{(1)}(q_1^2) = -iZ_\lambda \lambda_R + i(Z_\lambda \lambda_R)^2 \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] \,, \tag{8.14}$$

$$= -i\lambda_R \left(Z_\lambda - \lambda_R \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] \right) + O(\lambda_R^3) \,, \tag{8.15}$$

where in the second line we have used that $Z_\lambda = 1$ to lowest order in λ. From this we can read off the $O(\lambda_R)$ contribution to Z_λ, consistently with (8.13). Then, when predicting the scattering amplitude (or any other scattering amplitude) at some other scale as above, one simply applies (8.12) and (8.13).

Counterterms

Alternatively, we can achieve the same result by introducing so–called *counterterms* into the Lagrangian, with

$$\mathcal{L}_\lambda = -Z_\lambda \frac{\lambda_R}{4} (\phi(x)\phi^*(x))^2 = -\frac{\lambda_R}{4} (\phi(x)\phi^*(x))^2 - \delta_\lambda \frac{\lambda_R}{4} (\phi(x)\phi^*(x))^2 \,, \tag{8.16}$$

which defines the counterterm $\delta_\lambda = Z_\lambda - 1$. This again adds nothing new in principle, but in practice it is often useful to work with these. The effect is that we can calculate with a straightforward application of the Feynman rules, with $\lambda_0 = \lambda_R$, but we must add in additional diagrams due to the counterterm. In the above case this is straightforward: the Feynman rule corresponding to the counterterm is the same, but with $\lambda \to \delta_\lambda \lambda$. Then (8.14) becomes

$$i\mathcal{M}^{(1)}(q_1^2) = -i\lambda_R + i\lambda_R^2 \left[C \log\left(\frac{\Lambda^2}{q_1^2}\right) + D \right] - i\delta_\lambda \lambda_R + O(\lambda_R^3) \,, \tag{8.17}$$

which leads to the same result for Z_λ as above.

Renormalization schemes

Writing things in terms of Z_λ (or δ_λ) in fact reveals an additional freedom we have in performing our renormalization, in particular the fact that we only have to include the $\log \Lambda$ term in our definition of Z_λ for the procedure to work, while we are perfectly free to include or exclude any terms which are finite in the $\Lambda \to \infty$ limit. To see this, we consider an alternative procedure (known as a *renormalization scheme*) to defining a Z'_λ where for example we only include the logarithm above, i.e.

$$Z'_\lambda = 1 + \lambda'_R C \log\left(\frac{\Lambda^2}{q_1^2}\right) + O(\lambda_R^2) \; . \tag{8.18}$$

Note that this corresponds to a different definition of λ_R than in (8.6) (i.e. a different relation between the measured quantity $\mathcal{M}$ and the coupling λ_R), hence why we must add a λ'_R index above. In particular, as we must have

$$\lambda_0 = Z_\lambda \lambda_R = Z'_\lambda \lambda'_R \; , \tag{8.19}$$

this gives

$$\lambda'_R = \lambda_R(1 + \lambda_R D) + O(\lambda_R^3) \; , \tag{8.20}$$

i.e. λ_R and λ'_R do genuinely have different numerical values for the same measurement. In this scheme (8.11) becomes

$$i\mathcal{M}^{(1)}(q_2^2) = -i\lambda'_R + i(\lambda'_R)^2 \left[C \log\left(\frac{q_1^2}{q_2^2}\right) + D\right] \; , \tag{8.21}$$

which we can see after substitution via (8.20) is indeed equivalent up to (8.11) up to higher order terms in λ. The above results really express nothing more than the fact that the precise form of the original prescription (8.6) for relating the renormalized coupling to measurement was itself a choice. On the other hand, in any scheme the $\Lambda \to \infty$ divergent piece has to be the same, i.e.

$$\lambda'_R - \lambda_R = \text{finite} \; , \tag{8.22}$$

in the $\Lambda \to \infty$ limit. Thus a scheme independent result is that

$$Z_\lambda = 1 + \lambda_R C \log \Lambda^2 + \text{finite} + O(\lambda_R^2) \; . \tag{8.23}$$

Renormalization and dimensional regularization

As we will discuss below, it turns out that this method of introducing a cutoff Λ, while arguably more straightforward to interpret physically, is not the most appropriate way to deal with the apparent divergences that arise in QFTs. In dimensional regularization, discussed more below, (8.6) becomes

$$i\mathcal{M}^{(1)}(q^2) = -i\lambda + i\lambda^2\left[E\left(\frac{2}{\epsilon}+\log\left(\frac{\mu^2}{q^2}\right)\right)+F\right]\,, \tag{8.24}$$

where the fact that the coefficient E is the same between the ϵ and log terms is not a coincidence, but as will see in the following sections comes directly from the definition of the new arbitrary scale μ that has been introduced by the requirement that λ be kept dimensionless (one is not obliged to do this, but it makes it easier to keep track of things). This scale no longer plays the direct role of a cutoff on the loop momentum, as in Λ above. Instead, the divergence present in the original loop integration is implicit in the ϵ pole, which is divergent in the $D = 4-\epsilon = 4$ dimensional case. While the physical interpretation of λ_R discussed below (8.8) is lost here, one can still follow precisely the same renormalization procedure as before. In particular, the analogue of (8.14) becomes

$$i\mathcal{M}^{(1)}(q_1^2) = -iZ_\lambda\lambda_R + i(Z_\lambda\lambda_R)^2\left[E\left(\frac{2}{\epsilon}+\log\left(\frac{\mu^2}{q_1^2}\right)\right)+F\right]\,, \tag{8.25}$$

$$= -i\lambda_R\left(Z_\lambda - \lambda_R\left[E\left(\frac{2}{\epsilon}+\log\left(\frac{\mu^2}{q_1^2}\right)\right)+F\right]\right)+O(\lambda^2)\,, \tag{8.26}$$

and thus one can read off for example

$$Z_\lambda = 1+\lambda_R\left[E\left(\frac{2}{\epsilon}+\log\left(\frac{\mu^2}{q_1^2}\right)\right)+F\right]\,. \tag{8.27}$$

Subsituting this into the prediction of $\mathcal{M}$ at a scale q_2 we then get

$$\begin{aligned} i\mathcal{M}^{(1)}(q_2^2) &= -i\lambda_R + i\lambda_R^2\left[E\left(\frac{2}{\epsilon}+\log\left(\frac{\mu^2}{q_2^2}\right)\right)+F\right] - i\lambda_R^2\left[E\left(\frac{2}{\epsilon}+\log\left(\frac{\mu^2}{q_1^2}\right)\right)+F\right]\,,\\ &= -i\lambda_R + i\lambda_R^2 E\log\left(\frac{q_1^2}{q_2^2}\right)\,, \end{aligned} \tag{8.28}$$

exactly as in (8.11), and thus the result of this is the same, as promised. In fact, this precise result relies on the choice (8.27) of finite pieces (in the $\epsilon\to 0$ limit) to include in the definition of Z_λ, i.e. this is precisely a particular choice of renormalization scheme, as above (in fact, a rather unusual one for the case of dimensional regularization). We will discuss this issue of scheme choice more below, but note finally that as in (8.23) the divergent part, in this case the pole in ϵ, is scheme–independent, and so we have that

$$Z_\lambda = 1 + \frac{2E\lambda_R}{\epsilon} + \text{finite} + O(\lambda_R^2) \; . \tag{8.29}$$

We will make heavy use of this result for the case of scalar QED below, where we will only be interested in calculating the scheme–independent 'E' term above.

Running Coupling

Having discussed how to provide well–defined predictions from our theory we should ask what these actually are. To answer this, consider our final relationship (8.11) but written with $q_2 = q$ and $q_1 = \mu$ to keep things looking a little more general and make contact with later notation:

$$i\mathcal{M}^{(1)}(q^2) = -i\lambda_R(\mu) + i\lambda_R^2(\mu) C \log\left(\frac{\mu^2}{q^2}\right) \; , \tag{8.30}$$

where $\lambda_R(\mu)$ for the moment simply serves as a shorthand to indicate that the renormalization condition (8.8), with $q_1^2 = \mu^2$, has been imposed. However, returning to our initial expression (8.7), we know that impact of the one–loop correction leads to a difference in the scattering amplitudes at two different scales, q_1 and q_2. Thus the value of the renormalized coupling $\lambda_R(\mu)$ must also depend on the choice of scale μ, as implied by this notation. This dependence may be isolated by noting that the predicted amplitude on the left hand side of (8.30) should not depend on the choice of μ on the right hand side. Differentiating with respect to $\ln\mu$ we have

$$0 = -i\frac{\partial \lambda_R(\mu)}{\partial \ln\mu} + 2i\lambda_R(\mu)\frac{\partial \lambda_R(\mu)}{\partial \ln\mu} C \log\left(\frac{\mu^2}{q^2}\right) + 2i\lambda_R^2(\mu) C \; , \tag{8.31}$$

so that

$$\frac{\partial \lambda_R(\mu)}{\partial \ln\mu} = 2\lambda_R^2(\mu) C + O(\lambda_R^3) \; . \tag{8.32}$$

That is, due to the presence of these internal loop contributions the value of the coupling does indeed depend on the energy of the scattering process. This is known as the *running coupling*, and will lead to our theory interacting more weakly or strongly at increasing energy, depending on the sign of C. This is a genuinely observable effect that as we shall see has rather significant consequences in Nature. This scale dependence applies to other couplings in QFTs, as well as indeed to particle masses. Note that the value of C depends on the precise theory being considered, and thus the amount of scale variation expected (and indeed the sign of this) is a specific prediction of the theory.

Returning to (8.30), an alternative way to write this is then simply

$$i\mathcal{M}^{(1)}(q^2) = -i\lambda_R(q) \ , \tag{8.33}$$

where the argument on the left hand side implies we evaluate the coupling at scale $\mu = q$, as found by solving (8.32). Indeed, expanding the solution to this out, we get

$$\lambda_R(q) = \lambda_R(\mu) - \lambda_R^2(\mu) C \log\left(\frac{\mu^2}{q^2}\right) + O(\lambda_R^3) \ , \tag{8.34}$$

consistently with (8.30).

Finally, the above derivation also works equally well when applying dimensional regularization, by considering (8.28) as above. We in particular find that to consistently get the same result (8.32), independent of the regularization used, we should have $C = E$. A full calculation confirms this is indeed the case.

8.3 Scalar QED – renormalized Lagrangian

To renormalize our theory, we must rewrite our Lagrangian in terms of physical parameters and fields. For example, we can introduce a physical mass m via

$$m_0^2 = Z_m m^2 \ . \tag{8.35}$$

Here, m_0 is the bare mass that sits in our original Lagrangian, m corresponds to the 'renormalized' mass that is measured experimentally, and Z_m is the renormalization parameter, which this expression serves to define. We now associate m with the physical mass of the particle, and extract this from the measured value. Calculating the Feynman diagrams that contribute to the measured scalar mass then allows the Z_m to be calculated order by order in perturbation theory, by requiring that the result is finite and corresponds to this measured value. We will see how this works more explicitly below.

Redefining all terms in the scalar QED Lagrangian in this way, the renormalized Lagrangian is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, with

$$\mathcal{L}_0 = Z_2 \partial_\mu \phi^* \partial^\mu \phi - Z_m m^2 \phi^* \phi - Z_3 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ , \tag{8.36}$$

$$\mathcal{L}_1 = -iZ_1 e \left[\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right] A_\mu - Z_\lambda \frac{1}{4} \lambda (\phi^* \phi)^2 + Z_4 e^2 \phi^* \phi A^\mu A_\mu \ , \tag{8.37}$$

where this is now written purely in terms of the physical fields and couplings. It is convenient to define *counterterms* via

$$\mathcal{L}_0' = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \;, \tag{8.38}$$

$$\mathcal{L}_{ct} = \delta_2\partial_\mu\phi^*\partial^\mu\phi - \delta_m m^2\phi^*\phi - \delta_3\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \;, \tag{8.39}$$

where $\delta_i \equiv (Z_i - 1)$, so that $\mathcal{L} = \mathcal{L}_0' + \mathcal{L}_1 + \mathcal{L}_{ct}$. This has the benefit that the $\mathcal{L}_0'$ term (which we will generally simply write as $\mathcal{L}_0$ for simplicity) corresponds to the usual free Lagrangian, but now written in terms of the physical, i.e. renormalized, parameters. The usual Feynman rules then follow from these for the photon and scalar propagators, but we must now add in the contributions from the counterterms. This is simply a convenient way of writing things; the counterterms have not really been 'added', but rather correspond to a rewriting of the renormalized Lagrangian in such a way that the calculation follows more easily. The form of the Feynman rules corresponding to the above counterterms can be read off by considering the Lagrangian in momentum space as usual, with

$$\delta_2\partial_\mu\phi^*\partial^\mu\phi - \delta_m m^2\phi^*\phi : \qquad i(\delta_2 k^2 - \delta_m m^2) \;, \tag{8.40}$$

$$-\delta_3\frac{1}{4}F^{\mu\nu}F_{\mu\nu} : \qquad -i\delta_3(k^2 g^{\mu\nu} - k^\mu k^\nu) \;, \tag{8.41}$$

when considering the scalar and photon propagators, respectively. We will demonstrate how these enter explicitly below. We emphasise here that as we have now rewritten the Lagrangian completely in terms of the renormalized couplings and masses, any parameters such as m and e which appear below will correspond to the renormalized, and not bare, values.

Before moving on to this, we note that we have been quite fast and loose with the application of independent renormalization constants to the Lagrangian above, when in fact these terms are not completely independent. In particular, referring back to (8.1), the terms proportional to $Z_{1,2,4}$ come from

$$\mathcal{L} \ni (D_\mu\phi)^* D^\mu\phi = (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi \;. \tag{8.42}$$

Making the redefinitions

$$\phi_0 = Z_2^{1/2}\phi \;, \qquad A_0^\mu = Z_3^{1/2}A^\mu \;, \qquad e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} e \;, \tag{8.43}$$

we get

$$\mathcal{L} \ni (\partial_\mu + i\frac{Z_1}{Z_2}eA_\mu)Z_2^{1/2}\phi^*(\partial^\mu - i\frac{Z_1}{Z_2}eA^\mu)Z_2^{1/2}\phi \;, \tag{8.44}$$

which then reproduces the terms proportional to $Z_{1,2}$ correctly in the Lagrangian. However we can see that this also gives

$$Z_4 = \frac{Z_1^2}{Z_2} \,, \tag{8.45}$$

and thus indeed the renormalization constants are not independent. Moreover, we recall that gauge invariance of the Lagrangian requires the covariant derivative to enter as $D_\mu = \partial_\mu - ieA_\mu$, and would therefore seem to imply that we also need

$$Z_1 = Z_2 \,. \tag{8.46}$$

In fact, as the quantization of the Lagrangian requires us to fix a gauge, it is not completely clear if this naive argument will necessarily apply in the full quantum theory. In the sections which follow, we will concentrate on calculating the (scheme–independent) divergent pieces of the renormalization constants, and will show that this indeed is the case at 1–loop order. As we will discuss in more detail in Section 9, this result follows from the Ward–Takahashi identities of the theory, and holds in general.

8.4 Divergent Integrals and Regularization

A typical loop integral might have the form

$$I = \int \frac{\mathrm{d}^4 l}{l^4 - m^2} \sim \int \frac{\mathrm{d}l}{l} \,, \tag{8.47}$$

at high l. This therefore diverges in the UV $l \to \infty$ limit[8]. In particular, if we impose some arbitrary upper cutoff Λ on the integral we have

$$I_\Lambda = \int^\Lambda \frac{\mathrm{d}l}{l} \sim \log(\Lambda) \,, \tag{8.48}$$

and we say that this particularly integral is logarithmically divergent. Physically, we do not expect our description to be valid up to $l \to \infty$, and so it makes sense to cut things off at a scale Λ where we expect our theory to break down, and a new improved description to enter. Thus in a sense nothing is necessarily infinite in the calculation.

[8]Such integrals are often in addition divergent in the infrared $l \to 0$ limit, for example in the above case if there is a massless particle propagating ($m \to 0$). The nature of these divergences is quite distinct from the UV, and these will in particular cancel between diagrams when a sensible observable is defined, independent of the renormalization of the theory. We will not discuss the IR, and in particular what a 'sensible' observable actually is (in general quite a subtle question) further here.

On the other hand, our prediction would then seem to depend quite sensitively on the precise value we take for Λ, which does not appear to be very predicative at all. The process of renormalization, by expressing only finite observable quantities in terms of other finite observable quantities removes this Λ dependence entirely, albeit at the expense of introducing a new *renormalization scale* μ. However, physically we now associate this with the scale that we actually perform the observation at, rather than some (in principle unknown) scale at which new physics might enter.

One effect of renormalization, although not the only one, is that the final observables will certainly be finite. On the other hand, to get to that point it is still necessary to modify the theory in some way to make the divergent loop integrals that appear in intermediate steps tractable (and finite) in a well defined way. This process is known as *regularization*. One example of this is the simple cutoff we applied above, but this is in fact not the best way to do things. In particular, we have to be careful that our modification does not break some of the important underlying structure of our theory, on which the results may depend. In the case of a simple cutoff it turns out that gauge invariance is explicitly broken, which can be disastrous, as well as more prosaically translation invariance of the integral $l \to l + a$, which can be very unhelpful when manipulating loop integrals.

A better, and more popular choice, is known as dimensional regularization (DREG), which preserves both of the above properties. Here we analytically continue to D spacetime dimensions. Our toy integral above now becomes

$$I_D = \int \frac{\mathrm{d}^D l}{l^4 - m^2} \sim \int \frac{\mathrm{d}l}{l^{5-D}} \; , \tag{8.49}$$

at large l, which is now convergent for any $D < 4$. Note that D does not have to be an integer here: for example, we can see that the large l limit of the integral above is perfectly well defined for arbitrary D. This idea holds more generally, and in DREG we can treat our integrals as well–behaved functions of the continuous variable D. We can for example take

$$g^{\mu\nu} g_{\mu\nu} = D \; , \tag{8.50}$$

again for arbitrary D, without ever having to explicitly consider what a non–integer number of Lorentz indices might actually look like. For realistic loop integrals, we will make heavy use of the Euler–Gamma function, defined by

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, t^{z-1} e^{-t} \; , \tag{8.51}$$

such that $\Gamma(n) = (n-1)!$ for any positive integer n, but this takes a value for arbitrary

z. This is in particular obeys

$$\Gamma(z+1) = z\Gamma(z) \; . \tag{8.52}$$

Some algebraic manipulation then allows us to express the most general loop integral that we will encounter in terms of these functions. The master formula is

$$I(a,b) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^{2a}}{(k^2+X)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{(-X)^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a+\frac{D}{2}\right)\Gamma\left(b-a-\frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} \; . \tag{8.53}$$

Those integrals where the denominator does not take the simple form above can still be manipulated into this form via the method of *Feynman parameters*. In particular, with a bit of algebraic manipulation it is possible to show that

$$\frac{1}{A_1 \cdots A_n} = (n-1)! \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{\delta(x_1 + \cdots + x_n - 1)}{(x_1 A_1 + \cdots + x_n A_n)^n} \; . \tag{8.54}$$

We will only make use of the $n = 2, 3$ cases, for which we have explicitly

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{(Ax + B(1-x))^2} \; , \tag{8.55}$$

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{1}{(Ax + By + C(1-x-y))^3} \; . \tag{8.56}$$

These relations then bring things precisely into the form required to apply (8.53). Those integrals with uncontracted Lorentz indices inside the integration can still be brought into the above form by means of so–called *Passarino–Veltmann* (PV) reduction. For example, considering the integral

$$I_{\mu\nu} = \int \mathrm{d}^D k \, k_\mu k_\nu f(k^2, \cdots) \; , \tag{8.57}$$

where f is some function of k^2 and any other scalar products of Lorentz vectors that enter the integrand, but crucially is only dependent on the magnitude of k, i.e. k^2. Now, the final left hand result $I_{\mu\nu}$ must have the correct Lorentz covariant transformation properties. Moreover, any off–diagonal term, e.g. $I_{01} \sim \int \mathrm{d}^D k \, k_0 k_1$ will be antisymmetric in $\pm k_{0,1}$, and as the rest of the integrand depends only on k^2, will vanish. Hence $I_{\mu\nu}$ must be purely diagonal and transform as a tensor. Lorentz invariance therefore implies this must be proportional to the metric, and we have

$$I_{\mu\nu} = g_{\mu\nu} I' \; , \tag{8.58}$$

where I' is some undetermined scalar. We can then contract both sides with the (D–dimensional) metric to give

$$I_{\mu\nu} = \frac{g_{\mu\nu}}{D} \int \mathrm{d}^D k \, k^2 f(k^2, \cdots) \, , \tag{8.59}$$

This will again leave an integral that can be performed with the aid of the master formula (8.53), as required.

Using these results, we can then work in $D = 4 - \epsilon$ dimensions, where ϵ is a small arbitrary perturbation about the usual $D = 4$, and use

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) \, , \tag{8.60}$$

where $\gamma \approx 0.577$ is the Euler–Mascheroni constant. This will allow us to derive the singular $\sim 1/\epsilon$ behaviour of the integrals, which we then use to define our renormalization procedure.

Finally, we note that to give a dimensionless action, the Lagrangian $\mathcal{L}$ must have mass dimension $[D]$, as we now integrate over $\mathrm{d}^4x \to \mathrm{d}^D x$. From the kinetic terms for the scalar and gauge boson fields, we find $[\phi] = [A^\mu] = 1 - \epsilon/2$ and hence from the interaction terms this requires $[e] = \epsilon/2$. We would not like the dimension of our coupling to depend on the regularization procedure, and so to keep e dimensionless we will make the replacement $e \to e\tilde{\mu}^{\epsilon/2}$ below, where $\tilde{\mu}$ is an arbitrary scale with mass dimension. We note that there is strict requirement to keep e dimensionless, but it does simplify the analysis to do so.

Having discussed the generalities of this procedure, we will now consider the explicit calculations involved at the 1–loop level for scalar QED.

8.5 Photon propagator

Photon self–energy

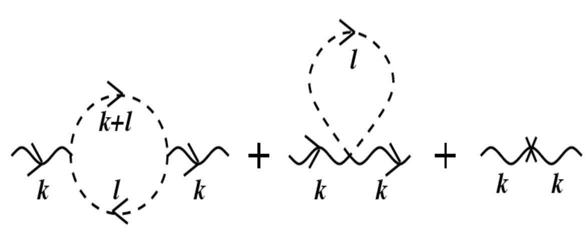

Figure 8: One–loop contributions to photon propagator.

At 1–loop order any photon line can receive contributions from the diagrams shown in Fig. 8; these will lead to corrections to the photon propagator. We denote the combined effect of these corrections as $\Pi^{\mu\nu}(k)$, known as the photon self–energy. The Ward identity tells us that we should have $k_\mu \Pi^{\mu\nu}(k) = 0$, and given this, the most general Lorentz invariant form it can have is

$$\Pi^{\mu\nu}(k) = \Pi(k^2)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) = k^2 \Pi(k^2) P^{\mu\nu} \;, \tag{8.61}$$

where $P_{\mu\nu}(k) = g_{\mu\nu} - k_\mu k_\nu / k^2$ as usual, and this defines the scalar term $\Pi(k^2)$. Now, let us explicitly compute the photon self–energy and confirm that it does indeed have this form. We have

$$\begin{aligned} i\Pi^{\mu\nu}(k) = &(-iZ_1 e)^2 (i)^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{(2l+k)^\mu (2l+k)^\nu}{((l+k)^2 - m^2)(l^2 - m^2)} \\ &+ (2iZ_4) e^2 g^{\mu\nu} i \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 - m^2} \\ &- i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \;, \end{aligned} \tag{8.62}$$

As we have $Z_i = 1 + O(e^2)$, and we are interested in the $O(e^2)$ corrections, we can simply set $Z_{1,4} = 1$ from now on. Combining the first two terms we have

$$\begin{aligned} i\Pi^{\mu\nu}(k) = e^2 &\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{N^{\mu\nu}}{((l+k)^2 - m^2)(l^2 - m^2)} \\ &- i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \;, \end{aligned} \tag{8.63}$$

where

$$N^{\mu\nu} = (2l+k)^\mu (2l+k)^\nu - 2\left((l+k)^2 - m^2\right) g^{\mu\nu} \;. \tag{8.64}$$

The above integrals are as expected divergent in the UV $l \to \infty$ limit, and so we will apply DREG to evaluate the above result. Applying Feynman parameters as in (8.55) this becomes

$$i\Pi^{\mu\nu}(k) = e^2 \tilde{\mu}^\epsilon \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{\mathrm{D}} q}{(2\pi)^D} \frac{N^{\mu\nu}}{(q^2 + X)^2} - i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \;, \tag{8.65}$$

where $q = l + xk$ and $X = x(1-x)k^2 - m^2$. As discussed above, we have also had to introduce a factor $\tilde{\mu}$ with dimensions of mass in order to keep e dimensionless. It will help in simplifying the result if we use

$$\left(\frac{2}{D}-1\right)\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{k^2}{(k^2+X)^2}=X\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{(k^2+X)^2}\;. \tag{8.66}$$

This follows from the master formula (8.53), with

$$I(0,2)=\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{(k^2+X)^2}=i\frac{1}{(4\pi)^{D/2}}\frac{1}{(-X)^{2-\frac{D}{2}}}\left(1-\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)\;, \tag{8.67}$$

$$I(1,2)=\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{k^2}{(k^2+X)^2}=-i\frac{1}{(4\pi)^{D/2}}\frac{1}{(-X)^{1-\frac{D}{2}}}\frac{D}{2}\Gamma\left(1-\frac{D}{2}\right)\;, \tag{8.68}$$

where we have used $\Gamma(n)=(n-1)\Gamma(n-1)$. Now, substituting for q we have

$$N^{\mu\nu}=(2q+(1-2x)k)^\mu(2q+(1-2x)k)^\nu-2((q+(1-x)k)^2-m^2)g^{\mu\nu}\;, \tag{8.69}$$

$$\stackrel{!}{=}4q^\mu q^\nu+(1-2x)^2k^\mu k^\nu-2(q^2+(1-x)^2k^2-m^2)g^{\mu\nu}\;, \tag{8.70}$$

$$\stackrel{!}{=}2q^2\left(\frac{2}{D}-1\right)g^{\mu\nu}+(1-2x)^2k^\mu k^\nu-2((1-x)^2k^2-m^2)g^{\mu\nu}\;, \tag{8.71}$$

$$\stackrel{!}{=}2Xg^{\mu\nu}+(1-2x)^2k^\mu k^\nu-2((1-x)^2k^2-m^2)g^{\mu\nu}\;, \tag{8.72}$$

$$=(1-2x)^2k^\mu k^\nu-2(1-2x)(1-x)k^2g^{\mu\nu}\;, \tag{8.73}$$

$$\stackrel{!}{=}-4y^2(k^2g^{\mu\nu}-k^\mu k^\nu)\;, \tag{8.74}$$

where the '!' indicates that these are not exact equalities, but only follow after dropping terms/making substitutions as described below. In particular, in the second line we have dropped all terms odd in q, which will give zero contribution to the otherwise even in q integration; in the third line we have used PV reduction (8.59); in the fourth line we have used (8.66); in the fifth line we have substituted the explicit expression for X back in; in the sixth line, we have made the change of variables $x=y+\frac{1}{2}$ and, noting that the denominator (with $X=(\frac{1}{4}-y^2)k^2-m^2$) is even in the $y=(-\frac{1}{2},\frac{1}{2})$ integration, discarded those terms odd in y in the numerator. Thus indeed, as required the photon self–energy is completely transverse, having the form implied by the Ward identity.

Thus we are interested in

$$
\begin{aligned}
-4e^2\tilde{\mu}^\epsilon \int_{-1/2}^{1/2} \mathrm{d}y y^2 \int \frac{\mathrm{d^D}q}{(2\pi)^D}\frac{1}{(q^2+X)^2} &= -4e^2\tilde{\mu}^\epsilon \int_{-1/2}^{1/2} \mathrm{d}y y^2 \frac{i}{(4\pi)^{D/2}}\frac{1}{(-X)^\epsilon}\Gamma\left(\frac{\epsilon}{2}\right) \ , \\
&= -4e^2\tilde{\mu}^\epsilon \int_{-1/2}^{1/2} \mathrm{d}y y^2 \left(i\frac{1}{8\pi^2}\frac{1}{\epsilon} + O(\epsilon^0)\right) \ , \\
&= -i\frac{e^2}{24\pi^2}\frac{1}{\epsilon} + O(\epsilon^0) \ , \qquad (8.75)
\end{aligned}
$$

where we substituted $D = 4 - \epsilon$, and have then expanded in ϵ, using (8.60). From (8.62) we therefore have

$$
\Pi(k^2) = -\frac{e^2}{24\pi^2}\frac{1}{\epsilon} + O(\epsilon^0) - (Z_3 - 1) \ . \qquad (8.76)
$$

Requiring that the renormalized result is finite then gives us the pole structure of the renormalization constant Z_3, with

$$
Z_3 = 1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon} + O(\epsilon^0) \ . \qquad (8.77)
$$

Interpretation: correction to photon propagator

How do we interpret the above result more broadly? At leading order, we associate a factor of $-i\tilde{\Delta}_{\mu\nu}$ with the photon propagator, see (6.58). As discussed above, this will receive contributions at the 1–loop order from the diagrams shown in Fig. 8. We write

$$
-i\tilde{\Delta}^{\mathrm{exact}}_{\mu\nu}(k) = -i\tilde{\Delta}_{\mu\nu}(k) + (-i\tilde{\Delta}_{\mu\rho}(k))(i\Pi^{\rho\sigma}(k))(-i\tilde{\Delta}_{\sigma\nu}(k)) + \cdots \ . \qquad (8.78)
$$

Now, it is then convenient to define *one–particle irreducible* (1PI) diagrams as those which cannot be divided in two by cutting through a single propagator, see Fig. 9 for one example in the case of corrections to the scalar propagator. In this case, we can

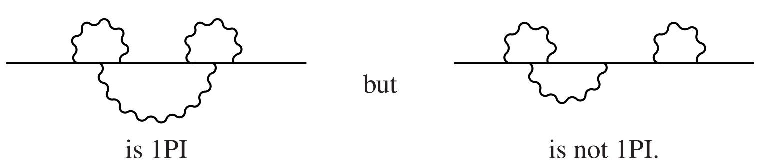

Figure 9: Example of 1PI and non–1PI diagrams.

sum the contributions of all diagrams of this sort via a simple geometric series. In particular, substituting (8.61) into (8.78) and using the projection operator property

$(P_{\mu\rho}P^{\rho\sigma} = P_\mu^{\ \sigma}...)$ we have

$$\tilde{\Delta}^{\rm exact}_{\mu\nu}(k) = \frac{P_{\mu\nu}}{k^2}\left(1+\Pi(k^2)+\cdots\right) + \xi\frac{k_\mu k_\nu}{k^4} , \tag{8.79}$$

$$= \frac{P_{\mu\nu}}{k^2}\sum_{n=0}^{\infty}\Pi(k^2)^n + \xi\frac{k_\mu k_\nu}{k^4} , \tag{8.80}$$

$$= \frac{P_{\mu\nu}}{k^2(1-\Pi(k^2))} + \xi\frac{k_\mu k_\nu}{k^4} , \tag{8.81}$$

where we have dropped the $i\epsilon$ in the denominator for simplicity, though it can straightforwardly be kept in place. To emphasise, here $\Pi(k^2)$ receives contributions to any arbitrary order in perturbation theory that we wish to calculate at, but only including those diagrams which are 1PI. At the 1–loop order considered above, all contributing diagrams are 1PI, but this is not the case beyond that. The point here is that the contribution of those diagrams which at some given order are not 1PI is simply given in terms of combinations of lower–order 1PI contributions to $\Pi(k^2)$, and these are already included in the geometric series above.

Now, this result shows that defined in this way these higher–order corrections preserve the original form of the photon propagator, up to this new factor of $\Pi(k^2)$ included in the denominator. In particular, provided that $\Pi(k^2)$ is regular as $k^2 \to 0$ (which indeed it is), the propagator continues to have a pole at $k^2 = 0$. That is, the photon will remain massless order–by–order in perturbation theory. In addition, we can see that the ξ dependent gauge–fixing contribution receives no renormalization.

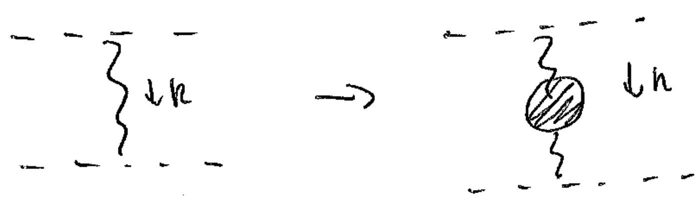

Figure 10: Impact of higher–order corrections to photon propagator on $\phi\phi \to \phi\phi$ scattering.

The above results also provide another, somewhat more intuitive, way to think about the renormalization constant Z_3, without introducing it via a counterterm. In particular, if in this case we consider the impact of the photon propagator corrections to e.g. $\phi\phi$ scattering in scalar QED (see Fig. 10) we can see that this leads to the replacement

$$\frac{e_0^2}{k^2}g_{\mu\nu} \to \frac{e_0^2}{k^2}g_{\mu\nu}\cdot\frac{1}{1-\Pi(k^2)} , \tag{8.82}$$

where we drop the term $\propto k^\mu k^\nu$, as this will give zero when contracted with the $\phi\phi\gamma$ vertex, by the same logic that lead us to (8.61). Thus it is natural to interpret the effect of these corrections as impacting on our assignment for, and corresponding experimental extraction of, the electric charge. One would in particular be led to impose a renormalization condition to relate the electric charge as extracted from a measurement of $\phi\phi$ scattering from the calculated result in terms of the bare parameter e_0, exactly as we did for the λ coupling in Fig. 7. One way to do this is to look at what happens for low energy scattering ($k^2 \to 0$), for which

$$\frac{e_0^2}{k^2} g_{\mu\nu} \to \frac{e_0^2}{k^2} g_{\mu\nu} \cdot \frac{1}{1-\Pi(0)} , \tag{8.83}$$

as the self–energy itself is regular as $k^2 \to 0$. That is, while we have a pole as $k^2 \to 0$, as discussed above, the residue of the pole is shifted from the LO result by an amount

$$\frac{1}{1-\Pi(0)} \equiv Z_3 , \tag{8.84}$$

which we define to be our renormalization constant. Then our renormalization condition is to define the measured value of the electric charge as that extracted from low energy scattering, i.e.

$$\frac{e_R^2}{k^2} g_{\mu\nu} \equiv \frac{e_0^2}{k^2} g_{\mu\nu} \cdot Z_3 , \tag{8.85}$$

and hence

$$e_0 = Z_3^{-1/2} e_R . \tag{8.86}$$

Now excluding the counterterm, we have

$$\Pi(k^2) = -\frac{e^2}{24\pi^2}\frac{1}{\epsilon} + O(\epsilon^0) . \tag{8.87}$$

which substituting into (8.84) gives exactly (8.77) to $O(e^2)$. Thus indeed the calculation in terms of counterterms is completely equivalent to the approach discussed above, and simply provides an often more straightforward way of keeping track of things.

The above discussion in fact corresponds to a particular renormalization scheme; one is in particular not obliged to fix things by looking at the case of low energy scattering. It is known as the 'on–shell' scheme, and will be discussed more in Section 14.6. As we have not bothered to keep track of the finite, $O(\epsilon^0)$ contributions, this gives us the result we need, but in general a bit more care is needed. In addition, we note that in general other types of loop correction can contribute to the RHS of Fig. 10, in particular scalar self–energy and vertex corrections discussed in Sections 8.7 and 8.8, respectively. These can also impact on our renormalization of the electric charge, however for the specific cases of scalar QED and full fermionic QED, their impact cancels and the above discussion does hold. This will be discussed further in Section 14.9.

8.6 $\overline{\text{MS}}$ renormalization scheme

What about the finite parts in (8.77)? As discussed in Section 8.2, in general we are free to absorb any finite pieces into the renormalization constants so long as this is performed consistently everywhere. This corresponds to moving finite pieces between the Z_i and the cross section calculation for a given process, and will leave any physical observables unchanged. However, a choice for how to do this, known as a renormalization scheme, must still be made.

One choice, known as *minimal subtraction* (MS), is to only absorb the poles in ϵ, and not finite pieces, in the renormalization constants. This however tends to leave finite factors of γ in the results for the renormalized observables, which are not particularly convenient and can even lead to some technical issues with convergence of the perturbative expansion in which we calculate observables. For this reason, a more commonly used scheme is modified minimal subtraction, or $\overline{\text{MS}}$. If we consider again the integral

$$
\begin{aligned}
I =& \tilde{\mu}^{\epsilon} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2+X)^2} \,, && (8.88)\\
=& \frac{i}{(4\pi)^2} \left(\frac{4\pi\tilde{\mu}^2}{-X}\right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) \,, && (8.89)\\
=& \frac{i}{(4\pi)^2} \left(\frac{4\pi\tilde{\mu}^2}{-X}\right)^{\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} - \gamma + O(\epsilon)\right) \,, && (8.90)\\
=& \frac{i}{(4\pi)^2} \left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma}}{-X}\right)^{\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} + O(\epsilon)\right) \,, && (8.91)\\
=& \frac{i}{(4\pi)^2} \left(\frac{\mu^2}{-X}\right)^{\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} + O(\epsilon)\right) \,, && (8.92)
\end{aligned}
$$

where we define $\mu^2 \equiv 4\pi\tilde{\mu}^2 e^{-\gamma}$. In the $\overline{\text{MS}}$ scheme, only the poles in ϵ are absorbed, but after we have performed this scale redefinition. This is equivalent to absorbing a finite factor of $\ln(4\pi e^{-\gamma})$ with the original $\tilde{\mu}$, which would not be absorbed in the MS case. Thus this scheme is minimal in the sense that only the ϵ pole is absorbed into the renormalization constant, and modified due to the μ redefinition which therefore removes the $\ln(4\pi e^{-\gamma})$ factor as well.

In the previous case, we therefore have

$$Z_3^{\overline{\rm MS}} = 1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon} + \cdots \ , \tag{8.93}$$

where the '$\cdots$' corresponds to higher–order terms in perturbation theory.

8.7 Scalar propagator

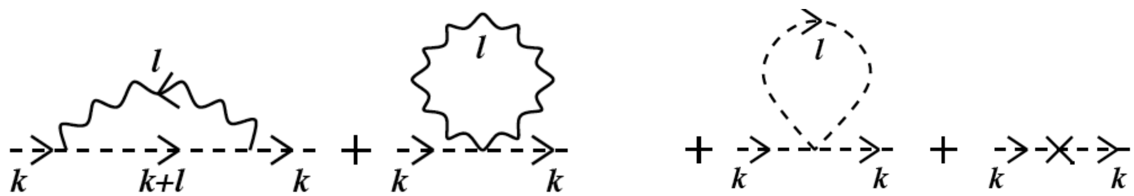

Figure 11: One–loop contributions to scalar propagator.

We first consider the photon corrections to the scalar propagator, shown in Fig. 11. Working in the Lorenz gauge[9], with

$$\tilde{\Delta}_{\mu\nu} = \frac{P_{\mu\nu}(l)}{l^2 - i\epsilon} \ , \tag{8.94}$$

we have

$$i\Pi_{\phi,\gamma}(k^2) = (-iZ_1 e)^2(-i)i\int \frac{{\rm d}^4 l}{(2\pi)^4}\frac{P_{\mu\nu}(l)(l+2k)^\mu (l+2k)^\nu}{l^2((l+k)^2 - m^2)} \ , \tag{8.95}$$

$$+ (2Z_4 i e^2 g^{\mu\nu})(-i)\int \frac{{\rm d}^4 l}{(2\pi)^4}\frac{P_{\mu\nu}}{l^2 - m_\gamma^2} + i(Z_2 - 1)k^2 \ . \tag{8.96}$$

We find that the Z_2 counterterm is all that is required to renormalize this contribution, and hence we have dropped the δ_m counterterm for simplicity. In the second line we have introduced a fictitious mass m_γ as an IR regulator. This removes the divergences present in this integral that are associated with the IR $l \to 0$ region. As discussed above these have nothing to do with the UV $l \to \infty$ region we are interested in for renormalization, and for sensibly defined observables this IR divergence will cancel with contributions from other diagrams. For our purposes however, we can simply remove

[9]In a more general gauge as in (6.58), the results for e.g. $Z_{1,2}$ will be ξ–dependent, though we still have $Z_1 = Z_2$ and all observables are as expected ξ independent.

them by hand by introducing this m_γ. We find

$$\tilde{\mu}^\epsilon \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{l^2 - m_\gamma^2} = \frac{i}{8\pi^2} \frac{1}{\epsilon} m_\gamma^2 \; , \tag{8.97}$$

and therefore this vanishes when we take $m_\gamma^2 \to 0$, with ϵ fixed. We can therefore omit this term.

The calculation of the second contribution, and that due to the 4–point scalar vertex, are left as an exercise in the problem classes. We find:

$$Z_2^{\overline{\mathrm{MS}}} = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\epsilon} + \cdots \; . \tag{8.98}$$

$$Z_m^{\overline{\mathrm{MS}}} = 1 + \frac{\lambda}{8\pi^2} \frac{1}{\epsilon} + \cdots \; . \tag{8.99}$$

8.8 $\gamma\phi\phi^*$ vertex

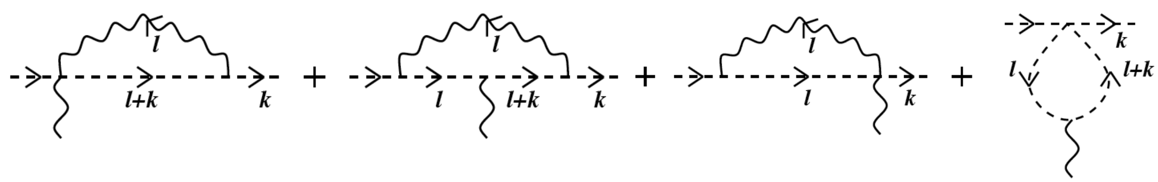

Figure 12: One–loop contributions to $\gamma\phi\phi^*$ vertex.

The contributing 1–loop diagrams are given in Fig. 12. For the renormalization procedure to work at all, we know that the divergent part of the vertex must have the same external momentum dependence as the leading order vertex, i.e.

$$i\mathbf{V}_3^\mu(k, k') \propto -ie(k + k')^\mu \frac{1}{\epsilon} + O(\epsilon^0) \; , \tag{8.100}$$

so that we may absorb the universal divergent part into the renormalization constant, which accompanies the leading–order renormalized vertex

$$-ieZ_1(k + k')^\mu \; . \tag{8.101}$$

This is indeed the case, and thus as we are only interested in the divergence structure here, we can simplify our calculation by taking a clever choice of external momenta; for a complete vertex function, including the full finite pieces, we would need to use a general set of external momenta. In particular, we take the incoming scalar to have zero momentum, and the incoming photon to have momentum k. By momentum conservation the outgoing scalar must also have momentum k. The corresponding diagrams are shown in Fig. 12.

Now, we can see that the left vertex in the second and third diagrams is $-iel^\mu$ and therefore, working again in the Lorenz gauge, these will give vanishing contributions, as $l^\mu P_{\mu\nu}(l) = 0$. For the fourth diagram, we have to evaluate

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(2l+k)^\mu}{(l^2-m^2)((l+k)^2-m^2)} \,, \tag{8.102}$$

however applying the usual Feynman parameterisation as above, the numerator will be

$$(2l+k)^\mu \to 2q^\mu + (1-2x)k^\mu \,, \tag{8.103}$$

where $q = l + xk$. The first term will vanish upon the q integration, and the second term will vanish upon x integration. Thus this diagram also gives no contribution, and we have

$$i\mathbf{V}_3^\mu(k,0) = (-ieZ_1)k^\mu + (-iZ_1 e)(2iZ_4 e^2)g^{\mu\nu} i(-i) \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{P_{\nu\rho}(l+2k)^\rho}{l^2((l+k)^2-m^2)} \,. \tag{8.104}$$

This integral can be performed by using $l^\mu P_{\mu\nu}(l) = 0$ and the replacement $P_{\mu\nu} \to (1-1/D)g_{\mu\nu}$. Using

$$\int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{(q^2+X)^2} = \frac{i}{8\pi^2}\frac{1}{\epsilon} + O(\epsilon^0) \,, \tag{8.105}$$

and setting $Z_i = 1 + O(e^2)$, we find

$$\mathbf{V}_3^\mu(k,0) = -ek^\mu \left(Z_1 - \frac{3e^2}{8\pi^2}\frac{1}{\epsilon} + O(\epsilon^0) \right) \,, \tag{8.106}$$

and we therefore have

$$Z_1^{\overline{\mathrm{MS}}} = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon} + \cdots \,. \tag{8.107}$$

8.9 $\gamma\gamma\phi\phi^*$ vertex

As the tree–level vertex factor $2iZ_4 e^2 g^{\mu\nu}$ does not depend on the external particle momenta we can simply set them to zero. Thus any diagram where an internal photon line attaches to an external scalar via a $\gamma\phi\phi^*$ vertex is zero as above. The remaining

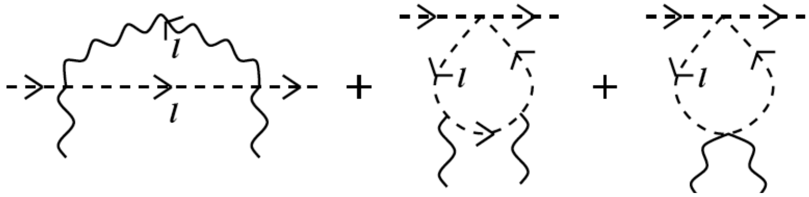

Figure 13: Non–vanishing one–loop contributions to $\gamma\gamma\phi\phi^*$ vertex in the Lorenz gauge.

diagrams are shown in Fig. 13. Here, we have also used that any diagrams with an external photon attached to the internal scalar line in the left hand diagram are zero due to the Lorenz gauge condition $l^\mu P_{\mu\nu} = 0$. It turns out that the divergent parts of the second and third diagrams in fact cancel each other, and so it is only the diagram of the form shown in the first case that contributes.

We end up with

$$\mathbf{V}_4^{\mu\nu}(0,0,0) = -e^2 g^{\mu\nu}\left(-2Z_4 + \frac{3e^2}{4\pi^2}\frac{1}{\epsilon} + O(\epsilon^0)\right) , \tag{8.108}$$

and so we have

$$Z_4^{\overline{\mathrm{MS}}} = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon} + \cdots . \tag{8.109}$$

8.10 $\phi\phi^*\phi\phi^*$ vertex

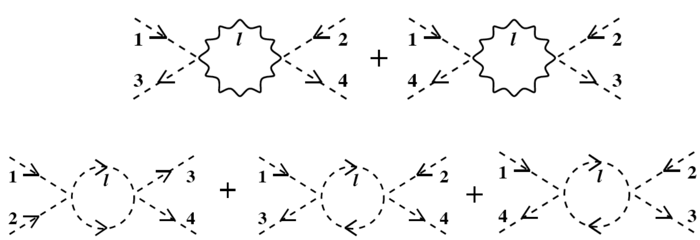

Figure 14: Non–vanishing one–loop contributions to $\phi\phi^*\phi\phi^*$ vertex.

Once again we can set the external momenta to zero, and all diagrams where an internal photon connects with an external scalar vanish. The remaining diagrams are shown in Fig. 14. There are only two independent diagrams, with

$$i\mathbf{V}_{4\phi}(0,0,0) = -iZ_\lambda\lambda + \left(\frac{1}{2}+\frac{1}{2}\right)(2iZ_4e^2)^2(-i)^2\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{g^{\mu\nu}P_{\nu\rho}(l)g^{\rho\sigma}P_{\sigma\mu}(l)}{(l^2-m_\gamma^2)^2}$$
$$+\left(\frac{1}{2}+1+1\right)(-iZ_\lambda\lambda)^2i^2\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{(l^2-m^2)^2}\;, \tag{8.110}$$

where the numerical prefactors sum over the contributing diagram topologies, with their corresponding symmetry factors (Exercise: confirm where these come from). These can be evaluated in the usual way and we end up with

$$Z_\lambda^{\overline{\mathrm{MS}}} = 1 + \left(\frac{3e^4}{2\pi^2\lambda}+\frac{5\lambda}{16\pi^2}\right)\frac{1}{\epsilon}+\cdots\;. \tag{8.111}$$

8.11 Summary

Combining the above results we have, in the $\overline{\mathrm{MS}}$ scheme,

$$Z_1 = Z_2 = Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}+\cdots\;, \tag{8.112}$$
$$Z_3 = 1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon}+\cdots\;. \tag{8.113}$$
$$Z_m = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}+\cdots\;, \tag{8.114}$$
$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda}+\frac{5\lambda}{16\pi^2}\right)\frac{1}{\epsilon}+\cdots\;, \tag{8.115}$$

and thus indeed renormalization at 1–loop order is consistent with the relations (8.45) and (8.46). We will motivate why this result holds from general consideration in the following section.

9. Ward–Takahashi Identities

n.b. This section is non–examinable

9.1 Scalar Field Theory

We saw in Section 4.3 that if a Lagrangian in classical field theory is invariant under a continuous transformation, then from Noether's theorem we have an associated conserved current. However, it is not immediately clear how this will apply the quantum case. In particular, the derivation of this conserved current assumes that classical equations of motion are satisfied. To see how this translates we consider the path integral

$$Z[J] = \int \mathcal{D}\phi \, e^{i[S+\int \mathrm{d}^4y J_a \phi_a]} \tag{9.1}$$

where we have a set of scalar fields labelled by a (summed over above). Considering a change of variables $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$, then we have

$$\delta Z[J] = i \int \mathcal{D}\phi \, e^{i[S+\int \mathrm{d}^4y J_a \phi_a]} \int \mathrm{d}^4x \left(\frac{\delta S}{\delta \phi_a(x)} + J_a(x) \right) \delta\phi_a(x) = 0 \; , \tag{9.2}$$

where we have assumed that this leaves the functional measure invariant. This is not guaranteed: if the measure is not invariant, then the classical symmetry will be broken by quantum corrections and is *anomalous*. Here, we will assume that this is not the case, and indeed for the explicit example of the Ward identity we will consider later, this is true.

Taking n functional derivatives with respect to $J_{a_j}(x_j)$, and then setting $J = 0$ we have

$$\begin{aligned} \int \mathcal{D}\phi \, e^{iS} \int \mathrm{d}^4x \Bigg[& i \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \\ & + \sum_{j=1}^{n} \phi_{a_1}(x_1) \cdots \delta_{a a_j} \delta^4(x - x_j) \cdots \phi_{a_n}(x_n) \Bigg] \delta\phi_a(x) = 0 \; , \end{aligned} \tag{9.3}$$

where the first term comes from acting on the exponent and in the second we have one contribution for each action on the source not in the exponent. Dropping the arbitrary $\delta\phi_a(x)$ and noting the path integral gives the vacuum expectation value of the time–ordered product of fields, we get

$$i \langle 0 | T \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) | 0 \rangle + \sum_{j=1}^{n} \big\langle 0 | T \phi_{a_1}(x_1) \cdots \delta_{a a_j} \delta^4(x - x_j) \cdots \phi_{a_n}(x_n) | 0 \big\rangle = 0 \; . \tag{9.4}$$

These are known as the *Schwinger–Dyson* equations of the theory. Now, consider the simplest case of a single real free scalar field, for which

$$\frac{\delta S}{\delta \phi(x)} = -(\partial_x^2 + m^2)\phi(x) \; . \tag{9.5}$$

From (9.4) with $n = 1$ we get

$$i(\partial_x^2 + m^2) \langle 0 | T \phi(x) \phi(x_1) | 0 \rangle = \delta^4(x - x_1) \; , \tag{9.6}$$

and thus the free field propagator $\Delta(x-x_1) = i\langle 0|T\phi(x)\phi(x_1)|0\rangle$ is a Green's function of the KG wave operator, as we expect. More generally, we have from (9.4) that

$$\langle 0|T\frac{\delta S}{\delta\phi_a(x)}\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)|0\rangle = 0 \quad \text{for} \quad x \neq x_{1,\cdots,n} \; , \tag{9.7}$$

and thus the classical equations of motion are satisfied by a quantum field inside a correlation function as long as its spacetime arguments differ from those of all of the other fields. When this is not the case, we have additional so–called contact terms.

Considering now a theory with a continuous symmetry and corresponding Noether current that leaves the Lagrangian invariant, we substitute (4.14) to get

$$\partial_\mu\langle 0|Tj^\mu(x)\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)|0\rangle + i\sum_{j=1}^n \left\langle 0|T\phi_{a_1}(x_1)\cdots\delta\phi_{a_j}(x_j)\delta^4(x-x_j)\cdots\phi_{a_n}(x_n)|0\right\rangle = 0 \; . \tag{9.8}$$

These are known as the *Ward–Takahashi* identities; they are more general than the Ward identity we have discussed earlier, but as we will now see this latter result follows directly from the former. In conclusion, the Noether current is indeed also conserved in the quantum theory, up to contact terms. The form of these terms depends on the specific transformation, through the $\delta\phi_{a_j}$.

9.2 QED

Consider a scattering process in QED that involves some external photon with momentum k. Recalling the discussion in Section 7.1, the LSZ formula reads

$$\langle f|i\rangle = i\epsilon^\mu \int \mathrm{d}^4x e^{-ikx}\left(\partial^2\right)\cdots\langle 0|TA_\mu(x)\cdots|0\rangle \; . \tag{9.9}$$

The equations of motion for the photon fields, including the renormalization constants, are

$$Z_3\partial^\mu F_{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial A^\mu} \; . \tag{9.10}$$

In the Lorenz gauge, for which the analysis below is simplest, this becomes

$$Z_3\partial^2 A_\mu = -\frac{\partial\mathcal{L}}{\partial A^\mu} = Z_1 j_\mu \; . \tag{9.11}$$

where j^μ is the electromagnetic current. Substituting this in the LSZ formula, we therefore have

$$\langle f|i\rangle = iZ_3^{-1}Z_1\epsilon^\mu \int \mathrm{d}^4x e^{-ikx}\cdots[\langle 0|Tj_\mu(x)\cdots|0\rangle + \text{contact terms}] \ , \tag{9.12}$$

where the contact terms arise, as above, because the classical equations of motion only hold inside quantum correlation functions up to these. Now (see chapter 67 of Srednicki for more details) it can be shown that these contact terms do not in fact contribute to the scattering amplitude, and can therefore be dropped. Recall from Section 5.8 that we required

$$k^\mu \mathcal{M}_\mu = 0 \tag{9.13}$$

for QED to make sense from the point of view of Lorentz invariance. We are now in a position to prove this from gauge invariance only. To see this, if we replace $\epsilon^\mu \to k^\mu$ in (9.12), we can write this as a derivative ∂^μ acting on the e^{-ikx}, and then integrate by parts to get this acting on the correlation function, giving

$$\langle f|i\rangle \to Z_3^{-1}Z_1 \int \mathrm{d}^4x e^{-ikx}\cdots\partial^\mu\langle 0|Tj_\mu(x)\cdots|0\rangle \ , \tag{9.14}$$

Now from the discussion in the previous section we know that the Ward identity gives us

$$\partial^\mu\langle 0|Tj_\mu(x)\cdots|0\rangle = \text{contact terms} \ . \tag{9.15}$$

which again do not contribute to the scattering amplitude. We have therefore proved (9.13) in the full quantum theory.

The Ward–Takahashi identities in fact place quite powerful constraints on our theory, which in addition to the result above lead to precise relations between the renormalization constants we introduced in the previous section. In particular, taking $n = 2$ and considering the explicit form for the Noether current j^μ as in (5.28) for the first term of (9.8) and relating the second term to exact scalar propagators, we can derive an expression

$$(p'-p)_\mu V^\mu_{3,\text{exact}}(p,p') = \frac{Z_1}{Z_2}e\left[\tilde{\Delta}^{\text{exact}}(p)^{-1} - \tilde{\Delta}^{\text{exact}}(p')^{-1}\right] \ , \tag{9.16}$$

where $V_{3,\text{exact}}$ is the exact $\gamma\phi\phi^*$ vertex. The exact vertex and propagators are certainly finite, and thus in the $\overline{\text{MS}}$ scheme (where all corrections to the Z_i are divergent), we indeed need

$$Z_1 = Z_2 \ , \tag{9.17}$$

as required. In fact, this result also holds for the on–shell renormalization scheme, though not necessarily for arbitrary schemes, where the finite contributions to Z_1 and Z_2 may in general be different. In addition, a very similar procedure can be applied to derive the same relation for full QED, see chapter 68 of Srednicki for further details.

10. Fermions

10.1 Representations of the Lorentz Group

A Lorentz transformation corresponds to the change of coordinates

$$(x^\mu)' = \Lambda^\mu_{\ \nu} x^\nu \ , \tag{10.1}$$

that preserves the interval $g_{\mu\nu} x^\mu x^\nu$, implying that the matrix $\Lambda^\mu_{\ \nu}$ must obey

$$g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} = g_{\rho\sigma} \ . \tag{10.2}$$

The set of all Lorentz transformations form a *group*: the product of any two Lorentz transformations is a Lorentz transformation; the product is associative; there is an identity transform $\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu}$; and every Lorentz transformation has an inverse. The $\Lambda^\mu_{\ \nu}$ matrices, written in terms of the usual boosts and rotations, correspond to one particular embedding, or *representation* of this more general group structure onto a vector space, in this case of 4×4 matrices acting on 4–vectors x^μ. This is however not the only possible representation.

We are now interested in finding the other available representations of the Lorentz group. Why? Because we are clearly only interested in writing down those theories which preserve Lorentz invariance, and this exercise will show us the possible ways to do this. More directly, a given type of particle in our universe will have mass, spin and various other quantum numbers. They also have some momenta and spin projection on a given axis. If we rotate or boost to another frame, the momentum and spin projection will change, but the other quantum numbers will not. A particle can therefore be defined as a set of states which mix amongst themselves under Lorentz transformations[10]. This exercise will therefore enable us to enumerate the possible classes of particle.

Generically we can write

$$\Phi'(x) = D(\Lambda)\Phi(\Lambda^{-1}x) \ , \tag{10.3}$$

[10]Or more completely, Poincaré invariance, which includes spacetime translations.

where D (with corresponding field Φ) is some general matrix representation of the group. Here the Λ^{-1} appears because we are dealing with an active transformation, i.e. we must invert the coordinate transformation (10.1) in order to express things in terms of the original field.

A given particle type will correspond to a given representation. For a scalar and spin–1 particle, these are explicitly

$$\phi'(x) = \phi(\Lambda^{-1}x) \;, \qquad A^{\mu'} = \Lambda^{\mu}_{\ \nu} A^{\nu}(\Lambda^{-1}x) \;. \tag{10.4}$$

While the scalar field transforms trivially (i.e. $D(\Lambda) = 1$), for the spin–1 field, which carries a Lorentz index, the transformation of the field itself is given in terms of Λ matrices. We now wish to find if there are any other representations of the Lorentz group, corresponding to other particle types. More specifically, we are interested in the case where no subset of states only transform amongst themselves, known as an *irreducible* representation.

To pursue this further, we start with the standard 4–vector representation that we are used to. We can then write any infinitesimal Lorentz transformation as a simple combination of rotations and boosts, given by

$$\delta x_\mu = i \sum_{i=1}^{3} \left[\theta_i (J_i)_{\mu\nu} + \chi_i (K_i)_{\mu\nu}\right] x^\nu \;, \tag{10.5}$$

where the J_i correspond to rotations around the three (x, y, z) axes, with angles θ_i, and the K_i correspond to boosts in the three (x, y, z) directions with boost parameters χ_i. These are the *generators* of the Lorentz group, in the sense that they form a complete basis from which we can construct (i.e. which generate) all Lorentz transformations. Explicitly we can write:

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \;, \qquad J_2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \;, \qquad J_3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;, \tag{10.6}$$

$$K_1 = i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;, \qquad K_2 = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;, \qquad K_3 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \;, \tag{10.7}$$

where the matrix indices (defined as $(K_i)^\mu{}_\nu$ and similarly for the rotations) are suppressed for clarity. A convenient, but completely equivalent, way to express this is to index the generators by $M^{\rho\sigma}$ instead of J_i and K_i, with $J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$ and $K_i \equiv M_{0i}$. This leads explicitly to

$$M^{\rho\sigma} = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{pmatrix} . \tag{10.8}$$

Here each value of the antisymmetric matrix $M^{\rho\sigma}$ is itself a 4×4 matrix, e.g.

$$(M_{01})_{\mu\nu} = (K_1)_{\mu\nu} \ , \tag{10.9}$$

where the μ, ν label as usual the indices of the matrix K_1. An explicitly covariant expression for M is given by

$$(M^{\rho\sigma})^{\mu\nu} = i\left(g^{\rho\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\rho\nu}\right) \ , \tag{10.10}$$

which we will make use of a little later, though not for now.

Written in terms of this (10.5) becomes

$$\delta x_\mu = i\theta_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu}x^\nu \ , \tag{10.11}$$

where the $\theta_{\rho\sigma}$ define six parameters (with $\theta_{\rho\sigma} = -\theta_{\sigma\rho}$) which again specify the Lorentz transformation in terms of boost and rotation angles. To be precise, we have $\chi_i = \theta^{0i}$ and $\theta_i = \frac{1}{2}\epsilon_{ijk}\theta^{jk}$. A general (non–infinitesimal) Lorentz transformation can then be written by exponentiating, with in this case

$$D(\Lambda) = \exp(i\theta_i J_i + i\chi_i K_i) = \exp(i\theta_{\rho\sigma}M^{\rho\sigma}) \ . \tag{10.12}$$

Now, we have the simple group constraint (which we of course expect physically be to be true) that the product of two Lorentz transformations should itself be a Lorentz transformations, i.e.

$$D(\Lambda'\Lambda) = D(\Lambda')D(\Lambda) \ . \tag{10.13}$$

Imposing this, and expanding (10.12) to linear order in the θ, it can be shown that the generators $M^{\mu\nu}$ have to satisfy the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}) \ . \tag{10.14}$$

These relations, known as the *Lie algebra*, are fundamental to the definition of the Lorentz group. They are indeed satisfied by (10.10), but as the requirement (10.13) must be satisfied irrespective of the representation, the generators for any representation must also satisfy these.

Writing the Lie algebra relation in terms of the J_i and K_i we find

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \ , \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \ , \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \ . \end{aligned} \tag{10.15}$$

The first of these is the usual set of commutators for angular momentum, i.e. $SO(3)$ rotations, as we would expect. The second says that $\mathbf{K}$ transforms as a 3–vector under rotations, while the third tells us that two boosts are equivalent to a rotation.

Having derived these expressions with the particular case of the 4–vector representation in mind, we would now like to be more general. In particular, we would like to find all possible sets of finite–dimensional matrices which satisfy these relations. To this end, we can define some new non–hermitian operators whose physical significance is not immediately clear, but which simplify the commutation relations greatly. In particular,

$$\begin{aligned} N_i^- &\equiv \frac{1}{2}(J_i - iK_i) \ , \\ N_i^+ &\equiv \frac{1}{2}(J_i + iK_i) \ , \end{aligned} \tag{10.16}$$

in terms of which the commutation relations become simply

$$\begin{aligned} \left[N_i^-, N_j^-\right] &= i\epsilon_{ijk} N_k^- \ , \\ [N_i^+, N_j^+] &= i\epsilon_{ijk} N_k^+ \ , \\ [N_i^-, N_j^+] &= 0 \ . \end{aligned} \tag{10.17}$$

These correspond to two independent Lie algebras of the $SU(2)$ group, which we denote $SU(2)_L \times SU(2)_R$. Such an identification greatly helps in finding the corresponding representations.

Of course we can immediately write down one representation of $SU(2)$, namely the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{10.18}$$

Although for our purposes this simplest case is sufficient, more generally while these correspond to the simplest, so–called fundamental representation of $SU(2)$, there exist alternative higher dimensional representations. In particular, for every half–integer n we have a set of three $(2n+1)\times(2n+1)$ hermitian matrices J_1, J_2, J_3 such that J_3 (say) has $(2n+1)$ eigenvalues from $-n,\cdots,n$. Indeed, this result follows from precisely the same procedure of identifying raising and lowering matrices that one would do in e.g. the standard analysis in quantum mechanics for identifying the eigenvalues of the spin/angular–momentum operator. We can label the corresponding representation by this index n, with the fundamental representation in terms of Pauli matrices above corresponding to the $n=\frac{1}{2}$ case, as one would expect from their appearance in the case of spin–$\frac{1}{2}$ particles.

Now, in (10.17) we have two such representations, and so we label the corresponding representations with two integers or half–integers n and m. A more physical direct way to label this comes from noting that the original angular momentum operator is given by $J_i = N_i^- + N_i^+$, and hence finding the allowed values of j becomes a standard problem in the addition of angular momenta, with $j=|n-m|,\cdots,n+m$.

Denoting the $J=n+m$ in the usual way, and writing the corresponding breakdown as (n,m), the four simplest and most common cases are

$$(0,0):\ \text{scalar},\ J=0\ , \tag{10.19}$$

$$\left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right):\ \text{left}-\text{handed and right}-\text{handed spinors},\ J=\frac{1}{2}\ , \tag{10.20}$$

$$\left(\frac{1}{2},\frac{1}{2}\right):\ 4-\text{vector},\ J=0\oplus J=1\ . \tag{10.21}$$

The $J=0$ case corresponds to the usual scalar case, while the final representation contains both $J=0$ and $J=1$, which is precisely what we expect for a 4–vector for which the time component is a scalar under rotations (i.e. '$J=0$') and the remaining three $J=1$ degrees of freedom are the usual spatial components. The two remaining $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ cases are completely new. We will call these left and right handed *spinor* representations.

10.2 Spinors

What do the representations for the $J=\frac{1}{2}$ case look like? As discussed above, the vector space they act on has $2J+1=2$ degrees of freedom, and therefore we need a set of 2×2 matrices which satisfy (10.17), which are precisely the Pauli matrices(10.18). For these we have

$$\left[\frac{\sigma_i}{2},\frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}\ , \tag{10.22}$$

and therefore these suitably normalized matrices form the representation we need. Thus we set $N_i^- = \frac{\sigma_i}{2}$ and $N_i^+ = 0$ for $(\frac{1}{2}, 0)$ and $N_i^- = 0$ and $N_i^+ = \frac{\sigma_i}{2}$ for $(0, \frac{1}{2})$. For the Lorentz transformations, from the definitions (10.16) we then get

$$\begin{aligned} \left(\frac{1}{2}, 0\right): \quad & J_i = \frac{\sigma_i}{2} \quad K_i = i\frac{\sigma_i}{2} \ , \\ \left(0, \frac{1}{2}\right): \quad & J_i = \frac{\sigma_i}{2} \quad K_i = -i\frac{\sigma_i}{2} \ . \end{aligned} \tag{10.23}$$

We call the elements of the vector space on which the spin-$\frac{1}{2}$ representations act spinors. We say that the $(\frac{1}{2}, 0)$ representation acts on left–handed Weyl spinors, denoted ψ_L, while the $(0, \frac{1}{2})$ representation acts on right–handed Weyl spinors, ψ_R. For rotation angles θ_i and boost parameters χ_j, from (10.12) we can see that these transform as

$$\psi'_L = e^{\frac{1}{2}(i\theta_j\sigma_j - \chi_j\sigma_j)}\psi_L \ , \qquad \psi'_R = e^{\frac{1}{2}(i\theta_j\sigma_j + \chi_j\sigma_j)}\psi_R \ . \tag{10.24}$$

We can combine the left and right handed spinors to form a four–component quantity

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ , \tag{10.25}$$

known as a *Dirac spinor*. We will identify these objects (suitably quantized) below with the spin–$\frac{1}{2}$ fermion fields we need to build up the full theory of QED and the Standard Model.

How do we write down the Lorentz transformation properties of these spinors? To start off we define the seemingly unrelated 4×4 matrices γ^μ, which satisfy the so–called *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1} \ . \tag{10.26}$$

Here the $(\gamma^\mu)_{ij}$ correspond to a set of 4 matrices, labelled by $\mu = (0 \cdots 3)$, carrying (unrelated) indices $i, j = (1 \cdots 4)$. (Exercise: show that it is not possible to satisfy these relations with lower–dimensional matrices). The $\mathbb{1}$ on the right hand side is the unit matrix in spinor space. One representation of these, given in the so–called Weyl basis is

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ , \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \tag{10.27}$$

where the I and σ_i are 2×2 matrices. It is easy to verify that the Clifford algebra indeed holds for this case. Now, if we define[11]

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \ , \tag{10.28}$$

then by application of (10.26) it can be shown that this forms a representation of the Lorentz algebra, i.e. it satisfies the commutation relation (10.14). In fact, without resorting to this we can show this more simply by considering the corresponding boost and rotation generators in this (claimed) representation. Using (10.28) we find that these should be given by

$$K_i \equiv S_{0i} = \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \ , \qquad J_i \equiv \frac{1}{2}\epsilon_{ijk}S^{jk} = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \ . \tag{10.29}$$

and thus the transformations (10.23) for the left and right handed spinors are completely equivalent to the Dirac spinor transformation

$$\psi' = S[\Lambda]\psi \ , \tag{10.30}$$

where

$$S[\Lambda] = \exp\left(i\theta_i J_i + i\chi_i K_i\right) = \exp\left(iS_{\mu\nu}\theta^{\mu\nu}\right) \ , \tag{10.31}$$

with the J_i and K_i given as in (10.29), and the $\chi_i = \theta^{0i}$ and $\theta_i = \frac{1}{2}\epsilon_{ijk}\theta^{jk}$ as before. Thus the Dirac spinor indeed transforms according to the representation of the Lorentz group defined by $S_{\mu\nu}$. Note also the block diagonal form of these generators (10.29), which makes it clear that this representation is reducible in terms of the two (irreducible) left and right handed spinor representations.

Now, consider the action of a rotation around the z–axis on a Dirac spinor by an angle θ. From (10.29) this corresponds to

$$S[\Lambda] = \begin{pmatrix} e^{i\frac{\theta}{2}\sigma_3} & 0 \\ 0 & e^{i\frac{\theta}{2}\sigma_3} \end{pmatrix} = \begin{pmatrix} I\cos\frac{\theta}{2} + i\sigma_3\sin\frac{\theta}{2} & 0 \\ 0 & I\cos\frac{\theta}{2} + i\sigma_3\sin\frac{\theta}{2} \end{pmatrix} \ . \tag{10.32}$$

Thus for a 2π rotation (which should get us back to where we started!) we have

$$\psi' = -\psi \ , \tag{10.33}$$

which is certainly not what we would get when rotating a vector. This demonstrates that the spinor representation is definitely something completely distinct from the

[11] Often the notation $\sigma^{\mu\nu} \equiv 2S^{\mu\nu}$ is used.

vector representation we are used to. Moreover, it is this additional minus sign which results in the spin–statistics connection; it is the same minus sign which arises when we interchange identical fermions, an action which can be performed by exactly such a rotational transformation.

Finally, if we define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} , \tag{10.34}$$

then we can form the left/right handed projector

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma_5) , \tag{10.35}$$

in terms of which we have

$$\psi_{L,R} = P_{L,R}\psi . \tag{10.36}$$

Here, we have been slightly sloppy with notation and interpret these as 4–component Dirac spinors but with only the left or right–handed components non–zero. In other words, we have

$$\gamma_5\psi_{L,R} = \mp\psi_{L,R} . \tag{10.37}$$

The γ_5 is known as the *chirality operator*, and the left and right handedness of the $\psi_{L,R}$ representations is known as *chirality*.

10.3 The Dirac Equation

We wish to build up an appropriate Lagrangian from these spinor fields. What kind of Lorentz invariants can we construct? We have

$$\psi_L^\dagger \to \psi_L^\dagger e^{\frac{1}{2}(-i\theta_j\sigma_j - \chi_j\sigma_j)} , \qquad \psi_R^\dagger \to \psi_R^\dagger e^{\frac{1}{2}(-i\theta_j\sigma_j + \chi_j\sigma_j)} . \tag{10.38}$$

as the Pauli matrices are Hermitian. It follows from this and (10.24) that $\psi^\dagger_{L,R}\psi_{R,L}$ is Lorentz invariant, while $\psi^\dagger_{L,R}\psi_{L,R}$ is not. We can therefore make use of the Lorentz scalar

$$\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R = \psi^\dagger\gamma^0\psi \equiv \overline{\psi}\psi , \tag{10.39}$$

where we have written the combination in Dirac spinor notation, and defined $\overline{\psi} \equiv \psi^\dagger\gamma^0$. An alternative way to demonstrate this, using the Dirac spinor transformation properties alone, comes from noting that

$$(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0 . \tag{10.40}$$

which follows from $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. This leads to

$$S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0 \ , \tag{10.41}$$

from which it follows that $\overline{\psi} \to \overline{\psi} S[\Lambda]^{-1}$ and hence $\overline{\psi}\psi$ is invariant.

Now consider the combination

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi} S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi \ , \tag{10.42}$$

where the γ^μ is unchanged as it does not itself transform covariantly (it is simply a fixed set of matrices). Thus if we have

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu_{\ \nu}\gamma^\nu \ , \tag{10.43}$$

then (10.42) will transform as a Lorentz vector. Considering infinitesimal transformations, we find this is equivalent to

$$[S^{\rho\sigma}, \gamma^\mu] = -(M^{\rho\sigma})^\mu_{\ \nu}\gamma^\nu \ . \tag{10.44}$$

After a little work with the explicit forms for $S^{\rho\sigma}$ and $M^{\rho\sigma}$, as given by (10.28) and (10.10) it can be shown that this is indeed the case.

Thus, a Lorentz invariant candidate for a Lagrangian describing a free Dirac field is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \ , \tag{10.45}$$

known as the Dirac Lagrangian. From the equations of motion we then arrive at the *Dirac Equation*

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \ . \tag{10.46}$$

where the 'm' is shorthand for '$m \cdot \mathbb{1}$', i.e. multiplying a unit matrix in spinor space. Introducing the *Feynman slash* notation, $a_\mu\gamma^\mu \equiv \not{a}$, we can write this in the slightly more compact form

$$(i\not{\partial} - m)\psi = 0 \ . \tag{10.47}$$

Now, we have

$$(i\not{\partial} + m)(i\not{\partial} - m)\psi = -(\partial^2 + m^2)\psi = 0 \ , \tag{10.48}$$

and thus ψ satisfies the Klein–Gordon equation (5.3) for a field with mass m. This justifies our association of (10.45) with a mass term. Note in particular that this is linear in the mass, in contrast to the scalar and vector cases, which are quadratic $\sim m^2$. This is directly related to the fact that the Dirac Lagrangian (and hence Dirac equation) is linear in derivatives; indeed, from dimensional counting alone, we know that we must associate a factor of m with $\overline{\psi}\psi$.

How does the Dirac Lagrangian look in two–component form? We find

$$\mathcal{L} = i\psi_L^\dagger \overline{\sigma}_\mu \partial^\mu \psi_L + i\psi_R^\dagger \sigma_\mu \partial^\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \ , \tag{10.49}$$

where we have defined $\sigma = (1, \sigma^i)$ and $\overline{\sigma} = (1, -\sigma^i)$. Thus, a massive fermion requires both left and right handed components, as these couple through the mass term, while a massless fermion can be described with either of these alone.

The corresponding Dirac equations take the form

$$i\overline{\sigma}_\mu \partial^\mu \psi_L = (E + \vec{p}\cdot\vec{\sigma})\psi_L = m\psi_R \ , \tag{10.50}$$

$$i\sigma_\mu \partial^\mu \psi_R = (E - \vec{p}\cdot\vec{\sigma})\psi_R = m\psi_L \ , \tag{10.51}$$

where we have moved to momentum space (substituting $p^\mu \to i\partial^\mu$). In the massless limit these decouple, as expected, and we find

$$\hat{h}\psi_{L,R} \equiv \frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|}\psi_{L,R} = \mp\psi_{L,R} \tag{10.52}$$

where we have used that $|\vec{p}| = E$ for massless particles. Here $\hat{h}$ corresponds to the projection of the spin on the direction of the momentum, and is known as the *helicity*. Thus, comparing with (10.37) we can see that in the massless limit the helicity and chirality eigenstates coincide. This is not true away from the massless limit, and indeed here helicity is no longer a good quantum number; we can always reverse the direction of $\vec{p}$ by a suitable Lorentz boost, swapping the sign of the h. However it is still a useful observable to consider, in particular in the relativistic $E \gg m$ limit, where chirality and helicity will still closely coincide.

10.4 Solutions of the Dirac equation

Recalling from (10.48) that the spinor solution to the Dirac equation also obeys the Klein–Gordon equation, we know that this will allow plane–wave solutions. We therefore consider the (classical) trial solution of the form

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0}\left[u(\mathbf{p})e^{-ipx} + v(\mathbf{p})e^{ipx}\right] \ , \tag{10.53}$$

where with $p^0 = (\mathbf{p}^2 + m^2)^{1/2}$ we see that this indeed satisfies the KG equation, as required. As ψ is itself a 4–component spinor, we must introduce corresponding spinor coefficients, which we denote $u(\mathbf{p})$, $v(\mathbf{p})$, in the expansion. To satisfy the KG equation, these must be independent of x. As ψ must satisfy the Dirac equation, we can extract solutions for these. Plugging (10.53) in the Dirac equation we find

$$
\begin{aligned}
(\not{p} - m)u(\mathbf{p}) = 0 \; , \\
(\not{p} + m)v(\mathbf{p}) = 0 \; .
\end{aligned} \tag{10.54}
$$

As we will confirm explicitly below, these each admit two linearly independent solutions, that we will label $u_\pm(\mathbf{p})$ and $v_\pm(\mathbf{p})$. In the particle rest frame we have simply $\not{p} = m\gamma_0$ and (10.54) reduce to

$$
\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u_s = \begin{pmatrix} I & I \\ I & I \end{pmatrix} v_s = 0 \; , \tag{10.55}
$$

which have solutions

$$
u_s = \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \; , \qquad v_s = \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix} \; , \tag{10.56}
$$

for arbitrary two–component spinors ξ_s, η_s. Choosing a particular normalization and spin labelling for the spinors, four linearly independent solutions are

$$
u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \; u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} , \; v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} , \; v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{10.57}
$$

We will associate these four degrees of freedom with the $\pm$ spin states of the particle, $u_\pm$, and its anti–particle, $v_\pm$.

The $\pm$ labelling is derived by noting that, given the two–fold degeneracy in the solutions for $u(\mathbf{p})$, $v(\mathbf{p})$ there must be some operator which commutes with the energy operator and whose eigenvalues label the two solutions. This is in fact given by considering the projection of the angular momentum along a given axis defined by the unit vector $\hat{n}$

$$
S_n = \hat{n}^i J_i = \frac{1}{2}\epsilon_{ijk}\hat{n}^i S^{jk} = \frac{1}{2}\hat{n}_i \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \; , \tag{10.58}
$$

which if $\hat{n}$ is taken as the direction of the particle momenta this corresponds to the helicity of the particle introduced in (10.52). Considering the projection along the

z–axis we find

$$S_z u_\pm(\mathbf{0}) = \pm\frac{1}{2}u_\pm(\mathbf{0}) \;, \tag{10.59}$$

$$S_z v_\pm(\mathbf{0}) = \mp\frac{1}{2}v_\pm(\mathbf{0}) \;, \tag{10.60}$$

where the reason for the swapped sign in the second case relates to their association with an antiparticle. Thus, we are justified in associating the $\pm$ solutions with the two spin states of the particle and antiparticle. Moreover, we can see the eigenvalues of this operator are $1/2$, confirming our association of these states as spin–$\frac{1}{2}$ particles (after quantization). In particular, we are considering the angular momentum projection for a particle *at rest*, and which therefore certainly has no angular momentum by virtue of its movement. This angular momentum is therefore intrinsic, and is precisely the spin of the particle.

The solutions (10.57) in the particle rest frame can then be converted to general solutions by applying the appropriate boost and rotations, or alternatively by solving the Dirac equation for general momentum p. In either case, we get

$$u_\pm = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_\pm \\ \sqrt{p\cdot\overline{\sigma}}\xi_\pm \end{pmatrix} \;, \qquad v_\pm = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta_\pm \\ -\sqrt{p\cdot\overline{\sigma}}\eta_\pm \end{pmatrix} \;. \tag{10.61}$$

In the relativistic $E \gg m$ limit, it can be shown that the helicity and chirality eigenstates again coincide, with $\gamma_5 u_\pm = \pm u_\pm$ and $\gamma_5 v_\pm = \mp v_\pm$.

An identity that will be particularly useful when calculating cross section summed (or averaged) over the spins of the interacting fermions can then be derived from these expressions and noting that $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s \eta_s^\dagger = I$, i.e. the usual completeness relation. We then find that

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \not{p} + m \;, \qquad \sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \not{p} - m \;, \tag{10.62}$$

which are known as the spin sum relations.

10.5 Aside: Photon Polarization Vectors and Spin

As an aside, we note that the above formalism also confirms for us the spin–1 nature of the photon field A_μ introduced before. To see this, we consider a photon travelling along the z axis for simplicity, for which the corresponding circular polarization vectors are given by (5.63) and (5.64). In the same way as for (10.58) we then calculate the projection of the photon angular momentum along the z axis to be

$$J_3 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{10.63}$$

for which we indeed have

$$(J_3)^\mu{}_\nu \epsilon^\nu_\pm(k) = \pm\epsilon^\mu_\pm(k) , \tag{10.64}$$

where we have kept track of Lorentz indices for completeness but they are not really relevant here, as we only interested in pure spatial rotation. Thus indeed the photon corresponds to a spin–1 state, with this choice of polarization vectors giving the ± 1 helicity eigenstates. Note that though we have considered the case of a photon moving along the z axis for simplicity, this will hold for an arbitrary directions provided we suitably choose our n in (10.58) to lie in the direction of the photon momentum.

11. Path Integrals for Fermions

11.1 Grassmann Variables

When we quantize, our expansion for the Dirac field will have the form

$$\psi(x) = \sum_{s=\pm} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ipx} \right] . \tag{11.1}$$

where $b_s(b_s^\dagger)$ and $d_s(d_s^\dagger)$ are the annihilation (creation) operators for the particle and antiparticle, respectively. Now, using this expansion it turns out that we cannot construct a sensible theory, with in particular a Hamiltonian that is bounded from below, by imposing the usual equal–time commutation relations. Instead we are led to impose *anticommutation* relations, which written in terms of the fields are

$$\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} = 0 , \tag{11.2}$$

$$\{\psi_\alpha(\mathbf{x},t), \overline{\psi}_\beta(\mathbf{y},t)\} = (\gamma_0)_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}) , \tag{11.3}$$

where we have kept the spinor indices explicit for clarity. The reason for this is directly related to the fermionic nature of the fields and the spin–statistics connection. We will not discuss this in detail here, but refer the student to chapter 12 of Schwartz and chapter 3 of Peskin and Schroeder for more information.

This anticommuting nature of the fermion fields leads to some added complication if we wish to pursue the path integral approach. In particular, in our equivalent of (6.4) for fermionic fields we must account for the fact that on the left hand side we have field operators which anticommute, while on the right hand side we deal with classical field values. For this to work consistently we must therefore introduce a new type of anticommuting number, known as a *Grassmann* variable, which we will use when dealing with the partition function on the right hand side. For a set of such variables η_i, with $i = 1, ..., n$, these obey

$$\{\eta_i, \eta_j\} = 0 \; , \tag{11.4}$$

which commute with the usual (commuting!) numbers, $a\eta_i = \eta_i a$. We thus have

$$\eta_i^2 = 0 \; , \qquad \eta_1(\eta_2\eta_3) = (\eta_2\eta_3)\eta_1 \; . \tag{11.5}$$

Where the second identity tells us that a combination of two such Grassmann variables behaves like a normal commuting number.

These identities actually limit quite significantly the non–zero objects we can write down. Thus, considering the simplest case of one Grassmann variable η, we can define a function $f(\eta)$ via the Taylor expansion

$$f(\eta) = a + \eta\mathbf{b} \; , \tag{11.6}$$

where due to $\eta^2 = 0$ all higher terms vanish. We will be interested in the case that the function $f(\eta)$ is itself commuting, for which the variable $\mathbf{b}$ must be anticommuting (we will use boldface in the this section to denote this), so that $\mathbf{b}f(\eta) = f(\eta)\mathbf{b}$. Thus

$$f(\eta) = a + \eta\mathbf{b} = a - \mathbf{b}\eta \; . \tag{11.7}$$

We can then derive two types of derivative. The *left derivative* with respect to η is given by the coefficient of η when $f(\eta)$ is written with η on the far left, and the *right derivative* with it on the far right. Explicitly,

$$\partial_\eta f(\eta) = +\mathbf{b} \; , \qquad f(\eta)\overleftarrow{\partial}_\eta = -\mathbf{b} \; . \tag{11.8}$$

When we consider the derivatives with respect to Grassmann variables, we will always refer to the left derivative. Note that this left derivative rule implies that the derivative itself should be treated as a Grassmann valued object, so that

$$\partial_\eta(\mathbf{b}\eta) = -\mathbf{b}\partial_\eta\eta = -\partial_\eta(\eta\mathbf{b}) = -\mathbf{b}\ . \tag{11.9}$$

We would also like to define a definite integral, in analogy to integrating over a real commuting variable x over $\pm\infty$. To give sensible results, we demand that this integral (when it converges) obeys linearity

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, cf(x) = c\int_{-\infty}^{+\infty} \mathrm{d}x\, f(x)\ , \tag{11.10}$$

and shift invariance

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, f(x+a) = \int_{-\infty}^{+\infty} \mathrm{d}x\, f(x)\ . \tag{11.11}$$

Again the nature of Grassmann variables restricts things quite a bit. For shift invariance, under $\eta \to \eta + \xi$, we have

$$\int_{-\infty}^{+\infty} \mathrm{d}\eta\,(a+\eta\mathbf{b}) = \int_{-\infty}^{+\infty} \mathrm{d}\eta\,(a+\eta\mathbf{b}+\xi\mathbf{b})\ , \tag{11.12}$$

and thus we require

$$\int_{-\infty}^{+\infty} \mathrm{d}\eta\,\xi\mathbf{b} = \xi\mathbf{b}\int_{-\infty}^{+\infty} \mathrm{d}\eta\, 1 = 0\ , \tag{11.13}$$

where we have used linearity in the last step. Thus we require

$$\int \mathrm{d}\eta\, 1 = 0\ , \tag{11.14}$$

where the integration limits are implied, and we have

$$\int \mathrm{d}\eta\, f(\eta) = \mathbf{b}\int \mathrm{d}\eta\,\eta\ . \tag{11.15}$$

This right hand integral should then be some number, and we conventionally define

$$\int \mathrm{d}\eta\,\eta = 1\ . \tag{11.16}$$

Note it is important to keep track of ordering here, and whether we are dealing with

Grassmann or ordinary variables. Thus we have

$$\int \mathrm{d}\eta\, \alpha(a + \mathbf{b}\eta) = -\alpha\mathbf{b} \ , \tag{11.17}$$

$$\int \mathrm{d}\eta\, \alpha(a + \eta\mathbf{b}) = \alpha\mathbf{b} \ , \tag{11.18}$$

$$\int \mathrm{d}\eta\, \alpha(\mathbf{a} + b\eta) = \alpha b \ . \tag{11.19}$$

We also have

$$\int \mathrm{d}\eta\, f(c\eta) = c\int \mathrm{d}\eta\, \eta\mathbf{b} = c\int \mathrm{d}\eta\, f(\eta) \ , \tag{11.20}$$

to be contrasted with the ordinary case

$$\int \mathrm{d}x\, f(cx) = \frac{1}{c}\int \mathrm{d}x\, f(x) \ . \tag{11.21}$$

We can readily generalise to the multi–dimensional case, with $\mathrm{d}\eta \to \mathrm{d}\eta_n \cdots \mathrm{d}\eta_1 \equiv \mathrm{d}^n\eta$ (note the ordering). Then, considering some linear change of variables

$$\eta_i = J_{ij}\eta'_j \ , \tag{11.22}$$

where J_{ij} is a matrix of commuting numbers (and therefore can be written on either side of η'_j), one can readily show that the analogue of (11.20) becomes

$$\mathrm{d}^n\eta \to \frac{\mathrm{d}^n\eta'}{(\mathrm{det}J)} \ , \tag{11.23}$$

which we contrast with the usual coordinate transformation $\mathrm{d}^n x \to (\mathrm{det}J)\mathrm{d}^n x'$.

We now use this to develop the concept of a Gaussian integral with respect to Grassmann variables. We consider the integral

$$\int \mathrm{d}^n\eta\ \exp(\frac{1}{2}\eta^T M\eta) = \int \mathrm{d}^n\eta\ \exp(\frac{1}{2}\eta_i M_{ij}\eta_j) \ , \tag{11.24}$$

where M_{ij} is an antisymmetric matrix of commuting (possibly complex) numbers. For $n = 2$ we have

$$M = \begin{pmatrix} 0 & +m \\ -m & 0 \end{pmatrix} \ , \tag{11.25}$$

and $\eta^T M\eta = 2m\eta_1\eta_2$. We therefore have

$$\int \mathrm{d}^n\eta \ \exp(\frac{1}{2}\eta^T M\eta) = \int \mathrm{d}^2\eta\,(1 + m\eta_1\eta_2) = m \ . \tag{11.26}$$

For general n, it can be shown that (see Srednicki chapter 44)

$$\int \mathrm{d}^n\eta \ \exp(\frac{1}{2}\eta^T M\eta) = (\det M)^{1/2} \ , \tag{11.27}$$

which it is instructive to compare with the usual Gaussian integral of commuting numbers

$$\int \mathrm{d}^n x \exp(-\frac{1}{2}x^T M x) = (2\pi)^{n/2}(\det M)^{-1/2} \ . \tag{11.28}$$

Finally, we introduce the concept of *complex* Grassmann variables via

$$\chi \equiv \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2) \ , \qquad \overline{\chi} \equiv \frac{1}{\sqrt{2}}(\eta_1 - i\eta_2) \ , \tag{11.29}$$

for which one readily finds that $\chi\chi = \overline{\chi\chi} = 0$ and $\{\chi, \overline{\chi}\} = 0$, as we would like. The above relation has inverse

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \overline{\chi} \\ \chi \end{pmatrix} \ , \tag{11.30}$$

which has determinant $-i$. Thus we have

$$\mathrm{d}^2\eta = \mathrm{d}\eta_2\mathrm{d}\eta_1 = \frac{1}{-i}\mathrm{d}\chi\mathrm{d}\overline{\chi} \ , \tag{11.31}$$

and so

$$\int \mathrm{d}\chi\mathrm{d}\overline{\chi}\,\overline{\chi}\chi = (-i)(-i)^{-1}\int \mathrm{d}\eta_2\mathrm{d}\eta_1\,\eta_1\eta_2 = 1 \ . \tag{11.32}$$

Thus for some function

$$f(\chi, \overline{\chi}) = a + \chi\mathbf{b} + \overline{\chi}\mathbf{c} + \overline{\chi}\chi d \ , \tag{11.33}$$

then we have $\int \mathrm{d}\chi\mathrm{d}\overline{\chi} f(\chi, \overline{\chi}) = d$ and in particular

$$\int \mathrm{d}\chi\mathrm{d}\overline{\chi}\exp(m\overline{\chi}\chi) = m \ . \tag{11.34}$$

For n complex Grassmann variables χ_i and their complex conjugates $\overline{\chi}_i$ we define

$$\mathrm{d}^n\chi \mathrm{d}^n\overline{\chi} \equiv \mathrm{d}\chi_n \mathrm{d}\overline{\chi}_n \cdots \mathrm{d}\chi_1 \mathrm{d}\overline{\chi}_1 \ , \tag{11.35}$$

and it follows that

$$\int \mathrm{d}^n\chi \mathrm{d}^n\overline{\chi} \exp(\overline{\chi} M \chi) = \det M \ , \tag{11.36}$$

where now χ is understood as a column with entries χ_i $(i = 1, \cdots, n)$, and similarly for $\overline{\chi}$. Finally, we can generalize our results by changing the integration variables and using shift invariance. We have

$$\int \mathrm{d}^n\eta \exp(\eta^T M \eta + \xi^T \eta) = (\det M)^{1/2} \exp(\frac{1}{2}\xi^T M^{-1}\xi) \ , \tag{11.37}$$

$$\int \mathrm{d}^n\chi \mathrm{d}^n\overline{\chi} \exp(\frac{1}{2}\overline{\chi} M \chi + \overline{\xi}\chi + \overline{\chi}\xi) = (\det M) \exp(-\overline{\xi} M^{-1}\xi) \ , \tag{11.38}$$

where in the latter case for example we make the shifts $\chi \to \chi - M^{-1}\xi$ and $\overline{\chi} \to \overline{\chi} - \overline{\xi} M^{-1}$, giving

$$(\overline{\chi} - \overline{\xi} M^{-1}) M (\chi - M^{-1}\xi) = \overline{\chi} M \chi - \overline{\xi}\chi - \overline{\chi}\xi + \overline{\xi} M^{-1}\xi \ , \tag{11.39}$$

as expected. It is these expressions that we make use of in deriving the path integral formulation for fermions. In particular, the first expression (11.37) for real Grassmann variables is relevant for Majorana fermions, while the latter expression (11.38) is relevant for Dirac fermions. We note that in the latter case there is in fact no requirement that χ and $\overline{\chi}$ are related by complex conjugation; the derivation of (11.38) applies equally well for arbitrary independent field variables χ and $\overline{\chi}$.

11.2 Fermion Propagator

For fermion fields ψ we introduce the path integral

$$Z_0(\eta, \overline{\eta}) = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left[iS_0 + i \int \mathrm{d}^4x \left(\overline{\eta}_\alpha(x)\psi_\alpha(x) + \overline{\psi}_\beta(x)\eta_\beta(x) \right) \right] \ , \tag{11.40}$$

where both ψ, $\overline{\psi}$ and η, $\overline{\eta}$ are complex Grassmann–valued spinors, i.e. each component ψ_α and η_α is a complex Grassmann function. Note the above pairings of two Grassmann valued spinors is consistent with the fact that the action overall is a standard commuting number. To be more precise, we can imagine defining the fermion fields ψ, $\overline{\psi}$ via a sum over some orthonormal basis

$$\psi(x) = \sum_i \psi_i \phi_i(x) \ , \tag{11.41}$$

where the coefficients ψ_i are Grassmann numbers, while the $\phi_i(x)$ corresponds to the usual Dirac spinors, which are ordinary complex numbers, and which appear as in Section 10.4 and in the Feynman rules of our theory; we will leave this decomposition implicit in what follows, though it always implied. As an aside, we note that the formal definition of the integral over Grassmann variables in the previous section means that we lose the straightforward interpretation of integration (and hence the above path integral) as the continuum limit of a sum over all possible field values at each spacetime point x. However, this is in fact qualitatively consistent with what we know about fermionic fields. Due to the Pauli exclusion principle, we cannot create an arbitrary number of fermionic field excitations at a given spacetime point x, as we could for bosonic fields. Thus the path integral is indeed restricted with respect to the bosonic case, and it can be shown that the formal definition of the Grasmann integration given above accounts for this effect correctly.

To be explicit, we have included the corresponding spinor indices in the above expression, but we will for brevity suppress these in what follows. Thus for a Dirac field with

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi \ , \tag{11.42}$$

we have

$$Z_0(\eta,\overline{\eta}) = \int \mathcal{D}\psi\mathcal{D}\overline{\psi} \exp\left[i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(-\tilde{\overline{\psi}}(-k)(k\!\!\!/ + m)\tilde{\psi}(k) + \tilde{\overline{\eta}}(-k)\tilde{\psi}(k) + \tilde{\overline{\psi}}(-k)\tilde{\eta}(k) \right) \right] . \tag{11.43}$$

Changing variables in the usual way, we then arrive at

$$\begin{aligned} Z_0(\eta,\overline{\eta}) &= Z_0(0,0) \exp\left[i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\overline{\eta}}(k)(-k\!\!\!/ + m)^{-1}\tilde{\eta}(-k) \right] \ , & (11.44) \\ &\equiv Z_0(0,0) \exp\left[i \int \mathrm{d}^4 x \mathrm{d}^4 y \, \overline{\eta}(x) S(x-y) \eta(y) \right] \ , & (11.45) \end{aligned}$$

where $S(x-y)$ is the Feynman propagator, given by

$$S(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{(-k\!\!\!/ + m)} \ . \tag{11.46}$$

Here $(-k\!\!\!/ + m)$ is a matrix in spinor space, and therefore while formally a solution, we still need to invert this. Using (10.26), we get

$$S(x-y)_{\alpha\beta} = -\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\epsilon} e^{-ik(x-y)} , \tag{11.47}$$

where we keep the spinor indices α, β explicit and have introduced the usual ϵ term to keep the path integral well defined. This satisfies

$$-(i\not{\partial}_x - m)_{\alpha\beta} S(x-y)_{\beta\gamma} = -\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(-\not{k}+m)_{\alpha\beta}(\not{k}+m)_{\beta\gamma}}{k^2-m^2+i\epsilon} e^{-ik(x-y)} , \tag{11.48}$$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(k^2-m^2)\delta_{\alpha\gamma}}{k^2-m^2-i\epsilon} e^{-ik(x-y)} , \tag{11.49}$$

$$= \delta^4(x-y)\delta_{\alpha\gamma} , \tag{11.50}$$

as required; recall that in the scalar case the propagator is given in terms of the inverse of the KG equation, $(\partial_x^2 - m^2)\Delta(x-y) = \delta^4(x-y)$.

To define a more general correlation function, we note that

$$\begin{aligned} \frac{\delta}{\delta\eta_\alpha(x)} \int \mathrm{d}^4y \left[\overline{\eta}(y)\psi(y) + \overline{\psi}(y)\eta(y)\right] &= -\overline{\psi}_\alpha(x) , \\ \frac{\delta}{\delta\overline{\eta}_\alpha(x)} \int \mathrm{d}^4y \left[\overline{\eta}(y)\psi(y) + \overline{\psi}(y)\eta(y)\right] &= \psi_\alpha(x) , \end{aligned} \tag{11.51}$$

and therefore the generalization of (6.4) becomes

$$\langle 0|T\psi_{\alpha_1}(x_1)\cdots\overline{\psi}_{\beta_1}(y_1)\cdots|0\rangle = \frac{1}{Z[0,0]} \frac{1}{i} \frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)} \cdots i \frac{\delta}{\delta\eta_{\beta_1}(y_1)} \cdots Z(\eta,\overline{\eta})|_{\eta=\overline{\eta}=0} , \tag{11.52}$$

where the factors of i are set to be consistent with the minus sign difference between the η and $\overline{\eta}$ derivatives, as in (11.51). We thus have

$$\langle 0|T\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle = -iS(x-y)_{\alpha\beta} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\epsilon} e^{-ik(x-y)} . \tag{11.53}$$

Finally, we note that while we have not explicitly referred to it, this derivation relies on the formal developments of the preceding section, and in particular the validity of the expression (11.38), suitably generalized to the path integral case. In particular, the $Z_0(0,0)$ plays the role of the $\det M$. While this will cancel when calculating the field expectation values we will be interested in, it is nonetheless important that such a thing can be suitably defined in the Grassmannian case.

12. Feynman Rules for Fermions

12.1 Yukawa Theory

To keep things as simple as possible, before considering full QED we will first look at the case of *Yukawa* theory, which describes the interaction of a Dirac field ψ and a real scalar ϕ. The interaction part of the Lagrangian is given by

$$\mathcal{L}_I = g\phi\overline{\psi}\psi \ , \tag{12.1}$$

for a coupling constant g. This is exactly the form of interaction that we will find between the Higgs Boson and massive fermions in the Standard Model.

Recalling the results of Section 7.2, the relevant path integral is therefore given by

$$Z(J,\overline{\eta},\eta) = \exp\left[ig\int \mathrm{d}^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\right)\right] Z_0(J,\eta,\overline{\eta}) \ , \tag{12.2}$$

with

$$Z_0(J,\eta,\overline{\eta}) = Z_0(0,0,0)\exp\left[i\int \mathrm{d}^4x\mathrm{d}^4y\,\overline{\eta}(x)S(x-y)\eta(y)\right]\exp\left[\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right] \ ,$$

where $\Delta(x-y)$ and $S(x-y)$ are the usual scalar and Dirac fermion propagators, given by (6.13) and (11.47), respectively. The straightforward generalization of (11.52) is then

$$\begin{aligned}&\langle 0|T\psi_{\alpha_1}(x_1)\cdots\overline{\psi}_{\beta_1}(y_1)\cdots\phi(z_1)\cdots|0\rangle = \\ &\frac{1}{Z[0,0,0]}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)}\cdots i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(z_1)}\cdots Z(J,\eta,\overline{\eta})|_{\eta=\overline{\eta}=J=0} \ ,\end{aligned} \tag{12.3}$$

and as usual we can define a connected generating functional $iW = \log Z$ that generates the connected diagrams we are interested in.

Thus, for the basic 3–point $\phi\overline{\psi}\psi$ vertex we will have the contribution

$$iW(J,\eta,\overline{\eta}) = \log\left[1 + ig\int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z\mathrm{d}^4w\,[\overline{\eta}(x)S(x-y)S(y-z)\eta(z)]\,\Delta(y-w)J(w)\right] + \cdots \ . \tag{12.4}$$

where the remaining terms will not give connected contributions when acted on with $\delta/\delta J(x)$ etc in the usual way.

12.2 $e^{\pm}\phi \to e^{\pm}\phi$ scattering

Considering as a more complicated example the $e^-\phi \to e^-\phi$ scattering process, we have

$$\langle 0|T\psi_\alpha(x)\overline{\psi}_\beta(y)\phi(z_1)\phi(z_2)|0\rangle = \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}i\frac{\delta}{\delta\eta_\beta(y)}\frac{1}{i}\frac{\delta}{\delta J(z_1)}\frac{1}{i}\frac{\delta}{\delta J(z_2)}iW(J,\eta,\overline{\eta})|_{\eta,\overline{\eta},J=0} \ , \tag{12.5}$$

where $Z = e^{iW}$ as usual. The first non–vanishing contribution to this corresponds to expanding (12.2) to second order, with

$$\begin{aligned} Z(J,\eta,\overline{\eta}) &\sim \frac{1}{2}\left[ig\int \mathrm{d}^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\right)\right]^2 Z_0(J,\eta,\overline{\eta}) + O(g^4)\ , \\ &= -i(ig)^2\int \mathrm{d}^4x'\mathrm{d}^4y'\mathrm{d}^4z_1'\mathrm{d}^4z_2'\mathrm{d}^4w_1\mathrm{d}^4w_2\overline{\eta}(x')S(x'-w_1)S(w_1-w_2)S(w_2-y')\eta(y') \\ &\cdot \Delta(w_1-z_1')J(z_1')\Delta(w_2-z_2')J(z_2')Z_0(J,\eta,\overline{\eta}) + \cdots\ , \end{aligned} \tag{12.6}$$

where as usual we drop terms that will not give a connected contribution in the end. When deriving expressions such as this, some care is needed in keeping track of minus signs when the $\delta/\delta\eta$ derivatives anticommute through the Grassmann fields. The logic is a bit easier to follow if we first anticommute the derivative w.r.t. $\eta(w_{1,2})$ and $\overline{\eta}(w_{1,2})$, reversing the order in (12.6). Breaking down the fermionic part of the above result, each derivative gives:

$$i\frac{\delta}{\delta\eta(w_1)} : \int \mathrm{d}^4x'\overline{\eta}(x')S(x'-w_1)Z_0(J,\overline{\eta},\eta)\ , \tag{12.7}$$

$$\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(w_1)} : -\int \mathrm{d}^4x'\mathrm{d}^4z'\overline{\eta}(x')S(x'-w_1)S(w_1-z')\eta(z')Z_0(J,\eta,\overline{\eta})\ , \tag{12.8}$$

$$i\frac{\delta}{\delta\eta(w_2)} : i\int \mathrm{d}^4x'\overline{\eta}(x')S(x'-w_1)S(w_1-w_2)Z_0(J,\eta,\overline{\eta})\ , \tag{12.9}$$

$$\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(w_2)} : -i\int \mathrm{d}^4x'\mathrm{d}^4y'\overline{\eta}(x')S(x'-w_1)S(w_1-w_2)S(w_2-y')\eta(y')Z_0(J,\eta,\overline{\eta})\ , \tag{12.10}$$

In the second line we pick up a minus sign as we pass through the $\overline{\eta}(x')$ to act on Z_0; in the third line we again pick up a minus sign, and a factor of i as we do not act on a Z_0; in the final line again we pick up a minus sign as we pass through $\overline{\eta}(x')$. In the third line we could instead have acted on Z_0, and then on the $\overline{\eta}$ term brought down from Z_0 in the last line. This gives the same result, and hence cancels the factor of 1/2 from the Taylor expansion.

Acting on this with the remaining functional derivatives according to (12.5) in order to extract the correlation function, we arrive at

$$\langle 0|T\psi_\alpha(x)\overline{\psi}_\beta(y)\phi(z_1)\phi(z_2)|0\rangle_C = -i(ig)^2\int \mathrm{d}^4w_1\mathrm{d}^4w_2$$
$$[S(x-w_1)S(w_1-w_2)S(w_2-y)]_{\alpha\beta}\,\Delta(z_1-w_1)\Delta(z_2-w_2)+(z_1\leftrightarrow z_2)\ . \tag{12.11}$$

Due to freedom to act on either the $J(z_1')$ or $J(z_2')$ terms in (12.6) with the scalar source derivative, we have two contributions, depending on the ordering of the $z_{1,2}$. These can be represented diagrammatically as in Fig. 15.

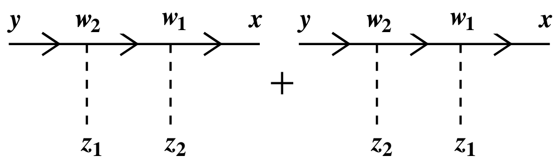

Figure 15: Diagram corresponding to (12.11).

To derive an expression for the corresponding scattering amplitude, we need to write down the LSZ reduction formula for fermionic fields. To do this, we recall that the Dirac fields can be expanded as

$$\psi(x) = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{-ipx}+d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ipx}\right]\ , \tag{12.12}$$

$$\overline{\psi}(x) = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\left[b_s^\dagger(\mathbf{p})\overline{u}_s(\mathbf{p})e^{ipx}+d_s(\mathbf{p})\overline{v}_s(\mathbf{p})e^{-ipx}\right]\ , \tag{12.13}$$

where we associate $b_s^{(\dagger)}$ with the annihilation (creation) of an electron and $d_s^{(\dagger)}$ with the annihilation (creation) of a positron. We then find that

$$b_s(\mathbf{p}) = \int \mathrm{d}^3x e^{ipx}\overline{u}_s(\mathbf{p})\gamma^0\psi(x)\ , \tag{12.14}$$

$$b_s^\dagger(\mathbf{p}) = \int \mathrm{d}^3x e^{-ipx}\overline{\psi}(x)\gamma^0 u_s(\mathbf{p})\ , \tag{12.15}$$

where we have used that

$$\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2E\delta_{s,s'}\ , \tag{12.16}$$

$$\overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) = 0\ , \tag{12.17}$$

which can immediately be seen to hold for particles at rest from (10.57), and does hold in general.

Given the above, the analogy of (7.2) for e.g. the b operator then becomes

$$b_1(+\infty) - b_1(-\infty) = i \int \mathrm{d}^4 x e^{ipx}\, \overline{u}_{s_1}(\mathbf{p})(-i\partial\!\!\!/ + m)\psi(x) \,, \tag{12.18}$$

$$b_1^\dagger(+\infty) - b_1^\dagger(-\infty) = -i \int \mathrm{d}^4 x\, \overline{\psi}(x)(+i\overleftarrow{\partial\!\!\!/} + m)u_s(\mathbf{p})e^{-ipx} \tag{12.19}$$

and thus for electrons we must make the following replacements

$$b_s(\mathbf{p})_{\text{out}} \to +i \int \mathrm{d}^4 x\, e^{ipx}\overline{u}_s(\mathbf{p})(-i\partial\!\!\!/ + m)\psi(x) \,, \tag{12.20}$$

$$b_s^\dagger(\mathbf{p})_{\text{in}} \to +i \int \mathrm{d}^4 x\, \overline{\psi}(x)(+i\overleftarrow{\partial\!\!\!/} + m)u_s(\mathbf{p})e^{-ipx} \,, \tag{12.21}$$

where 'in' and 'out' denote incoming ($t \to -\infty$) and outgoing ($t \to +\infty$) states, and the second relation follows after Hermitian conjugation. When we apply these replacements, after the dust has settled the u and $\overline{u}$ spinors will be left over in the scattering amplitude. This leads to the following Feynman rules for fermion fields: for an incoming and outgoing fermion particle, we associate the spinors $u_s(\mathbf{p})$ and $\overline{u}_s(\mathbf{p})$ respectively.

For the scalar fields we have as before

$$a^\dagger(\mathbf{k})_{\text{in}} \to i \int \mathrm{d}^4 x e^{-ikx}(\partial^2 + m^2)\phi(x) \,, \tag{12.22}$$

$$a(\mathbf{k})_{\text{out}} \to i \int \mathrm{d}^4 x e^{ikx}(\partial^2 + m^2)\phi(x) \,. \tag{12.23}$$

To evaluate the scattering amplitude

$$\langle f|i\rangle = \left\langle 0|T a(\mathbf{k}')_{\text{out}} b_{s'}(\mathbf{p}')_{\text{out}} b_s^\dagger(\mathbf{p})_{\text{in}} a^\dagger(\mathbf{k})_{\text{in}}|0\right\rangle \,, \tag{12.24}$$

we then apply these replacements and the usual identities $(\partial_x^2 + m^2)\Delta(x-y) = \delta^4(x-y)$ and $(-i\partial\!\!\!/_x + m)S(x-y) = \delta^4(x-y)$, as well as replacing $S(w_1 - w_2)$ by its momentum space version, to eliminate all integrals up to an overall momentum conserving delta function. We find

$$i\mathcal{M}(e^-\phi \to e^-\phi) = i(ig)^2 \overline{u}_{s'}(\mathbf{p}') \left[\frac{p\!\!\!/ + k\!\!\!/ + m}{(p+k)^2 - m^2} + \frac{p\!\!\!/ - k\!\!\!/' + m}{(p-k')^2 - m^2} \right] u_s(\mathbf{p}) \,. \tag{12.25}$$

The corresponding Feynman diagrams are shown in Fig. 16. The Feynman rules for drawing these are that we must, similar to the scalar QED case, draw a solid line with an arrow pointed towards (away from) the vertex for an incoming (outgoing) particle. The allowed 3–point vertex must then have one arrow pointing towards and one pointing away from the vertex; this corresponds to the only combination of fermion fields allowed by the $g\overline{\psi}\psi\phi$ interaction term, and ensures electric charge is preserved at the vertex. We associate a factor of ig with every such vertex.

For internal fermion lines we also associate an arrow, which must be consistent with the allowed 3–point vertex. As given by the derivation in Section 11.2, we then associate a factor

$$i\frac{\not{p}+m}{p^2-m^2} \,, \tag{12.26}$$

where p is the momentum flowing *in the direction* of the arrow. Putting these together we can see that this is consistent with our result for the amplitude above.

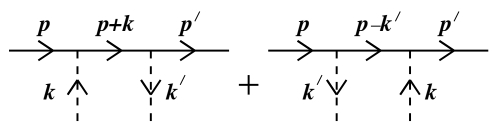

Figure 16: Diagram corresponding to (12.25).

What about for positron scattering, $e^+\phi \to e^+\phi$? We are now interested in

$$\langle f|i\rangle = \langle 0|Ta(\mathbf{k}')_{\rm out}d_{s'}(\mathbf{p}')_{\rm out}d_s^\dagger(\mathbf{p})_{\rm in}a^\dagger(\mathbf{k})_{\rm in}|0\rangle \,, \tag{12.27}$$

and the equivalent inversion to (12.14) becomes

$$d_s^\dagger(\mathbf{p}) = \int {\rm d}^3x e^{-ipx}\overline{v}_s(\mathbf{p})\gamma^0\psi(x) \,, \tag{12.28}$$

$$d_s(\mathbf{p}) = \int {\rm d}^3x e^{ipx}\overline{\psi}(x)\gamma^0 v_s(\mathbf{p}) \,, \tag{12.29}$$

where we have used

$$\overline{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) = 2E\delta_{s,s'} \,, \tag{12.30}$$

$$\overline{v}_s(\mathbf{p})\gamma^0 u_{s'}(-\mathbf{p}) = 0 \,. \tag{12.31}$$

The equivalent LSZ replacements are therefore

$$d_s^\dagger(\mathbf{p})_{\rm in} \to -i \int \mathrm{d}^4x e^{-ipx} \overline{v}_s(\mathbf{p})(-i\not\partial + m)\psi(x) \ , \tag{12.32}$$

$$d_s(\mathbf{p})_{\rm out} \to -i \int \mathrm{d}^4x \overline{\psi}(x)(+i\overleftarrow{\not\partial} + m)v_s(\mathbf{p})e^{ipx} \ . \tag{12.33}$$

This leads to the following Feynman rules for fermion fields: for outgoing (incoming) fermion antiparticles, we draw a line with an arrow pointing towards (away from) the vertex, and associate the spinors $v_s(\mathbf{p})$ ($\overline{v}_s(\mathbf{p})$). Note the difference in exponentials for the Fourier transforms, which now have a minus sign relative to the electron case, $py \to -py$ etc. For the Feynman rules this leads us to label the external anti–fermions with *minus* the four–momenta that flows in the direction of the fermion arrow we draw, consistently with the anti–particle interpretation, i.e. that the fermion arrow points in the opposite direction to the corresponding particle[12]. We therefore arrive at

$$i\mathcal{M}(e^+\phi \to e^+\phi) = i(ig)^2\overline{v}_s(\mathbf{p})\left[\frac{-\not p + \not k' + m}{(p-k')^2 - m^2} + \frac{-\not p - \not k + m}{(p+k)^2 - m^2}\right] v_{s'}(\mathbf{p}') \ , \tag{12.34}$$

The corresponding Feynman diagrams are shown in Fig. 17.

To emphasise, the incoming positron carries momentum p, which we label as $-p$ simply as this is defined as flowing in the direction of the fermion arrow, however in the end we associate a spinor $\overline{v}_s(\mathbf{p})$ associated with the momentum carried physically by the anti–particle. Thus, in the left hand figure we have momenta $p + k$ flowing through the internal propagator, which we label as $-p - k$ as the momentum is flowing in the opposite direction to the fermion line. Such labelling can help in keeping track of things, but is not essential. Rather we can simply remember to associate the spinors as defined above, and keep track of the fact that the fermion propagator is defined in (12.26) with the momentum flowing in the direction of the fermion arrow.

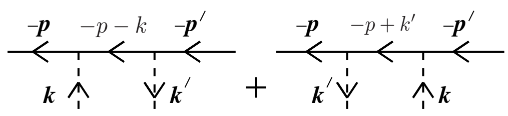

Figure 17: Diagram corresponding to (12.34).

[12]There is an additional overall minus sign in front of the integral, and so technically we should associate $-v_{s'}(\mathbf{p}')$ with an external fermion line etc, however this is attached to each v and $\overline{v}$ consistently and can only affect the overall sign of the amplitude (rather than the relative sign between amplitudes). This can therefore be dropped.

12.3 $e^{\pm}e^{\pm} \to e^{\pm}e^{\pm}$ scattering

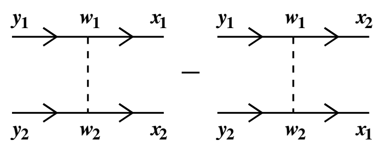

Figure 18: Diagram corresponding to (12.36).

For this process we are interested in

$$\begin{aligned}&\left\langle 0\right|T\psi_{\alpha_1}(x_1)\overline{\psi}_{\beta_1}(y_1)\psi_{\alpha_2}(x_2)\overline{\psi}_{\beta_2}(y_2)\left|0\right\rangle = \\ &\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)}i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_2}(x_2)}i\frac{\delta}{\delta\eta_{\beta_2}(y_2)}iW(\overline{\eta},\eta,J)|_{\eta,\overline{\eta},J=0}\ .\end{aligned} \tag{12.35}$$

To evaluate this, we will expand the exponent of (12.2) to second order as before. The result is

$$\begin{aligned}&\left\langle 0\right|T\psi_{\alpha_1}(x_1)\overline{\psi}_{\beta_1}(y_1)\psi_{\alpha_2}(x_2)\overline{\psi}_{\beta_2}(y_2)\left|0\right\rangle = \\ &-i(ig)^2\int \mathrm{d}^4w_1\mathrm{d}^4w_2\,[S(x_1-w_1)S(w_1-y_1)]_{\alpha_1\beta_1}\,\Delta(w_1-w_2)\,[S(x_2-w_2)S(w_2-y_2)]_{\alpha_2\beta_2} \\ &-((y_1,\beta_1)\leftrightarrow(y_2,\beta_2))\ ,\end{aligned} \tag{12.36}$$

and is shown diagrammatically in Fig. 18.

Notice we have a *relative minus sign* between the two contributions when we swap the fermion labelings. Where does this come from? To give the second contribution we swap the source terms acted on by the two $\overline{\eta}$ derivatives, which corresponds to evaluating the first contribution as before, but with the relabelling $(y_1,\beta_1)\leftrightarrow(y_2,\beta_2)$ in (12.35). However, these derivates anti–commute, and we have

$$\frac{1}{i}\frac{\delta}{\delta\eta_{\alpha_1}(x_1)}i\frac{\delta}{\delta\overline{\eta}_{\beta_1}(y_1)}\frac{1}{i}\frac{\delta}{\delta\eta_{\alpha_2}(x_2)}i\frac{\delta}{\delta\overline{\eta}_{\beta_2}(y_2)} = -\frac{1}{i}\frac{\delta}{\delta\eta_{\alpha_1}(x_1)}i\frac{\delta}{\delta\overline{\eta}_{\beta_2}(y_2)}\frac{1}{i}\frac{\delta}{\delta\eta_{\alpha_2}(x_2)}i\frac{\delta}{\delta\overline{\eta}_{\beta_1}(y_1)}\ . \tag{12.37}$$

Thus the second contribution can be written down by simply interchanging the y and β labels in the first term on the right hand side of (12.36), but with an additional overall minus sign included. Thus at the level of the Feynman rules the anticommuting nature of the fermionic fields leads directly to the spin–statistics connection; when we interchange identical fermions, we must include a relative minus sign. To account for this in the Feynman rules, we can draw each diagram in *standard form*, that is with all fermion lines drawn horizontal and their arrows pointing left to right, with the left end points in some fixed order. If the ordering of the labelling of the right endpoints of the fermion lines is an even (odd) permutation of some arbitrarily chosen ordering, the relative sign is positive (negative).

Thus, for $e^-e^- \to e^-e^-$ the two diagrams are shown in Fig. 19, and we can see that there is indeed a relative minus sign. Similarly for $e^+e^- \to e^+e^-$ the two diagrams are shown in Fig. 20, and we can again see that there is a relative minus sign.

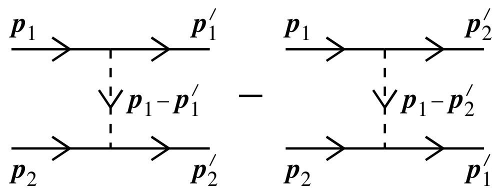

Figure 19: Feynman diagrams for $e^-e^- \to e^-e^-$.

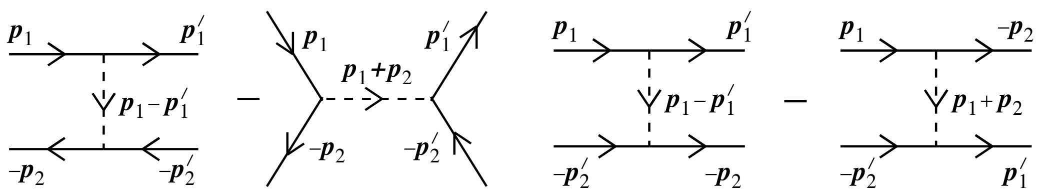

Figure 20: Feynman diagrams for $e^+e^- \to e^+e^-$, with the standard ordering shown in the right.

12.4 Fermion Loops

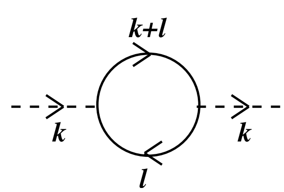

Figure 21: Fermion loop.

Another consequence of the anticommuting nature of the Dirac fields is that we must in fact introduce an additional minus sign for every Fermion loop. Writing

$$\frac{1}{i}\frac{\delta}{\delta J(x)} \equiv \delta_x^{\phi} \,, \quad i\frac{\delta}{\delta \eta_\alpha(x)} \equiv \delta_x^{\eta_\alpha} \,, \quad \frac{1}{i}\frac{\delta}{\delta \overline{\eta}_\alpha(x)} \equiv \delta_x^{\overline{\eta}_\alpha} \,, \tag{12.38}$$

then for the loop shown in Fig. 21 we are interested in

$$Z(\overline{\eta},\eta,J)\sim\frac{1}{2}(ig)^2\int \mathrm{d}^4x\mathrm{d}^4y(\delta_x^\phi\delta_x^{\eta_\alpha}\delta_x^{\overline{\eta}_\alpha})(\delta_y^\phi\delta_y^{\eta_\beta}\delta_y^{\overline{\eta}_\beta})Z_0[J,\overline{\eta},\eta]\;, \tag{12.39}$$

If we consider first the derivatives with respect to the spinor source terms, we find

$$(\delta_x^{\eta_\alpha}\delta_x^{\overline{\eta}_\alpha})(\delta_y^{\eta_\beta}\delta_y^{\overline{\eta}_\beta})e^{i\int \mathrm{d}^4x_1\mathrm{d}^4y_1\overline{\eta}(x_1)S(x_1-y_1)\eta(y_1)}=\mathrm{Tr}\left[S(x-y)S(y-x)\right]e^{i[\cdots]}\;, \tag{12.40}$$

where the trace is performed over the spinor indices, and as usual we drop terms $\sim S_F(0)$, which will give disconnected contributions. On the other hand, if we were to consider the equivalent expression for a bosonic loop, we would have a term proportional to

$$(\delta_x^\phi\delta_x^\phi)(\delta_y^\phi\delta_y^\phi)e^{\frac{i}{2}\int \mathrm{d}^4x_1\mathrm{d}^4y_1J(x_1)\Delta(x_1-y_1)J(y_1)}=-\Delta(x-y)\Delta(y-x)e^{i[\cdots]}\;. \tag{12.41}$$

The crucial point is the relative minus sign between these two expressions, which comes from the anticommutation of the η and $\overline{\eta}$ fields, and is absent in the scalar case. Thus we have that there is a relative minus sign between bosonic and fermionic loops. This leads to the Feynman rule that we should introduce (-1) for every fermion loop, again encoding the spin–statistics connection.

12.5 Summary of Feynman Rules

The momentum–space Feynman rules for fermionic fields are summarised below:

- For an incoming fermion particle, draw a line with an arrow directed towards the vertex, and associate the spinor $u_s(\mathbf{p})$, where p is directed along the arrow.

- For an outgoing fermion particle, draw a line with an arrow directed away from the vertex, and associate the spinor $\overline{u}_s(\mathbf{p})$, where p is directed along the arrow.

- For an incoming fermion antiparticle, draw a line with an arrow directed away from the vertex, and associate the spinor $\overline{v}_s(\mathbf{p})$, where $-p$ is directed along the arrow.

- For an outgoing fermion antiparticle, draw a line with an arrow directed towards the vertex, and associate the spinor $v_s(\mathbf{p})$, where $-p$ is directed along the arrow.

- For Yukawa theory, we associate a factor of ig with every scalar–fermion–antifermion vertex.

- For each internal fermion, associate the propagator

$$i\frac{\not{p}+m}{p^2-m^2+i\epsilon} \; , \tag{12.42}$$

  where p is the momentum pointing along the direction of the fermion arrow (this must be drawn so that these consistently match with the fermion arrows of the external states).

- Account for relative minus signs when swapping identical fermions in the final state by writing diagrams in standard form (or otherwise).

- Associate an additional factor of -1 with each fermion loop.

13. Spinor Technology

In the previous section we derived the Feynman rules for fermions and the corresponding amplitudes for some of the basic processes, such as $e^{\pm}\phi \to e^{\pm}\phi$, within our toy Yukawa theory. We will now consider in a little more detail how to relate these to the sort of observables we might actually measure.

13.1 $e^-\phi \to e^-\phi$ and spin sums

From Section 12.3, see Fig. 16, we found that the amplitude for $e^-(p)\phi(k) \to e^-(p')\phi(k')$ scattering is given at leading order by

$$\mathcal{M}(e^-\phi \to e^-\phi) = -g^2\overline{u}_{s'}(\mathbf{p}')\left[\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}+\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\right]u_s(\mathbf{p}) \equiv \overline{u}'Au \; , \tag{13.1}$$

where we use the shorthand $u_{s'} = u'$, $u_s = u$, and

$$\begin{aligned} A &\equiv -g^2\left[\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}+\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\right] \; , \\ &= -g^2\left[\frac{\not{k}+2m}{s-m^2}+\frac{-\not{k}'+2m}{u-m^2}\right] \; , \end{aligned} \tag{13.2}$$

in this particular instance, but expressions of this type will occur quite generically. Here in the second step we have also introduced the usual Mandelstam variables $s = (p+k)^2$ and $u = (p-k')^2$, and used the Dirac equation $\not{p}u(p) = mu(p)$ to simplify things a little. The first step to calculating an observable is of course to square the amplitude above. Considering

$$\mathcal{M}^* = (u')^T(\gamma^0)^* A^* u^* = u^\dagger A^\dagger \gamma^0 u' = \overline{u}\overline{A}u' \ , \tag{13.3}$$

where we have defined $\overline{A} \equiv \gamma^0 A^\dagger \gamma^0$. Now, using $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$ and the usual anticommutation relation, we find

$$\gamma^0(\gamma^\mu)^\dagger = \gamma^\mu\gamma^0 \ , \tag{13.4}$$

and thus in general we find that

$$\overline{\not{a}_1 \cdots \not{a}_n} = \not{a}_n \cdots \not{a}_1 \ , \tag{13.5}$$

that is, this operation simply corresponds to reversing the ordering of the gamma matrices within A. For our particularly simple case we therefore have $\overline{A} = A$. In general, the squared amplitude will take the form

$$\begin{aligned} |\mathcal{M}|^2 &= (\overline{u}' A u)(\overline{u}\overline{A}u') \ , &(13.6)\\ &= \overline{u}'_\alpha A_{\alpha\beta} u_\beta \overline{u}_\gamma \overline{A}_{\gamma\delta} u'_\delta \ , &(13.7)\\ &= (u'_\delta \overline{u}'_\alpha) A_{\alpha\beta} (u_\beta \overline{u}_\gamma) \overline{A}_{\gamma\delta} \ , &(13.8)\\ &= \mathrm{Tr}\left[(u'\overline{u}')A(u\overline{u})\overline{A}\right] \ , &(13.9) \end{aligned}$$

where we have included the spinor indices in the intermediate steps for clarity, and where in the last line the trace is then performed over these.

Thus we are left having to calculate the $u'\overline{u}'$ and $u\overline{u}$, which are in general non–trivial matrices in spinor space that depend on the momenta and spins of the particles. However, in practice we are often not interested in measuring, or find it difficult to measure, the individual spin states of the particles. In a collider, for example, it is quite common for the incoming particle beams to be unpolarized, while in the detector for many observables we are not interested in the spin of the produced particles, if these can be measured at all. Our observable is therefore defined by summing over the spins in the final state and averaging over those in the initial state (i.e. summing and dividing by the number of possible spin states). In the present case, we therefore have

$$\left\langle |\mathcal{M}|^2 \right\rangle \equiv \frac{1}{2}\sum_{s,s'} |\mathcal{M}|^2 \ , \tag{13.10}$$

where the 2 corresponds to the spin states of the initial–state electron; if we were instead considering $e^+e^- \to e^+e^-$ this would become a 4.

At this point, the spin sums (10.62) we wrote down in the previous section become very useful. We can directly apply them to the above, giving

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{2}\mathrm{Tr}\left[(\not{p}' + m)A(\not{p} + m)A\right] \ , \tag{13.11}$$

where we have substituted $A = \overline{A}$ for concreteness. This is now given purely as a trace over gamma matrices, without any need to write down an explicit form for the spinors u; this is always the case when we consider spin summed/average results, and greatly simplifies things. Writing everything out explicitly, we have

$$\left\langle|\mathcal{M}|^2\right\rangle = g^4\left[\frac{\langle\Phi_{ss}\rangle}{(s-m^2)^2} + \frac{\langle\Phi_{su}\rangle + \langle\Phi_{us}\rangle}{(s-m^2)(u-m^2)} + \frac{\langle\Phi_{uu}\rangle}{(u-m^2)^2}\right] \ , \tag{13.12}$$

where

$$\langle\Phi_{ss}\rangle = \frac{1}{2}\mathrm{Tr}\left[(\not{p}' + m)(\not{k} + 2m)(\not{p} + m)(\not{k} + 2m)\right] \ , \tag{13.13}$$

$$\langle\Phi_{us}\rangle = \frac{1}{2}\mathrm{Tr}\left[(\not{p}' + m)(-\not{k}' + 2m)(\not{p} + m)(\not{k} + 2m)\right] \ , \tag{13.14}$$

$$\langle\Phi_{su}\rangle = \frac{1}{2}\mathrm{Tr}\left[(\not{p}' + m)(\not{k}' + 2m)(\not{p} + m)(-\not{k}' + 2m)\right] \ , \tag{13.15}$$

$$\langle\Phi_{uu}\rangle = \frac{1}{2}\mathrm{Tr}\left[(\not{p}' + m)(-\not{k}' + 2m)(\not{p} + m)(-\not{k}' + 2m)\right] \ . \tag{13.16}$$

To go any further, we must derive some useful identities for the traces of gamma matrices, known as *Trace Theorems*.

13.2 Trace Theorems

Using the basic results

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\cdot\mathbb{1} \ , \qquad \mathrm{Tr}(\mathbb{1}) = 4 \ , \tag{13.17}$$

along with the properties of the γ_5 matrix

$$\{\gamma_5, \gamma^\mu\} = 0 \ , \qquad \gamma_5^2 = \mathbb{1} \ , \tag{13.18}$$

combined with the cyclicity of the trace, it is possible to derive some very useful identities for the trace of gamma matrices. First, we have

$$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma_5^2\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma_5) = \mathrm{Tr}(-\gamma_5^2\gamma^\mu) = -\mathrm{Tr}(\gamma^\mu) \ , \tag{13.19}$$

where in the first step we have inserted the identity $\gamma_5^2 = \mathbb{1}$, in the second we have used the cyclicity of the trace, in the third we have anticommuted the γ_5 and γ^μ and then in the last line used $\gamma_5^2 = \mathbb{1}$ again. Thus the gamma matrices must be traceless. In fact, we have seen that this is the case for the explicit representation (10.27) we have used; as other representations are related by a unitary transformation $U^{-1}\gamma^\mu U$, then if this holds in one representation, it must hold in all of them. This result can readily be generalised, and we find

$$\mathrm{Tr}(\text{odd no. of } \gamma^\mu \text{s}) = 0 \; . \tag{13.20}$$

So the first non–trivial trace we can consider is

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu} \cdot \mathrm{Tr}\mathbb{1} = 4g^{\mu\nu} \; , \tag{13.21}$$

where we have used the cyclicity of the trace in the first step, substituted the usual anticommutation relation in the second and then evaluated the trace of the unit matrix. Quite trivially, this leads to

$$\mathrm{Tr}(\not{a}\not{b}) = 4(ab) \; , \tag{13.22}$$

for arbitrary 4–vectors a and b. By repeated application of the anticommutation relation, it is possible to determine expressions for any arbitrary even number of gamma matrices. For example, we have

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \; , \tag{13.23}$$

and thus

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4((ab)(cd) - (ac)(bd) + (ad)(bc)) \; . \tag{13.24}$$

Writing $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, then this immediately implies that

$$\mathrm{Tr}(\gamma_5) = 0 \; . \tag{13.25}$$

It can also be shown that

$$\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0 \; , \qquad \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma} \; . \tag{13.26}$$

Finally, we will sometimes encounter the contraction

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}g_{\mu\nu} \cdot \mathbb{1} = D \cdot \mathbb{1} \; , \tag{13.27}$$

where we quote the general D–dimensional result, which will be useful when evaluating loop calculations. In a similar way we then have

$$\gamma^\mu \not{a} \gamma_\mu = (2-D)\not{a} \ , \tag{13.28}$$

and so on.

13.3 $e^-\phi \to e^-\phi$ revisited

We now have the tools to calculate the spin–averaged amplitude squared (13.12) for the $e^-\phi \to e^-\phi$ process. We can in particular use (13.21) and (13.23) to evaluate the traces (13.13)–(13.16). To do this, we make use of the following results

$$(pk) = (p'k') = \frac{1}{2}(s - m^2 - M^2) \ , \tag{13.29}$$

$$(pp') = \frac{1}{2}(2m^2 - t) \ , \tag{13.30}$$

$$(kk') = \frac{1}{2}(2M^2 - t) \ , \tag{13.31}$$

$$(p'k) = (pk') = \frac{1}{2}(m^2 + M^2 - u) \ , \tag{13.32}$$

where m (M) are the fermion (scalar) particle masses. Then after some fairly tedious algebra, we arrive at

$$\langle \Phi_{ss} \rangle = -su + m^2(9s + u) + 7m^4 - 8m^2M^2 + M^4 \ , \tag{13.33}$$

$$\langle \Phi_{uu} \rangle = -su + m^2(9u + s) + 7m^4 - 8m^2M^2 + M^4 \ , \tag{13.34}$$

$$\langle \Phi_{su} \rangle = \langle \Phi_{us} \rangle = su + 3m^2(s + u) + 9m^4 - 8m^2M^2 - M^4 \ . \tag{13.35}$$

We can see that $\langle \Phi_{uu} \rangle$ is directly related to $\langle \Phi_{ss} \rangle$ upon the simple interchange of $u \leftrightarrow s$, as we would expect from simple observation of the explicit expressions, which are related by $k \to -k'$. This sort of observation can be very helpful when calculating matrix elements such as these, as it avoids us having to do essentially the same calculation twice.

We can then use these, combined with (7.51) to calculate the differential cross section for unpolarized $e^-\phi \to e^-\phi$ scattering, with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2 s}\left[\frac{\langle \Phi_{ss} \rangle}{(s-m^2)^2} + \frac{\langle \Phi_{su} \rangle + \langle \Phi_{us} \rangle}{(s-m^2)(u-m^2)} + \frac{\langle \Phi_{uu} \rangle}{(u-m^2)^2}\right] \ . \tag{13.36}$$

13.4 Second example: $e^+e^- \to e^+e^-$ scattering

As another example of the application of trace theorems, we consider the slightly more complicated case of $e^+(p_1)e^-(p_2) \to e^+(p_1')e^-(p_2')$ scattering, shown in Fig. 20. In a similar way, we arrive at

$$\left\langle |\mathcal{M}|^2 \right\rangle = g^4 \left[\frac{\langle \Phi_{ss} \rangle}{(s-M^2)^2} - \frac{\langle \Phi_{st} \rangle + \langle \Phi_{ts} \rangle}{(s-M^2)(t-M^2)} + \frac{\langle \Phi_{tt} \rangle}{(t-M^2)^2} \right] . \tag{13.37}$$

Note the relative minus sign, which is due to the interchange of identical fermions between the s and t–channel diagrams. We have

$$\langle \Phi_{ss} \rangle = \frac{1}{4} \mathrm{Tr}\left[(\not{p}_1 + m)(\not{p}_2 - m)\right] \mathrm{Tr}\left[(\not{p}_2' - m)(\not{p}_1' + m)\right] \ , \tag{13.38}$$

$$\langle \Phi_{tt} \rangle = \frac{1}{4} \mathrm{Tr}\left[(\not{p}_1 + m)(\not{p}_1' + m)\right] \mathrm{Tr}\left[(\not{p}_2' - m)(\not{p}_2 - m)\right] \ , \tag{13.39}$$

$$\langle \Phi_{st} \rangle = \frac{1}{4} \mathrm{Tr}\left[(\not{p}_1 + m)(\not{p}_1' + m)(\not{p}_2' - m)(\not{p}_2 - m)\right] \ , \tag{13.40}$$

$$\langle \Phi_{ts} \rangle = \frac{1}{4} \mathrm{Tr}\left[(\not{p}_1 + m)(\not{p}_2 - m)(\not{p}_2' - m)(\not{p}_1' + m)\right] \ , \tag{13.41}$$

where the factor of 4 is due to the 2 spin states of both incoming fermions. To give one explicit example of the kind of thing we need to do, we find

$$\mathrm{Tr}\left[(\not{p}_1 + m)(\not{p}_2 - m)\right] = \mathrm{Tr}\left[\not{p}_1 \not{p}_2\right] - m^2 \mathrm{Tr}\mathbb{1} \ , \tag{13.42}$$

$$= 4(p_1 p_2) - 4m^2 = 2s - 8m^2 \ . \tag{13.43}$$

The other traces can be calculated in a similar way, and here we will simply quote the result:

$$\left\langle |\mathcal{M}|^2 \right\rangle = g^4 \left[\frac{(s-4m^2)^2}{(s-M^2)^2} + \frac{st - 4m^2 u}{(s-M^2)(t-M^2)} + \frac{(t-4m^2)^2}{(t-M^2)^2} \right] \ , \tag{13.44}$$

which can readily be used to calculate the corresponding cross section, as in (13.36). These calculations can be somewhat lengthy when done by hand, but as promised no explicit form for the spinors themselves is required. In addition, the rules for evaluating these traces are simple and readily amenable to implementation in a computer, via symbolic manipulation programs such as `FORM`.

14. Quantum Electrodynamics

14.1 Lagrangian

In Section 5 we introduced the concept of a local $U(1)$ gauge symmetry, and demonstrated that by requiring the Lagrangian for a free complex scalar field obey such a symmetry, we are led to the introduction of a photon field and hence the theory of scalar QED. Having now discussed the case of fermionic fields, we are ready to build up the full theory of QED, the theory of fermionic and photon fields, in an exactly analogous way. In what follows we will only consider one type (known as a flavour) of fermion, which we will take to be the electron for concreteness. In the Standard Model, there are in fact three flavours of such fermions (the e, μ, τ leptons), identical up to their differing masses, but the results which follow readily generalise to this case.

We start as before with the free Lagrangian, given by

$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi \; . \tag{14.1}$$

Considering the global transformations (with $q = -e$)

$$\psi \to e^{-ie\alpha}\psi \; , \qquad \overline{\psi} \to e^{ie\alpha}\overline{\psi} \; , \tag{14.2}$$

we can immediately see that these leave the Lagrangian invariant, and it us straightforward to derive the corresponding conserved current

$$j^\mu = e\overline{\psi}\gamma^\mu\psi \; , \tag{14.3}$$

where we have as before dropped the factor of α in defining our normalization. In exactly the same way as for the scalar QED case we then promote this global symmetry to a local symmetry, $\alpha \to \alpha(x)$, and introduce the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu \; , \tag{14.4}$$

with

$$A_\mu \to A_\mu + \partial_\mu\alpha(x) \; , \tag{14.5}$$

to ensure that the Lagrangian is invariant under this local $U(1)$ gauge symmetry. We thus get

$$\mathcal{L}_{\rm QED} = i\overline{\psi}D\!\!\!/\psi - m\overline{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \; , \tag{14.6}$$

$$= i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi - A_\mu j^\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \; , \tag{14.7}$$

where we have added in the required kinetic term for the photon field. This is the full Lagrangian for QED, and as you can see, it is quite simple! We can in particular see that, due to the simple linear form of the kinetic term for the fermion field, imposing gauge symmetry has introduced a single interaction term between the photon and fermion fields

$$\mathcal{L}_I = -A_\mu j^\mu = -e\overline{\psi}\not{A}\psi \; . \tag{14.8}$$

Thus the fields couple directly to the current j^μ, often known for this reason as the EM current. This is in contrast to the scalar QED case, for which the kinetic term in the fields was quadratic in derivatives and a contact $\gamma\gamma\phi\phi^*$ term was present; the equivalent interaction for fermion fields would not be gauge invariant, and is absent here.

14.2 Discrete Symmetries of QED

As well as an overall Lorentz symmetry, there are a set of discrete transformations under which our Lagrangian can (or cannot) be invariant, namely charge conjugation (C), parity (P) and time reversal (T). We will discuss P and C explicitly here, and will in particular show that QED is both C and P invariant. A treatment of T reversal can in found in many textbooks, and will be omitted for brevity here. Moreover, the *CPT Theorem* tells us that any basically sensible quantum field theory (which obeys Lorentz invariance, is local and has a Hermitian Hamiltonian) must be invariant under the combined CPT transformation. Thus the T symmetry of any such theory is directly connected to its CP symmetry.

Parity

This is a discrete symmetry also contained within the Lorentz group, corresponding to a simple spatial reflection, i.e.

$$(x_0, \mathbf{x}) \to (x_0, -\mathbf{x}) \; . \tag{14.9}$$

In addition to this, fields can have their own transformation properties. For example, for a real scalar field

$$\phi(t, \mathbf{x}) \to \eta\phi(t, -\mathbf{x}) \; , \tag{14.10}$$

where $\eta = \pm 1$ are both options consistent with the basic requirement that two parity transformations should return us the identity (i.e. $P^2 = 1$). Here η is known as the intrinsic parity of the field. $\eta = +1$ corresponds to a *scalar* state, with even intrinsic parity (such as the SM Higgs Boson), and $\eta = -1$ corresponds to a *pseudoscalar* state with odd parity (such as a neutral pion).

For a vector field we again have two possibilities

$$A_0(t, \mathbf{x}) = \pm A_0(t, -\mathbf{x}) \; , \qquad A_i(t, \mathbf{x}) = \mp A_i(t, -\mathbf{x}) \; , \tag{14.11}$$

In this case, if $A_i \to -A_i$ we say that A_μ has odd parity $P = -1$ and transforms as a vector, as this is how a standard 4–vector such as ∂_μ transforms. For $A_i \to A_i$ we say that A_μ has even parity $P = +1$ and transforms as an axial–vector. Given that our photon field transforms as $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ under a gauge transformation, then for a scalar function $\alpha(x)$, for consistency we require the photon to transform in the same way as the vector ∂_μ and therefore to have odd parity (this is also observed experimentally to be the case).

How does a spinor transform under parity? Such a reflection leaves the direction of rotation unchanged but swaps the boost direction, with

$$J_i \to J_i \;, \qquad K_i \to -K_i \;. \tag{14.12}$$

A more precise statement of the above result is that indeed in the 4–vector representation, the matrix $P = \mathrm{diag}(1,-1,-1,-1)$, which produces a parity transformation, commutes with the rotation generators, J_i, but not with the boost generators, K_i. We therefore have from (10.16) that a parity transformation give $N_i^+ \leftrightarrow N_i^-$ and therefore this interchanges $\psi_L \leftrightarrow \psi_R$ (see also (10.24)). In terms of a Dirac spinor, we can see that this simply corresponds to

$$P: \qquad \psi \to \gamma_0 \psi \;, \tag{14.13}$$

which indeed swaps $\psi_L \leftrightarrow \psi_R$, and is also consistent with the fact that two parity transformations should return us the identity, as $\gamma_0^2 = \mathbb{1}$.

From (14.13), we find

$$P: \qquad \overline{\psi}\psi \to \psi^\dagger \gamma_0 \gamma_0 \gamma_0 \psi = \overline{\psi}\psi \;, \tag{14.14}$$

$$P: \qquad \overline{\psi}\gamma^\mu \psi \to \psi^\dagger \gamma_0 \gamma_0 \gamma^\mu \gamma_0 \psi = \overline{\psi}(\gamma^\mu)^\dagger \psi \;, \tag{14.15}$$

where we have suppressed the inversion of the $\mathbf{x}$ arguments after the transformation for simplicity, and in the second case we have used (13.4). Thus in the second case this transforms as a vector, with odd parity. Thus, when we combine these this with the photon fields to give the QED interaction term, we have that the Dirac Lagrangian is even under a parity transformation. Physically, this tells us that QED preserves parity, i.e. it treats left and right handed fermion fields identically. On the other hand we have

$$P: \qquad \overline{\psi}\gamma^\mu \gamma_5 \psi \to \psi^\dagger \gamma_0 \gamma_0 \gamma^\mu \gamma_5 \gamma_0 \psi = -\overline{\psi}(\gamma^\mu)^\dagger \gamma_5 \psi \;, \tag{14.16}$$

and therefore this axial–vector current $J_A^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$ transforms in the opposite way to the vector current $J_V^\mu = \overline{\psi}\gamma^\mu\psi$. As a result the weak interaction, which involves the

$J_V^\mu - J_A^\mu$, or $V - A$ current, does not conserve parity.

Charge Conjugation

Another discrete operation of interest, not directly related to spacetime, is charge conjugation. This corresponds simply to the interchange of a particle with its anti–particle. Taking

$$C: \qquad \psi \to -i\gamma^2\psi^* \ , \tag{14.17}$$

we find that this indeed corresponds to interchanging $u_\pm \leftrightarrow v_\pm$, as required, with the complex conjugation needed to be consistent with the relative minus sign in the e^{ipx} for the field decomposition (10.53). After some algebra, we find that

$$C: \qquad \overline{\psi}\psi \to \overline{\psi}\psi \ , \tag{14.18}$$

$$C: \qquad \overline{\psi}\partial\!\!\!/\psi \to \overline{\psi}\partial\!\!\!/\psi \ . \tag{14.19}$$

Note that to show these results we must make use of the fact that these fields ψ are Grassmann valued, i.e. they anticommute. Thus the free Dirac Lagrangian is C invariant. On the other hand we have

$$C: \qquad \overline{\psi}\gamma^\mu\psi \to -\overline{\psi}\gamma^\mu\psi \ , \tag{14.20}$$

where the relative minus sign compared to the expression above is due to the derivative acting on the complex conjugated ψ^* in the former case. Thus QED will only be invariant under charge conjugation, as it is observed to be, if the photon has odd C–parity, i.e.

$$C: \qquad A_\mu \to -A_\mu \ . \tag{14.21}$$

This may seem strange for a real field, which is its own antiparticle, however the point is that the photon couples to the electric charge of the fermionic fields. As charge conjugation flips the sign of this, to keep everything invariant under C the photon must transform in this way. Finally, the axial–vector current again transforms in the opposite way to the vector, with

$$C: \qquad \overline{\psi}\gamma^\mu\gamma_5\psi \to \overline{\psi}\gamma^\mu\gamma_5\psi \ . \tag{14.22}$$

Once again, this plays an important role in the case of the weak interaction.

14.3 Feynman Rules

To derive the Feynman rules for QED, we can apply the same approach described in

Sections 7.2 and 11.2. In particular, (7.16) becomes

$$Z[J,\eta,\overline{\eta}] = \exp\left[-ie\int \mathrm{d}^4x \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\gamma^\mu_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\right] Z_0[J,\eta,\overline{\eta}]\ , \tag{14.23}$$

where the generating functional for the free theory is given by

$$\begin{aligned} Z_0[J,\eta,\overline{\eta}] = Z_0[0,0,0]\exp\left[i\int \mathrm{d}^4x\mathrm{d}^4y\, \overline{\eta}(x)S(x-y)\eta(y)\right] \\ \cdot\exp\left[\frac{i}{2}\int d^4xd^4yJ_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right]\ , \end{aligned} \tag{14.24}$$

with the free propagators given by (11.47) and (6.26).

In fact, we have already written down the required Feynman rules from our analysis of scalar QED and Yukawa theory, with the single exception of the $\gamma\overline{\psi}\psi$ interaction vertex. This could be derived in the usual way, but up to overall factors can be simply read off from the Lagrangian. We find that we should associate a factor of

$$-ie\gamma^\mu\ , \tag{14.25}$$

with every interaction vertex (recall here that e is defined as being positive).

The Feynman rules are summarised below:

- For an incoming fermion particle, draw a line with an arrow directed towards the vertex, and associate the spinor $u_s(\mathbf{p})$, where p is directed along the arrow.

- For an outgoing fermion particle, draw a line with an arrow directed away from the vertex, and associate the spinor $\overline{u}_s(\mathbf{p})$, where p is directed along the arrow.

- For an incoming fermion antiparticle, draw a line with an arrow directed away from the vertex, and associate the spinor $\overline{v}_s(\mathbf{p})$, where $-p$ is directed along the arrow.

- For an outgoing fermion antiparticle, draw a line with an arrow directed towards the vertex, and associate the spinor $v_s(\mathbf{p})$, where $-p$ is directed along the arrow.

- For each internal fermion, associate the propagator

$$i\frac{\not{p}+m}{p^2-m^2+i\epsilon}\ , \tag{14.26}$$

where p is the momentum pointing along the direction of the fermion arrow (this must be drawn so that these consistently match with the fermion arrows of the external states).

- Account for relative minus signs when swapping identical fermions in the final state by writing diagrams in standard form (or otherwise).
- Associate an additional factor of -1 with each fermion loop.
- For every incoming (outgoing) photon, associate a polarization vector $\epsilon^{\mu}_{\lambda_i}$ ($\epsilon^{\mu *}_{\lambda_i}$).
- For every internal photon, associate a factor

$$-i\frac{g^{\mu\nu} - (1-\xi)\,\frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} \ . \tag{14.27}$$

where p is the momentum carried by the photon, for general gauge parameter ξ.

- The only allowed vertex connects a photon to a fermion line with one arrow pointing towards the vertex and one pointing away. We associate a factor

$$-ie\gamma^\mu \ , \tag{14.28}$$

with this.

We now move on to consider the application of these Feynman rules to specific processes.

14.4 Scattering in QED: $e^+e^- \to \gamma\gamma$

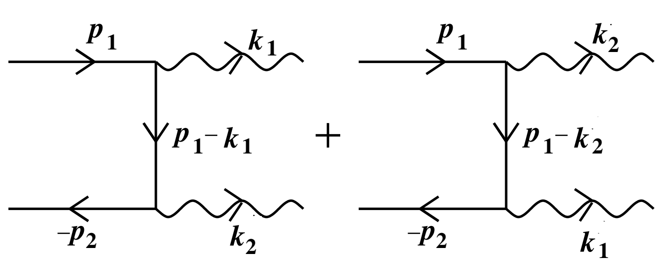

Figure 22: Leading–order Feynman diagrams for $e^+e^- \to \gamma\gamma$.

To demonstrate the techniques we will need to calculate observables in QED, we start by considering the tree–level $e^+(p_1)e^-(p_2) \to \gamma(k_1)\gamma(k_2)$ scattering process. The two contributing diagrams are shown in Fig. 22. We have

$$\mathcal{M} = -e^2 \epsilon_1^{\mu *} \epsilon_2^{\nu *} \overline{v}(p_2) \left[\gamma_\nu \frac{\not{p}_1 - \not{k}_1 + m}{(p_1 - k_1)^2 - m^2} \gamma_\mu + \gamma_\mu \frac{\not{p}_1 - \not{k}_2 + m}{(p_1 - k_2)^2 - m^2} \gamma_\nu \right] u(p_1) \ . \tag{14.29}$$

Squaring the amplitude (see Section 13.1), we find

$$|\mathcal{M}|^2 = \epsilon_1^{\mu *} \epsilon_2^{\nu *} \epsilon_1^{\rho} \epsilon_2^{\sigma} (\overline{v}_2 A_{\mu\nu} u_1)(\overline{u}_1 \overline{A}_{\rho\sigma} v_2) \ , \tag{14.30}$$

where we define

$$A_{\mu\nu} = -e^2 \left[\gamma_\nu \frac{\not{p}_1 - \not{k}_1 + m}{(p_1 - k_1)^2 - m^2} \gamma_\mu + \gamma_\mu \frac{\not{p}_1 - \not{k}_2 + m}{(p_1 - k_2)^2 - m^2} \gamma_\nu \right] \ . \tag{14.31}$$

Now, from (13.5) we have that $\overline{A}_{\mu\nu} = A_{\nu\mu}$, and averaging over the spin states of the incoming fermions we arrive at

$$\frac{1}{4} \sum_{s_1, s_2} |\mathcal{M}|^2 = \frac{1}{4} \epsilon_1^{\mu *} \epsilon_2^{\nu *} \epsilon_1^{\rho} \epsilon_2^{\sigma} \mathrm{Tr} \left[A_{\mu\nu} (\not{p}_1 + m) A_{\sigma\rho} (\not{p}_2 - m) \right] \ . \tag{14.32}$$

This can be used to calculate the cross section for e^+e^- annihilation into two photons with arbitrarily defined polarization states. However, as with the fermion spins, we are most often interested in the case where the polarization is not measured and so we simply sum over these. In this case, we can once again simplify the above expression.

Armed with (7.71), we can write down an expression for the squared amplitude averaged over the fermion spins, and summed over the photon polarization states. We have

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \mathrm{Tr} \left[A_{\mu\nu} (\not{p}_1 + m) A^{\nu\mu} (\not{p}_2 - m) \right] \ , \tag{14.33}$$

$$= e^4 \left[\frac{\langle \Phi_{tt} \rangle}{(t - m^2)^2} + \frac{\langle \Phi_{tu} \rangle + \langle \Phi_{ut} \rangle}{(t - m^2)(u - m^2)} + \frac{\langle \Phi_{uu} \rangle}{(u - m^2)^2} \right] \ , \tag{14.34}$$

where

$$\langle \Phi_{tt} \rangle = \frac{1}{4} \mathrm{Tr} \left[\gamma_\nu (\not{p}_1 - \not{k}_1 + m) \gamma_\mu (\not{p}_1 + m) \gamma^\mu (\not{p}_1 - \not{k}_1 + m) \gamma^\nu (\not{p}_2 - m) \right] \ , \tag{14.35}$$

$$\langle \Phi_{ut} \rangle = \frac{1}{4} \mathrm{Tr} \left[\gamma_\mu (\not{p}_1 - \not{k}_2 + m) \gamma_\nu (\not{p}_1 + m) \gamma^\mu (\not{p}_1 - \not{k}_1 + m) \gamma^\nu (\not{p}_2 - m) \right] \ , \tag{14.36}$$

$$\langle \Phi_{uu} \rangle = \langle \Phi_{tt} \rangle : \ (k_1 \leftrightarrow k_2) \ , \tag{14.37}$$

$$\langle \Phi_{tu} \rangle = \langle \Phi_{ut} \rangle : \ (k_1 \leftrightarrow k_2) \ , \tag{14.38}$$

where the last expressions denote that we should simply interchange $k_1 \leftrightarrow k_2$ in Φ_{tt} and Φ_{ut} to give the corresponding results. Making use of the results

$$\gamma^\mu \gamma_\mu = 4 \ , \tag{14.39}$$
$$\gamma^\mu \not{a} \gamma_\mu = -2\not{a} \ , \tag{14.40}$$
$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4(ab) \ , \tag{14.41}$$
$$\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu = -2\not{c}\not{b}\not{a} \ , \tag{14.42}$$

which can readily be derived from the basic properties of the gamma matrices, see Section 13.2, and expressing the results in terms of the usual Mandelstam variables we eventually get

$$\langle \Phi_{tt} \rangle = 2\left[ut - m^2(3t+u) - m^4\right] \ , \tag{14.43}$$
$$\langle \Phi_{ut} \rangle = \langle \Phi_{tu} \rangle = 2m^2(s-4m^2) \ , \tag{14.44}$$
$$\langle \Phi_{uu} \rangle = \langle \Phi_{tt} \rangle : \ (u \leftrightarrow t) \ , \tag{14.45}$$

where the last two results follow as the interchange $k_1 \leftrightarrow k_2$ corresponds precisely to $u \leftrightarrow t$.

14.5 Renormalized Lagrangian

We now consider the calculation of 1–loop radiative corrections in QED. To do so, we must consider as in the scalar QED case the renormalised Lagrangian

$$\mathcal{L}_0 = iZ_2 \overline{\psi} \not{\partial} \psi - Z_m m \overline{\psi}\psi - Z_3 \frac{1}{4} F^{\mu\nu}F_{\mu\nu} \ , \tag{14.46}$$
$$\mathcal{L}_I = -Z_1 e \overline{\psi} \not{A} \psi \ , \tag{14.47}$$

or, in the counterterm language

$$\mathcal{L}_0' = i\overline{\psi} \not{\partial} \psi - m \overline{\psi}\psi - \frac{1}{4} F^{\mu\nu}F_{\mu\nu} \ , \tag{14.48}$$
$$\mathcal{L}_I = -Z_1 e \overline{\psi} \not{A} \psi \ , \tag{14.49}$$
$$\mathcal{L}_{ct} = i\delta_2 \overline{\psi} \not{\partial} \psi - \delta_m m \overline{\psi}\psi - \delta_3 \frac{1}{4} F^{\mu\nu}F_{\mu\nu} \ , \tag{14.50}$$

where $\delta_i = Z_i - 1$ as usual.

We now derive the 1–loop corrections to the photon and fermion propagators, and to the QED vertex. In contrast to the scalar QED case, we will take more care to keep track of the finite pieces, enabling us to e.g. demonstrate the different renormalization schemes that we may choose.

14.6 1–Loop Correction to Photon propagator

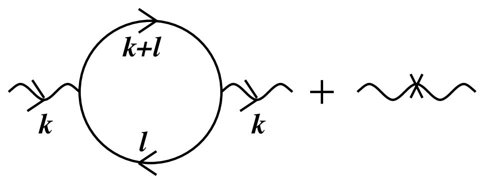

Figure 23: Feynman diagrams contributing at 1–loop to photon propagator in QED.

We first consider the 1–loop correction to the photon propagator. The overall form is similar to the scalar QED case considered in Section 8.5, with the added complication that we must now deal with the gamma matrix structure in the numerator, but with the simplification that there is no diagram due to the contact interaction. There is therefore one contributing diagram at 1–loop, shown in Fig. 23.

As in the scalar QED case, the most general form the photon self–energy can have is

$$\Pi^{\mu\nu}(k) = \Pi(k^2)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \ , \tag{14.51}$$

and we will demonstrate below that this is indeed the case at 1–loop order.

The corresponding amplitude is

$$i\Pi^{\mu\nu}(k) = (-1)(-iZ_1 e)^2 i^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\left[(\not{l}+\not{k}+m)\gamma^\mu(\not{l}+m)\gamma^\nu\right]}{((l+k)^2-m^2)(l^2-m^2)} - i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \ , \tag{14.52}$$

where the first (-1) is due the presence of the fermion loop. Introducing Feynman parameters, the integrand becomes

$$\frac{\mathrm{Tr}\left[(\not{l}+\not{k}+m)\gamma^\mu(\not{l}+m)\gamma^\nu\right]}{((l+k)^2-m^2)(l^2-m^2)} = \int_0^1 \mathrm{d}x \frac{N^{\mu\nu}}{(q^2+X)^2} \ , \tag{14.53}$$

where $q = l + xk$ and

$$X = x(1-x)k^2 - m^2 \ . \tag{14.54}$$

For the numerator we have

$$
\begin{aligned}
N^{\mu\nu} &= \mathrm{Tr}\left[(\not{l} + \not{k} + m)\gamma^\mu(\not{l} + m)\gamma^\nu\right] \ , &(14.55)\\
&= 4\left\{(l+k)^\mu l^\nu + l^\mu (l+k)^\nu - \left[l(l+k) - m^2\right] g^{\mu\nu}\right\} \ , &(14.56)\\
&\stackrel{!}{=} 4\left\{2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - \left[q^2 - x(1-x)k^2 - m^2\right] g^{\mu\nu}\right\} \ , &(14.57)\\
&\stackrel{!}{=} 4\left\{-2x(1-x)k^\mu k^\nu + \left[\left(\frac{2}{D} - 1\right) q^2 + x(1-x)k^2 + m^2\right] g^{\mu\nu}\right\} \ , &(14.58)\\
&\stackrel{!}{=} 4\left\{-2x(1-x)k^\mu k^\nu + \left[X + x(1-x)k^2 + m^2\right] g^{\mu\nu}\right\} \ , &(14.59)\\
&\stackrel{!}{=} 8x(1-x)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \ , &(14.60)
\end{aligned}
$$

Here, in the second line we have evaluated the trace; in the third line we have substituted $q = l + xk$ and dropped terms odd in q; in the fourth line we have moved to D dimensions and applied PV reduction; in the fifth line we have used the relation (8.66); in the last line we have substituted for X directly. Thus, as we expect the 1–loop correction to the photon self–energy is completely transverse.

Now, we are left with the integral

$$
\begin{aligned}
\tilde{\mu}^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{(q^2 + X)^2} &= \tilde{\mu}^\epsilon \frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{(-X)^{\epsilon/2}} \Gamma\left(\frac{\epsilon}{2}\right) \ , &(14.61)\\
&= \frac{i}{(4\pi)^2} \left(\frac{4\pi\tilde{\mu}^2}{-X}\right)^{\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} - \gamma + O(\epsilon)\right) \ , &(14.62)\\
&= \frac{i}{(4\pi)^2} \left(\frac{\mu^2}{-X}\right)^{\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} + O(\epsilon)\right) \ , &(14.63)\\
&= \frac{i}{8\pi^2} \left(\frac{1}{\epsilon} - \frac{1}{2} \ln\left(\frac{-X}{\mu^2}\right)\right) \ , &(14.64)
\end{aligned}
$$

where we have made the $\overline{MS}$ definition $\mu^2 = 4\pi\tilde{\mu}^2 e^{-\gamma}$, and used that

$$
x^\epsilon = 1 + \epsilon \ln x + O(\epsilon^2) \ , \tag{14.65}
$$

with terms of $O(\epsilon)$ safely set to zero in the last line, as these will vanish when we set $\epsilon = 0$. Thus, taking $Z_1 = 1 + O(e^2)$ we have the final result

$$
\Pi(k^2) = -\frac{e^2}{\pi^2} \int_0^1 \mathrm{d}x\, x(1-x) \left[\frac{1}{\epsilon} - \frac{1}{2} \ln\left(\frac{-X}{\mu^2}\right)\right] - (Z_3 - 1) + O(e^4) \ , \tag{14.66}
$$

Which gives

$$Z_3^{\overline{\rm MS}} = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} + O(e^4) \;, \tag{14.67}$$

for which

$$\Pi^{\overline{\rm MS}}(k^2) = \frac{e^2}{2\pi^2}\int_0^1 {\rm d}x\, x(1-x)\ln\left(\frac{-X}{\mu^2}\right) + O(e^4) \;. \tag{14.68}$$

As an alternative, we can impose the *on–shell* renormalization scheme. Here, as discussed in Section 8.5, we fix things by demanding that our calculation for the exact photon propagator, which we recall is given (in the Feynman gauge) by

$$-i\tilde{\Delta}^{\rm exact}_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2(1-\Pi(k^2)) - i\epsilon} \tag{14.69}$$

has the same behaviour as the observed one at $k^2 = 0$, i.e. for an on–shell photon. From gauge invariance, we have shown that this automatically has a pole at $k^2 = 0$, as we expect, however we must also fix the residue at this pole to be the same, so that the analytic structure is preserved. We must therefore also impose $\Pi(k^2 = 0) = 0$. This gives

$$Z_3^{\rm on-shell} = 1 - \frac{e^2}{6\pi^2}\left[\frac{1}{\epsilon} - \ln\left(\frac{m}{\mu}\right)\right] + O(e^4) \;, \tag{14.70}$$

and thus

$$\Pi^{\rm on-shell}(k^2) = \frac{e^2}{2\pi^2}\int_0^1 {\rm d}x\, x(1-x)\ln\left(\frac{-X}{m^2}\right) + O(e^4) \;. \tag{14.71}$$

In both cases, we must input an observation at some scale in order to determine the self–energy. For the on–shell scheme we have done this directly, by taking $k^2 = 0$ as our reference points. For the $\overline{\rm MS}$ scheme, we can see that (14.68) contains a μ dependence, and so to implement this, we will need to choose some scale μ_0 at which to input an observation. We will see how this works concretely for the QED beta function below. Finally, we can see that

$$Z_3^{\rm on-shell} - Z_3^{\overline{\rm MS}} = \frac{e^2}{6\pi^2}\ln\left(\frac{m}{\mu}\right) + O(e^4) \;, \tag{14.72}$$

that is, these differ only by finite terms, as they must; we are free to interchange such pieces by changing renormalization scheme (so long as this is done consistently), but the divergence structure is unchanged.

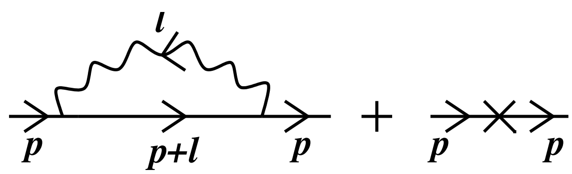

Figure 24: Feynman diagrams contributing at 1–loop to fermion propagator in QED.

14.7 1–Loop Correction to Fermion propagator

At leading order we associate a factor of $-i\tilde{S}_F$ with an internal fermion line, with

$$-i\tilde{S}_F(p) = i\frac{\not{p}+m}{p^2-m^2} \,, \tag{14.73}$$

dropping the $i\epsilon$ term for simplicity. At higher orders this will again receive corrections, with

$$-i\tilde{S}^{\rm exact}(p) = -i\tilde{S}_F(p) + (-i\tilde{S}_F(p))(i\Sigma(\not{p}))(-i\tilde{S}_F(p)) + \cdots \,. \tag{14.74}$$

Here this defines $\Sigma(\not{p})$, the electron self–energy, and at 1–loop the contributing diagram is shown in Fig. 24. These can be summed up in a similar way to the photon propagator, with after a little manipulation

$$-i\tilde{S}^{\rm exact}(p) = \frac{i}{\not{p}-m+\Sigma(\not{p})} \,, \tag{14.75}$$

where this is understood as the matrix inverse (in spinor space). Here, and in the following section we work in the Feynman gauge for simplicity; as in the case of scalar QED, if we work in a more general gauge (6.58), the results for e.g. $Z_{1,2}$ will be ξ–dependent, though we still have $Z_1 = Z_2$ and all observables are as expected ξ independent. At 1–loop we have

$$\begin{aligned} i\Sigma(\not{p}) &= (-iZ_1 e)^2 \int \frac{{\rm d}^4 l}{(2\pi)^4}\gamma_\mu \frac{(\not{p}+\not{l}+m)}{(l^2-m_\gamma^2)((p+l)^2-m^2)}\gamma^\mu + i(Z_2-1)\not{p} - i(Z_m-1)m \,, \\ &= -e^2\tilde{\mu}^\epsilon \int_0^1 {\rm d}x \frac{{\rm d}^D q}{(2\pi)^D}\frac{N}{(q^2+X)^2} + i(Z_2-1)\not{p} - i(Z_m-1)m + O(e^4) \,, \end{aligned} \tag{14.76}$$

where in the first line as in the case of scalar QED, we have had to introduce a fictional photon mass to regulate the IR divergence associated with the $l \to 0$ part of the integral, and which we are not interested in here. In the second line we have introduced Feynman parameters as usual, with $q = l + xp$, and $X = x(1-x)p^2 - xm^2 - (1-x)m_\gamma^2$. The numerator is given by

$$N = \gamma_\mu(\not{p} + \not{l} + m)\gamma^\mu = (2-D)(\not{p} + \not{l}) + Dm \stackrel{!}{=} (2-D)(1-x)\not{p} + Dm \;, \qquad (14.77)$$

where in the last equality we have dropped a term odd in q. Now, the D–dimensional integral is of exactly the same form as for the photon propagator correction, and so we get

$$\begin{aligned}\Sigma(\not{p}) = &-\frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}x\,((\epsilon-2)(1-x)\not{p} + (4-\epsilon)m)\left[\frac{1}{\epsilon} - \frac{1}{2}\ln\left(\frac{-X}{\mu^2}\right)\right] \;, \\ &+ (Z_2-1)\not{p} - (Z_m-1)m + O(e^4) \;. \end{aligned} \qquad (14.78)$$

Which gives

$$Z_2^{\overline{\mathrm{MS}}} = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + O(e^4) \;, \qquad (14.79)$$

$$Z_m^{\overline{\mathrm{MS}}} = 1 - \frac{e^2}{2\pi^2}\frac{1}{\epsilon} + O(e^4) \;. \qquad (14.80)$$

Now, we can see from (14.75) and (14.78) that the 1–loop correction to the fermion propagator has generated a contribution proportional to the fermion mass, with the ϵ pole piece absorbed into the renormalization constant Z_m. We therefore expect our renormalization condition to correspond to inputting the value of the fermion (i.e. electron, muon...) mass, as determined experimentally. In other words, our renormalized mass should correspond to this. However, as we will see the $\overline{\mathrm{MS}}$ scheme does not in fact make such a direct assignment, which is instead achieved in the on–shell scheme.

In the process of discussing this, we can also clarify how renormalization will proceed in the current context if we work with the Z_i's rather than counterterms. To do this, we recall from (11.53) that

$$\langle 0|T\psi(x)\overline{\psi}(y)|0\rangle = -i\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{S}_F(p)e^{-ip(x-y)} \;, \qquad (14.81)$$

where we have dropped the spinor indices for simplicity. So far this corresponds to the general expression for the propagator S_F, which can be defined at a particular order in perturbation theory, or indeed as the exact one in the same way as above. The impact of renormalizing the fields in our QED Lagrangian, i.e. $\psi_0 = Z_2^{1/2}\psi$, therefore simply corresponds to

$$\tilde{S}_{F,0}(p) = Z_2\tilde{S}_F(p) \;, \qquad (14.82)$$

and we therefore have

$$-i\tilde{S}^{\rm exact}(p) = -\frac{1}{Z_2} i\tilde{S}_0^{\rm exact}(p) = \frac{1}{Z_2}\frac{i}{\not{p} - m_0 + \Sigma(\not{p})} , \tag{14.83}$$

where $\Sigma(\not{p})$ now includes no counterterm contribution. Now, returning to our Lagrangian (14.46), we are interested in

$$\mathcal{L}_0^F = iZ_2\overline{\psi}\not{\partial}\psi - Z_m m\overline{\psi}\psi = Z_2\overline{\psi}\left(i\not{\partial} - Z_m' m\right)\psi , \tag{14.84}$$

where we have defined $Z_m' \equiv Z_m/Z_2$. Indeed, in many discussions of this topic it is precisely Z_m' that is defined to be the renormalization constant Z_m, and hence care is needed to keep track of which notation is being used; one can of course always use the above relation to translate between the two conventions. In particular, we have

$$(Z_m^{\overline{\rm MS}})' = \frac{Z_m^{\overline{\rm MS}}}{Z_2^{\overline{\rm MS}}} = 1 - \frac{3e^2}{8\pi^2}\frac{1}{\epsilon} + O(e^4) . \tag{14.85}$$

Here, we simply introduce this notation to emphasise the fact that in our notation we have

$$m_0 = Z_m' m = \frac{Z_m}{Z_2} m , \tag{14.86}$$

and hence (14.83) becomes

$$-i\tilde{S}^{\rm exact}(p) = \frac{i}{Z_2\not{p} - Z_m m + \Sigma(\not{p}) + \cdots} = \frac{i}{\not{p} - m + \delta_2\not{p} - \delta_m m + \Sigma(\not{p})} , \tag{14.87}$$

where in the second line we have dropped terms of $O(e^4)$ (the $\cdots$) and higher, which is allowed as we are only working at 1–loop. This is precisely the same as (14.75), where the counterterm contributions are included in $\Sigma(\not{p})$, and thus we have indeed verified that the two approaches will achieve the same result, at least at the current order.

Now, our experimental input is that the particle propagator is observed to have a pole at a mass $m_{\rm pole}$, and this is what we physically associate with say the measured electron mass (i.e. $m_{\rm pole}^{\rm el.} = 0.511$ MeV). Thus it is natural to assign

$$m^{\rm on-shell} = m_{\rm pole} . \tag{14.88}$$

To achieve this, we need that

$$\left[\delta_2\not{p} - \delta_m m + \Sigma(\not{p})\right]\big|_{\not{p}=m_{\rm pole}} = m_{\rm pole}(\delta_2 - \delta_m) + \Sigma(\not{p})\big|_{\not{p}=m_{\rm pole}} = 0 . \tag{14.89}$$

We in addition, as for the photon propagator require that the residue of this pole to match that in the LO propagator, i.e. to be i. This particular choice is certainly a sensible one, but is strictly speaking conventional, with other choices in principle possible provided one keeps track of the suitable normalization of things in the LSZ formula. Indeed, this no longer holds in the $\overline{\rm MS}$ scheme. Making this choice for the on–shell scheme, we require

$$i = \lim_{\not{p}\to m_{\rm pole}} (\not{p} - m_{\rm pole}) \frac{i}{\not{p} - m + \delta_2 \not{p} - \delta_m m + \Sigma(\not{p})} = \lim_{\not{p}\to m_{\rm pole}} \frac{i}{1 + \delta_2 + \frac{{\rm d}}{{\rm d}\not{p}}\Sigma(\not{p})} \;, \quad (14.90)$$

where we have used L'Hôpital's rule. Finally, (14.89) and (14.90) are sufficient to fix Z_2 and Z_m in the on–shell scheme. We will not calculate these explicitly here, but we can see that they will certainly be different from the $\overline{\rm MS}$ values, as expected. However, it is straightforward to see from (14.78) that the $1/\epsilon$ terms will be the same in the two cases, as they must.

Given the values of Z_m are different in the two schemes, and recalling the discussion in Section 8.2, we therefore expect the numerical value we should assign to the renormalized mass, $m^{\overline{\rm MS}}$, to be different from the physical pole mass, and indeed it will be. To see this, we note that in any scheme the denominator of (14.87) must have a pole at the physical pole mass $\not{p} = m^{\rm pole}$. Hence, in the $\overline{\rm MS}$ scheme we have

$$m^{\rm pole} - m^{\overline{\rm MS}} + \delta_2 m^{\rm pole} - \delta_m m^{\overline{\rm MS}} + \Sigma(\not{p})|_{\not{p}=m_{\rm pole}} = 0 \;. \quad (14.91)$$

Rearranging, and using that $m^{\rm pole} = m^{\overline{\rm MS}} + O(e^2)$ (i.e. at LO all schemes coincide), we have

$$m^{\overline{\rm MS}} = m^{\rm pole}(1 + \delta_2 - \delta_m) + \Sigma(\not{p})|_{\not{p}=m_{\rm pole}} \;, \quad (14.92)$$

which it is a straightforward exercise to show is non–zero as well as finite; the $1/\epsilon$ poles in the $\delta_{2,m}$ precisely cancel those in the $\Sigma(\not{p})|_{\not{p}=m_{\rm pole}}$, as required. We note in addition that the $\overline{\rm MS}$ mass in (14.92) depends on the scale μ via the $\Sigma(\not{p})$ term, in a completely analagous way to the dependence of the photon self–energy $\Pi(k^2)$, and as well see the coupling α.

In summary, by applying the standard $\overline{\rm MS}$ renormalization scheme we end up with a definition of the renormalized fermion mass that does not (beyond LO) correspond precisely to the value one would most straightforwardly assign it be experimentally, i.e. the pole mass $m^{\rm el.}_{\rm pole} = 0.511$ MeV in the case of an electron. Note however that provided we work systematically in either scheme the final result will be the same at the order we work at. So, in the $\overline{\rm MS}$ scheme we have seen that (14.87) still has a pole at $\not{p} = m^{\rm pole}$, as it does in the on–shell scheme. At each order in perturbation theory, the $\overline{\mathrm{MS}}$ mass will receive corrections as in (14.92), while at any order in perturbation theory the pole mass is fixed. Nonetheless, it may seem a little strange to work in such a scheme, and in a sense it is less intuitive. However, for essentially technical reasons the $\overline{\mathrm{MS}}$ scheme is easier to work with, and hence tends to be what is used, though this is not always true. Moreover in QCD, where free quarks are not observed in nature, the concept of a pole mass for a quark is not such an obviously intuitive or useful concept.

14.8 1–Loop Correction to Vertex Function

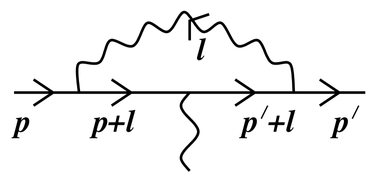

Figure 25: Feynman diagrams contributing at 1–loop to QED vertex.

The Feynman diagram contributing to the QED vertex at 1–loop is shown in Fig. 25. Defining $iV^\mu(p',p)$ as the sum of all 1PI diagrams with an incoming fermion of momentum p and an outgoing fermion with momentum p', in the Feynman gauge we have

$$\begin{aligned} iV^\mu(p',p) = &-iZ_1 e\gamma^\mu \\ &+ (-ie)^3 i^2(-i)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\gamma^\nu(\not{p}'+\not{l}+m)\gamma^\mu(\not{p}+\not{l}+m)\gamma_\nu}{(l^2-m_\gamma^2)((p'+l)^2-m^2)((p+l)^2-m^2)} + O(e^4)\ , \end{aligned} \tag{14.93}$$

where as before we have introduced an artificial photon mass to regulate the associated IR divergence. After introducing Feynman parameters, performing some spinor manipulation and D–dimensional momentum integration, we finally arrive at

$$V^\mu(p',p) = -Z_1 e\gamma^\mu - \frac{e^3}{8\pi^2}\left[\left(\frac{1}{\epsilon}-1-\frac{1}{2}\int \mathrm{d}F_3 \ln\left(\frac{-X}{\mu^2}\right)\right)\gamma^\mu - \frac{1}{4}\int \mathrm{d}F_3\frac{N^\mu}{X}\right]\ , \tag{14.94}$$

where

$$\int \mathrm{d}F_3 \equiv 2\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(x_1+x_2+x_3-1)\ , \tag{14.95}$$

and

$$X = x_1(1-x_1)p^2 + x_2(1-x_2)(p')^2 - 2x_1x_2 p\cdot p' - (x_1+x_2)m^2 - x_3 m_\gamma^2 \;, \tag{14.96}$$

$$N^\mu = \gamma_\nu\left[x_1 \not{p} - (1-x_2)\not{p}' - m\right]\gamma^\mu\left[(1-x_1)\not{p} - x_2\not{p}' + m\right]\gamma^\nu \;. \tag{14.97}$$

Which gives

$$Z_1^{\overline{\rm MS}} = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + O(e^4)\;. \tag{14.98}$$

14.9 The QED β Function

From the kinetic terms of the renormalized QED Lagrangian, we can read off that we have made the field redefinitions

$$\psi_0 = Z_2^{1/2}\psi\;, \qquad A_0^\mu = Z_3^{1/2}A^\mu\;, \tag{14.99}$$

where the '0' refers to the bare fields. Thus for the interaction term we must have

$$e_0 Z_2 Z_3^{1/2}\overline{\psi}\not{A}\psi = Z_1 \tilde{\mu}^{\frac{\epsilon}{2}} e \overline{\psi}\not{A}\psi\;, \tag{14.100}$$

where we have introduced the usual dimensionful parameter $\tilde{\mu}$ required when continuing to D dimensions. Defining the renormalized charge via $e_0 = Z_e\tilde{\mu}^{\frac{\epsilon}{2}}e$, we then get

$$Z_e = Z_1 Z_2^{-1} Z_3^{-1/2}\;. \tag{14.101}$$

Stated physically, we have that in principle all three classes of correction we have considered above will contribute to the renormalization of the electric charge. This makes perfect sense, as if we consider the possible higher order corrections to the QED vertex, then we would expect the self–energy corrections to the photon and fermion legs, as well as the vertex correction itself, to all contribute. However, we have seen above that

$$Z_1^{\overline{\rm MS}} = Z_2^{\overline{\rm MS}} = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + O(e^4)\;, \tag{14.102}$$

and thus these corrections exactly cancel, leaving us with

$$e_0 = Z_3^{-1/2}\tilde{\mu}^{\frac{\epsilon}{2}} e \Rightarrow \alpha_0 = Z_3^{-1}\tilde{\mu}^{\epsilon}\alpha\;, \tag{14.103}$$

where we have expressed things in terms of the fine–structure parameter $\alpha = \frac{e^2}{4\pi}$, as is conventional. This therefore tells us that for QED only the correction to the photon propagator contributes to the renormalization of the electric charge. We note that (14.102) is of course not a coincidence, but is rather a direct result of the Ward–

Takahashi identity for QED described in Section 9.

Thus, using (14.67) we have to 1–loop order

$$\alpha_0 = \left(1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon}\right)^{-1} \tilde{\mu}^\epsilon \alpha \ . \tag{14.104}$$

Now, the important point here is that the bare α_0 corresponds to the input parameter in the Lagrangian before any regularization or renormalization has been performed. It therefore has to be independent of the scale $\tilde{\mu}$. When we continue to D dimensions, and take $\mathrm{d}^4x \to \mathrm{d}^Dx$ in the Lagrangian, we introduced *by hand* this scale in order to keep the renormalized coupling dimensionless. At leading order we have $Z_i = 1$, and so this simply corresponds to

$$\alpha_0 \equiv \tilde{\mu}^\epsilon \alpha \ . \tag{14.105}$$

As the parameter on the left hand side is by definition independent of $\tilde{\mu}$, the renormalized coupling must depend on the scale in such way as to exactly cancel the impact of varying it. We see this sort of thing all of the time when we introduce some system of units: if we take $\epsilon = 1$, for example, then we can imagine writing e_0, as a number, e, multiplied by the unit $\tilde{\mu}$. If we change our unit $\tilde{\mu}$ from eV to keV (say), then e must be reduced by a factor of a 1000, for consistency.

Pursuing this more generally, writing

$$\ln \alpha_0 = \epsilon \ln \tilde{\mu} + \ln \alpha(\mu^2) \ , \tag{14.106}$$

where we now explicitly write the μ^2 (or equivalently μ, this is just a choice of notation, both of which are used in the literature) argument of α. Then, differentiating the left hand side, we must have

$$\frac{\partial \ln \alpha_0}{\partial \ln \mu} = 0 = \epsilon + \frac{\partial \ln \alpha(\mu^2)}{\partial \ln \mu} \ , \tag{14.107}$$

where we have taken $\tilde{\mu} \to \mu$, which we are free to do as these differ by a constant prefactor. Hence we have

$$\frac{\partial \alpha(\mu^2)}{\partial \ln \mu} = -\epsilon\alpha \ , \tag{14.108}$$

giving

$$\beta(\alpha(\mu^2)) \equiv \left.\frac{\partial \alpha(\mu^2)}{\partial \ln \mu}\right|_{\epsilon=0} = 0 \ , \tag{14.109}$$

where we have defined the β–function, which expresses the variation of the coupling with scale μ. For $D = 4$, at leading order we have simply $\alpha = \alpha_0$, and so this indeed vanishes.

So far, this is all a relatively trivial result of the definition (14.105). However, if we now consider the 1–loop correction to this (14.104) something interesting happens. In particular, applying the same procedure as before, we find

$$\frac{\partial \alpha(\mu^2)}{\partial \ln \mu} = -\epsilon\alpha(\mu^2) + \frac{2\alpha^2}{3\pi} \;, \tag{14.110}$$

and thus

$$\beta(\alpha(\mu^2)) = \frac{2\alpha^2}{3\pi} \;. \tag{14.111}$$

In other words, by including the effect of quantum corrections, the measured coupling α (i.e. after setting $D = 4$) has developed a *scale dependence*.

Although the above derivation is completely general and correct, this can be seen somewhat more transparently if we consider the 1–loop correction (14.68) to the photon vacuum polarization, which we can see explicitly has a dependence on the photon virtuality k^2. In particular, motivated by the way in which this appears in the correction to the photon propagator, we can define an *effective coupling* via

$$\alpha_{\rm eff}(k^2) = \frac{\alpha}{1-\Pi(k^2)} \;, \tag{14.112}$$

where again we see that the quantum correction will induce a dependence on the photon virtuality k^2 via $\Pi(k^2)$. The cross section for e.g. electron–electron scattering will be $\propto \alpha_{\rm eff}(k^2)^2$ and hence we can relate this directly to the observed scattering cross section at energy scale k^2. It is related to, but not the same as, the QED coupling α. We in particular have

$$\begin{aligned}
\alpha_{\rm eff}(k^2) &= \alpha(\mu^2)\left(1+\frac{2\alpha(\mu^2)}{\pi}\int_0^1 {\rm d}x\, x(1-x)\ln\left(\frac{-X}{\mu^2}\right)\right) + O(\alpha^3)\;,\\
&\approx \alpha(\mu^2)\left(1+\frac{2\alpha(\mu^2)}{\pi}\int_0^1 {\rm d}x\, x(1-x)\left[\ln x(1-x)+\ln\left(\frac{-k^2}{\mu^2}\right)\right]\right) + O(\alpha^3)\;,\\
&= \alpha(\mu^2)\left(1+\frac{\alpha(\mu^2)}{3\pi}\left[-\frac{5}{3}+\ln\left(\frac{-k^2}{\mu^2}\right)\right]\right) + O(\alpha^3)\;,
\end{aligned} \tag{14.113}$$

where in the second line we assume $-k^2 \gg m^2$, and we have explicitly included the μ argument of α on the RHS, following the discussion above. Thus physically we indeed expect our measured coupling to vary with the energy scale $\sim k^2$ of the interaction, something we have motivated from more general grounds above. To be concrete, if we consider the effective coupling measured at two values of $k_f^2, k_i^2 \gg m^2$, then we find simply

$$\alpha_{\rm eff}(k_f^2) - \alpha_{\rm eff}(k_i^2) = \frac{\alpha^2(\mu^2)}{3\pi} \ln\left(\frac{k_f^2}{k_i^2}\right) + O(\alpha^3) \ , \tag{14.114}$$

and hence we expect the effective charge to increase with increasing energy scale. We note that the above result can also be derived by imposing a renormalization condition at some scale k_i, by writing $\alpha(|k_i^2|)$ (introducing a modulus to keep the argument positive, as is conventional) in terms of $\alpha_{\rm eff}(k_i^2)$ at this scale, such that

$$\alpha(-k_i^2) = \alpha_{\rm eff}(k_i^2)\left(1 + \frac{5\alpha_{\rm eff}(k_i^2)}{9\pi}\right) + O(\alpha^3) \ , \tag{14.115}$$

and remembering that $\alpha(-k_i^2) = \alpha_{\rm eff}(k_i^2) + O(\alpha^2)$. Thus, one can certainly interpret μ in the same way as the discussion around (8.30), i.e. as the scale at which we input a measurement for α.

However, rather than picking one particular scale for μ in this way, we can be more general, and require that the observable $\alpha_{\rm eff}$ cannot depend on this arbitrary scale μ on the RHS of (14.113). The corresponding μ dependence in the coupling α should then be defined to ensure this. We get

$$\frac{\partial\alpha_{\rm eff}(k^2)}{\partial \ln\mu} = 0 = \frac{\partial\alpha(\mu^2)}{\partial \ln\mu} - \frac{2\alpha^2(\mu^2)}{3\pi} + O(\alpha^3) \ , \tag{14.116}$$

which precisely corresponds to the expression we have derived above for the QED β function.

Writing

$$\beta(\alpha) = b_0\alpha^2 \ , \tag{14.117}$$

we can solve (14.111), to give

$$\alpha(\mu_f^2) = \frac{\alpha(\mu_i^2)}{1 - b_0\alpha(\mu_i^2)\ln\left(\frac{\mu_f}{\mu_i}\right)} \ . \tag{14.118}$$

We therefore see that the scale dependence, or *running*, of the coupling α has two distinct possibilities, depending on the sign of b_0:

- $b_0 > 0$: *Infrared Free*. The coupling α *decreases* as the scale $\mu \to 0$ towards the IR, but *increases* as the scale $\mu \to \infty$ towards the UV.

- $b_0 < 0$: *Asymptotically Free*. The coupling α *increases* as the scale $\mu \to 0$ towards the IR, but *decreases* as the scale $\mu \to \infty$ towards the UV.

As we shall see later, the second case is relevant for QCD. The former case is true for QED, and indeed we know that at low scales we have $\alpha \to \alpha(0) \approx 1/137$. On other hand, as $\mu \to \infty$ the coupling increases and at some point we will enter a region where $\alpha \sim 1$. In such a situation the theory becomes strongly coupled, and a purely perturbative treatment is no longer applicable. From (14.118) we can in particular see that there is a pole at

$$\alpha(\mu_i^2) \ln\left(\frac{\mu_f}{\mu_i}\right) = \frac{1}{b_0} \,, \tag{14.119}$$

which is known as a *Landau pole*. Clearly upon reaching this point our theory will have broken down entirely. Should we worry about this for QED? Well, if we consider the QED coupling at the scale of the electron mass, $\alpha(m_e) = 1/137$, then we have

$$\mu_{\text{Landau}} = M_e e^{\frac{3\pi}{2\alpha(m_e)}} \approx 10^{286}\,\text{eV} \,, \tag{14.120}$$

which is quite a bit (!) higher than the Planck scale $m_P \sim 10^{28}$ eV. Actually in the Standard Model, we have 3 flavours of lepton, which leads b_0 to be a factor of 3 larger, as well as the impact of W boson loops to worry about. This brings the scale of the pole down quite a bit, but still safely a lot higher than m_P.

Now, there remains the question of what we should actually take for the scale μ in a given problem. From the discussion above it seems natural to associate this scale with the virtuality k^2 of the photon. Thus, in for example $e^+(p_1)e^+(p_2) \to e^+(p_1')e^+(p_2')$ scattering, we should evaluate the corresponding coupling α at the (positive) scale $Q^2 = -k^2 = -(p_1 - p_3)^2$. One might object that the derivation in terms of the effective coupling proceeded by demanding that the observable itself is independent of the renormalization scale μ. In other words, why does it matter at all what value we take for μ? This is a perfectly valid question, and indeed it is quite true that any observable, calculated to all orders in perturbation theory, must be independent of μ. However, the point is that to an arbitrary truncated order in a perturbative expansion, this will not be true. Indeed, looking at (14.113) we can see that if $\mu^2 \gg -k^2$ or $\mu^2 \ll -k^2$ then the logarithm will become arbitrarily large and the second term in our perturbative expansion may well become larger than the first, in particular if $\alpha(\mu^2)|\ln(-k^2/\mu^2)| \sim 1$ or larger. Thus, for such choices we can expect our perturbative expansion to be rather poorly behaved. One might still object that the scale dependence in α should compensate for this, but the point is that what we have in fact required is that the dependence on μ vanishes in (14.113) to the $O(\alpha^2)$ of the perturbative calculation. While the scale dependence of α is precisely determined so as to achieve this, the observable when truncated to this order will retain a $O(\alpha^3)$ dependence on μ. In particular, we find explicitly for (14.113) that

$$\frac{\partial \alpha_{\rm eff}(k^2)}{\partial \ln \mu} = \left(\frac{2}{3\pi}\right)^2 \alpha^3(\mu^2)\left[-\frac{5}{3}+\ln\left(\frac{-k^2}{\mu^2}\right)\right] \,, \tag{14.121}$$

at this order[13]. Thus, while formally zero at $O(\alpha^2)$, unless we take $\mu^2 \sim |k^2|$ the logarithm present here will also be large and $\alpha_{\rm eff}(k^2)$ will depend sensitively, and unphysically, on the choice we take. Stated another way, if we do take $\mu^2 \sim |k^2|$ then we can see that the perturbative expansion is well behaved and nicely convergent, i.e. our $O(\alpha^2)$ result will approximate the true 'all orders' one rather closely. Thus any choice such as $\mu^2 \gg -k^2$ or $\mu^2 \ll -k^2$ which takes us far away from this $O(\alpha^2)$ result is certainly unreliable. The running coupling, with $\mu^2 \sim |k^2|$, is said to *resum* these higher order logarithms, giving a convergent calculation that results in the scale dependence effects we describe above. In particular, taking $\mu^2 = |k^2|$ in (14.113) we have

$$\alpha_{\rm eff}(k^2) = \alpha(|k^2|)\left(1-\frac{5\alpha(|k^2|)}{9\pi}\right) \,, \tag{14.122}$$

and so up to the small $O(\alpha)$ correction the effective coupling is simply given by $\alpha(|k^2|)$, with its appropriate scale variation given by the QED β–function.

Finally, we emphasise that the RHS of (14.122), and the similar expressions above, corresponds to the QED *prediction* for the measured value of $\alpha_{\rm eff}$ at some scale k^2, given $\alpha(\mu^2)$ as an input. Thus we in general expect the RHS of (14.122) to agree with the measured value, but only within $O(\alpha^3)$ precision, as these are the terms we do not include in the perturbative expansion. Thus while we should take $\mu^2 \sim |k^2|$, the precise choice is open to us, and indeed can be considered to parameterise the perturbative uncertainty on our prediction, given the RHS of (14.113) depends on this choice at precisely the $O(\alpha^3)$ level we omit in our calculation. Typically, we therefore take a range of values for μ^2, typically $\mu^2 \in (-k^2/4, 4k^2)$, to give a spread of predictions that estimate the perturbative uncertainty on our prediction.

15. Non–Abelian Gauge Theory

15.1 Non–Abelian Groups

In the previous sections, we have discussed the $U(1)$ gauge symmetry possessed by the

[13]One might worry about what happens if we consider an s–channel exchange, for which $k^2 > 0$ and the argument of the logarithm becomes negative. This is in general not a problem, as we can analytically continue via $\ln(-k^2) = \ln(k^2) + i\pi$, and a careful treatment then shows that the $i\pi$ does not contribute.

complex scalar and spinor fields $\phi(x)$ and $\psi(x)$ in the case of scalar and regular QED, respectively. This corresponded to multiplication of our fields by a simple commuting phase, known as an *abelian* symmetry. We now wish to generalise this to the case where the symmetry does not involve commuting operations, corresponding to a *non–abelian* gauge symmetry. To do so, we first write things in a slightly different (but equivalent) way. Our local gauge transformation is

$$\phi(x) \to U(x)\phi(x) \; , \tag{15.1}$$

with $U^\dagger = U^{-1}$, and by ϕ we denote either a scalar or spinor field. Explicitly, we have

$$U(x) = e^{iq\alpha(x)} \; . \tag{15.2}$$

This corresponds to a one–dimensional unitary $U(1)$ transformation. We can generalise this by introducing a set of N fields, with a corresponding transformation

$$\phi_i(x) \to U_{ij}\phi_j(x) \; . \tag{15.3}$$

Two commonly occurring examples are $SU(N)$, where U_{ij} is a $N \times N$ unitary matrix, and $SO(N)$, where U_{ij} is a $N \times N$ orthogonal matrix:

$$SU(N): \qquad UU^\dagger = I, \qquad \det(U) = 1 \; , \tag{15.4}$$

$$SO(N): \qquad UU^T = I, \qquad \det(U) = 1 \; , \tag{15.5}$$

where the second requirement holds for the special groups, otherwise these are the larger $U(N)$ and $O(N)$ groups. These are however directly related to the special groups. For example, any unitary matrix can be written as

$$U = e^{i\phi}\tilde{U} \; , \tag{15.6}$$

where ϕ is a real parameter and $\det(\tilde{U}) = 1$. Mathematically, we say that $U(N)$ is the direct product of $U(1) \times SU(N)$, and $SU(N)$ is a non–trivial subgroup of $U(N)$. Thus $SU(N)$ represents a simpler case to $U(N)$, and indeed when it comes to e.g. QCD it is an experimental fact that the correct gauge group is $SU(N)$. Moreover, from the point of view of QFT it only really makes sense to consider these subgroups individually, as these behave in quite different ways. To gauge $U(N)$, we should gauge $U(1)$ and $SU(N)$ independently, with in general different gauge couplings.

Now, we can write some infinitesimal transformation θ_{ij} from the origin as

$$U_{jk}(x) = \delta_{jk} + \theta_{jk} \;, \tag{15.7}$$
$$= \delta_{jk} + i\theta^a(x)(T^a)_{jk} + O(\theta^2) \;, \tag{15.8}$$

where in the second line we have expanded in terms of a linearly independent set of basis matrices $(T^a)_{jk}$. The expansion parameters $\theta_a(x)$ are defined to be real, so that the unitarity of U_{jk} and $\det(U) = 1$ implies that these are hermitian and traceless. There are $N^2 - 1$ such matrices, labelled above by the index a, and they are known as the *generators* of the group; we have met precisely such objects before in Section 10, when we expanded a general Lorentz transformation in terms of the basis generators of rotations and boosts. Demanding that the product of two separate such transformations is still a $SU(N)$ transformation implies that the commutator of two generators must be a linear combination of generators

$$\left[T^a, T^b\right] = if^{abc}T^c \;, \tag{15.9}$$

where f^{abc} are real numerical factors known as the *structure constants* of the group. If the structure constants do not vanish, the group is *non–abelian*. For the $U(1)$ case we trivially have only one generator, which is a number, and so this is an *abelian* group. We are free to choose the generators so that they obey the conventional normalization

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \;. \tag{15.10}$$

For $SU(2)$ the basis of $N^2 - 1 = 3$ linearly independent hermitian traceless matrices are the Pauli matrices σ^a, with the above normalization implying $T^a = \sigma^a/2$; again we met these before in this context in Section 10. The corresponding structure constants are then given by the Levi–Civita symbols, $f^{abc} = \epsilon^{abc}$.

15.2 Non–Abelian Gauge Symmetry

Concentrating on the $SU(N)$ case, we have seen from QED that the corresponding Lagrangian is invariant under such a transformation provided we introduce a covariant derivative

$$D_\mu = \partial_\mu - igA_\mu \;, \tag{15.11}$$

where we now introduce a coupling g, and note that to follow the normal convention this corresponds to taking $g = -e$ in comparison to the QED case. We require this to transform as

$$D_\mu \to U(x) D_\mu U^\dagger(x) \;, \tag{15.12}$$

see (5.27). This is satisfied if A_μ transforms as

$$A_\mu \to U(x) A_\mu U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x) \ . \tag{15.13}$$

To be explicit, for a $U(1)$ symmetry the $U(x)$ factors in the first term cancel automatically and we indeed have

$$A_\mu \to A_\mu + \partial_\mu \alpha(x) \ , \tag{15.14}$$

which is consistent with the usual QED transformation.

In direct analogy to the $U(1)$ case, we can write some arbitrary $U(x)$ in terms of the generator matrices as

$$U(x) = \exp\left[ig\Gamma^a(x) T^a\right] \ , \tag{15.15}$$

where the real parameters $\Gamma^a(x)$ are not infinitesimal. We then define $A_\mu(x)$ as a matrix of fields; for the abelian gauge symmetry to generalise consistently, we require these to be Hermitian and traceless, i.e. to have the same general properties as the generator matrices. In particular, we can see that the second term in (15.13) corresponds to an arbitrary traceless and hermitian matrix, and thus $A_\mu(x)$ must have at least that number of degrees of freedom. Then, if we assume that $A_\mu(x)$ itself is traceless and hermitian, as would be natural, the overall transformation (15.13) preserves this.

The covariant derivative can be written as

$$D_\mu^{ij} = \partial_\mu \delta_{ij} - igA_\mu(x)_{ij} \ , \tag{15.16}$$

where $i, j = 1, \cdots, N$ are the group indices. The $A_\mu(x)$ transforms according to (15.13) as above. For the kinetic term, we define the field strength as for the $U(1)$ case by

$$F_{\mu\nu}(x) \equiv \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \ . \tag{15.17}$$

where we now pick up a final commutator of the fields for this non–abelian gauge group. Note here (and in many places below) we have dropped the gauge indices for simplicity. It follow that under a gauge transformation

$$F_{\mu\nu} \to U(x) F_{\mu\nu}(x) U^\dagger(x) \ , \tag{15.18}$$

which in contrast to the $U(1)$ case is not gauge invariant due to the fact that the $U(x)$ are now matrices that can not be simply pulled through F. However if we perform a trace of the group indices, then due to the cyclic property of traces this is gauge invariant. We are therefore led to the kinetic term

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\text{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \ , \tag{15.19}$$

where the normalization is conventional. We can expand the $A_\mu(x)$ and $F^{\mu\nu}$ in the basis of generator matrices

$$A_\mu(x) \equiv A^a_\mu(x)T^a \ , \qquad F_{\mu\nu} \equiv F^a_{\mu\nu}T^a \ , \tag{15.20}$$

in terms of which we find

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu \ , \tag{15.21}$$

using (15.9). For the kinetic term we have

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}F^{a,\mu\nu}F^b_{\mu\nu}\text{Tr}\left(T^aT^b\right) = -\frac{1}{4}F^{a,\mu\nu}F^a_{\mu\nu} \ , \tag{15.22}$$

using (15.10). Thus we can see that this leads to interactions between the gauge fields $A_\mu(x)$ themselves. Before considering the specific case of QCD, we first discuss some general aspect of group representations, that will play a role in the sections which follow.

15.3 Group Representations

Any set of $D(R) \times D(R)$ traceless hermitian matrices T^a_R (where R is part of the name, rather than being an index) that satisfies the same commutation relations as the original generators

$$\left[T^a_R, T^b_R\right] = i f^{abc} T^c_R \ , \tag{15.23}$$

forms a representation of the group. Here $D(R)$ is the dimension of the representation. The original $D(R) = N$ case is known as the *fundamental* representation. The matter fields ϕ, or in QCD, ψ, transform in this representation, with

$$\psi \to \exp\left[ig\Gamma^a(x)T^a\right]\psi \ . \tag{15.24}$$

Taking the complex conjugate of the above relation, we can see that the matrices $-(T^a_R)^*$ also obey the commutation relations. If $-(T^a_R)^* = T^a_R$ or we can find a unitary transformation $T^a_R \to U^{-1}T^a_R$ that makes $-(T^a_R)^* = T^a_R$ for every a then the representation R is *real*. If such a transformation does not exist, but we can find a unitary matrix $V \neq I$ such that $-(T^a_R)^* = V^{-1}T^a_RV$ for every a then the representation is *pseudoreal*.

If no such unitary matrix exists, then the representation is complex. In this case, the *complex conjugate representation* $\overline{R}$ is specified by

$$T^a_{\overline{R}} = -(T^a_R)^* \ , \tag{15.25}$$

An important representation for the case of non–abelian groups is the *adjoint representation*, given by

$$(T^a_A)^{bc} = -if^{abc} \ , \tag{15.26}$$

as the f^{abc} are real and completely antisymmetric, the generators are manifestly hermitian. With a little work one can show that these do indeed satisfy (15.9), as required. The adjoint representation is also real. Note the dimension of the adjoint representation is equal to the number of generators, so $D(A) = N^2 - 1$ for $SU(N)$. In the context of a gauge theories, we find that the gauge fields transform in the adjoint representation of the group. In particular, for a *global* gauge transformation the gauge field transforms as

$$A_\mu \to U A_\mu U^\dagger \ , \tag{15.27}$$

which infinitesimally becomes

$$A_\mu \to A^a_\mu (1 + ig\theta^b T^b) T^a (1 - ig\theta^c T^c) \ , \tag{15.28}$$

and thus

$$A^a_\mu \to A^a_\mu - g A^b_\mu f^{bac} \theta^c = A^a_\mu + ig\theta^b (T^b_A)^{ac} A^c_\mu \ . \tag{15.29}$$

Hence, the gauge fields transforms in the adjoint representation for this global symmetry, i.e

$$A_\mu \to \exp(ig\Gamma^b T^b_A) A_\mu \ , \tag{15.30}$$

in the non–infinitesimal case. The extension to a local symmetry is considered in the following section.

Finally, we note that two numbers usually characterise a representation. The *index* $T(R)$, defined via

$$\mathrm{Tr}(T^a_R T^b_R) = T(R)\delta^{ab} \tag{15.31}$$

and the *quadratic Casimir* C(R), defined via

$$\sum_a (T^a_R T^a_R)_{ij} = C(R)\delta_{ij} \ . \tag{15.32}$$

For $SU(N)$, in the fundamental representation we have

$$T(N) = \frac{1}{2} , \qquad C(N) = \frac{N^2 - 1}{2N} , \tag{15.33}$$

while in the adjoint representation we have

$$C(A) = T(A) = N . \tag{15.34}$$

15.4 QCD: A First Look

The theory of strong interactions, quantum chromodynamics (QCD) is a $SU(3)$ non–abelian gauge theory. The $N = 3$ group indices are known as colours, while there are six flavours of quark (up, down, strange, charm , bottom, top). There are $3^2 - 1 = 8$ generator matrices, which are given in terms of the so–called *Gell–Mann* matrices. The explicit form of these is not important for our purposes, but these can readily be written down if need be.

The Lagrangian is given in terms of Dirac fields $\psi_{iI}(x)$, where i is the colour index and I is the flavour index, with

$$\mathcal{L} = i\overline{\psi}_{iI} \not{D}_{ij} \psi_{jI} - m_I \overline{\psi}_{iI} \psi_{iI} - \frac{1}{2} \mathrm{Tr} \left(F^{\mu\nu} F_{\mu\nu} \right) . \tag{15.35}$$

Expanding out, and dropping the flavour indices for simplicity we have

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}+\psi} &= i\overline{\psi}_i \not{\partial} \psi_i - m \overline{\psi}_i \psi_i & (15.36) \\
&+ g T^a_{ij} \overline{\psi}_i \not{A}^a \psi_j & (15.37) \\
&- \frac{1}{2} \partial^\mu A^{a\nu} \partial_\mu A^a_\nu + \frac{1}{2} \partial^\mu A^{a\nu} \partial_\nu A^a_\mu & (15.38) \\
&- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu , & (15.39)
\end{aligned}$$

where the i, j run over the fundamental group indices, $i, j = 1, \cdots, 3$, and the $a, b...$ run over the adjoint indices, $a, b = 1, \cdots, 8$. The first line corresponds to the quark kinetic term; the second line to a $gq\overline{q}$ interaction vertex; the third line to the gluon kinetic term; the fourth line, two new interaction vertices between the gluons themselves. The latter are of course absent in QED, and are a direct result of the non–abelian nature of the gauge symmetry, as can be seen by the fact that these terms are proportional to the structure constants. However, before writing down the Feynman rules for QCD, we must address the issue that arose already in the case of QED, namely that if we naively apply the above expression to derive a gluon propagator, the overall gauge redundancy in our description will spoil things. To do this, we apply the same Faddeev–Popov procedure as before.

15.5 Faddeev–Popov Gauge Fixing

Recall that in the derivation of the photon propagator in Section 6.3 it was not possible by default to perform the necessary inversion to shift the photon field variable A_μ so that the propagator could be derived. We found however that due to the invariance of the Lagrangian under the QED gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \ , \tag{15.40}$$

we could exclude all components of A_μ that are parallel to the photon momentum k_μ, and are redundant in the corresponding momentum integral. This resulted in a path integral that could be evaluated.

We will have exactly the same problem in the case of non–Abelian gauge theory, however the simple argument of above no longer holds due to the non–linear nature of the gauge transformation. To see this, we consider an infinitesimal transformation

$$U(x) = I + ig\theta^a(x)T^a \ , \tag{15.41}$$

under which the gauge field transformation (15.13) becomes

$$\begin{aligned} A_\mu^a(x) \to A_\mu^a(x) + gf^{abc}A_\mu^b(x)\theta^c(x) + \partial_\mu\theta^a(x) \ , && (15.42) \\ = A_\mu^a(x) + \left[\delta^{ac}\partial_\mu + gf^{abc}A_\mu^b(x)\right]\theta^c(x) \ , && (15.43) \\ = A_\mu^a(x) + \left[\delta^{ac}\partial_\mu - igA_\mu^b(T_A^b)^{ac}\right]\theta^c(x) \ , && (15.44) \\ = A_\mu^a(x) + D_\mu^{ac}\theta^c(x) \ , && (15.45) \end{aligned}$$

where T_A are the generators in the adjoint representation, and comparing with (15.16) we can see that D_μ^{ac} is the covariant derivative, but in the adjoint representation; this generalises the statement made above about the gauge groups transforming in the adjoint representation for the global transformation. Now, this is very similar in form to the usual QED $U(1)$ gauge transformation, however instead of a simple ∂_μ we now have a more complicated covariant derivative. As this no longer simply involves ∂_μ (which becomes k_μ on Fourier transforming) we will no longer have that the components of A_μ parallel to k_μ can be simply excluded.

We must therefore pursue the more general Faddeev–Popov gauge fixing procedure described in Section 6.4. Recall that we could write the path integral as

$$Z = \int \mathcal{D}A\, e^{iS[A]}\Delta(A)\delta(f(A)) \;. \tag{15.46}$$

For the function we can simply generalise the previous $U(1)$ version to give

$$f(A^a) = \partial^\mu A^a_\mu - \sigma^a(x) \;, \tag{15.47}$$

as we will now be integrating over each of the $a = 1, ..., N^2 - 1$ gauge fields. We now have

$$\begin{aligned}
[\Delta(A)]^{-1} = \int \mathcal{D}\theta\, \delta(f(A_\theta)) &= \int \mathcal{D}\theta\, \delta(\partial^\mu A^a_\mu - \sigma^a(x) - \partial^\mu(g f^{abc}\theta^b A^c_\mu - \partial_\mu\theta^a)) \;, \\
&\overset{!}{=} \int \mathcal{D}\theta\, \delta(i\partial^\mu(g f^{abc}\theta^b A^c_\mu - \partial_\mu\theta^a)) \;, && (15.48) \\
&= \int \mathcal{D}\theta\, \delta(-i\partial^\mu D^{ab}_\mu\theta^b) \;, && (15.49)
\end{aligned}$$

where in the second line we drop the $f(A)$ terms as before, as this will be accompanied by a $\delta(f(A))$ above. We have also multiplied by an overall factor of i in the delta function, which are free to do as we are not interested in overall constants; this is just to maintain consistency with the conventional definition of the ghost Lagrangian we will introduce below.

Now, in the Abelian case the factor in the third line corresponded to an overall constant, which could be dropped, however here due to the non–Abelian nature of the transformation this depends on the gauge fields and so cannot be omitted from the path integral above. To account for this term, we can write formally

$$i\partial^\mu(g f^{abc}A^c_\mu(x) - \partial_\mu\delta^{ab})\theta^b(x) = \int \mathrm{d}^4y K^{ab}(x,y)\theta^b(y) \;, \tag{15.50}$$

which at this stage simply defines the operator

$$\begin{aligned}
K^{ab}(x,y) &= i\partial^\mu(g f^{abc}A^c_\mu - \partial_\mu\delta^{ab})\delta^4(x-y) \;, && (15.51) \\
&= -i\partial^\mu D^{ab}_\mu\delta^4(x-y) \;. && (15.52)
\end{aligned}$$

Now, returning to the meaning of our Faddeev–Popov determinant $\Delta(A)$, we recall this is introduced as an overall Jacobian factor. In particular, we have the elementary result

$$\int \mathrm{d}\theta \delta(K\theta) = \frac{1}{K} \tag{15.53}$$

for real numbers, which can be generalised to the case of some (non–singular) matrix K_{ij}, with

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \delta(K_{ij}\theta_j) = \frac{1}{\det K} \ , \tag{15.54}$$

where $i = 1, \cdots, n$. Now, in the path integral generalisation of this, the integer index i is replaced by a continuous real variable y (say), and the summation becomes an integral. Thus we can see that the RHS of (15.50) is precisely the path integral generalisation of $K_{ij}\theta_j$; the fact that we have an additional summation over the gauge indices, a, changes nothing in principle. Thus we can make the association

$$\Delta(A) = \det(K) \ . \tag{15.55}$$

At this point, such an expression is not of much use as we do not actually know how to evaluate the determinant of this continuous, infinite–dimensional, object. However, we already know how to write the determinant of some matrix as an integral over Grassmann variables (11.36), i.e.

$$\int \mathrm{d}^n\chi \mathrm{d}^n\overline{\chi} \exp\left(\overline{\chi}_i K_{ij} \chi_j\right) = \det K \ . \tag{15.56}$$

and we can generalize this to the case of functional integration, by as before replacing our finite–dimensional integral with a functional integral and the summation over i, j by an integral over continuous spacetime variables, i.e.

$$\int \mathcal{D}\chi \mathcal{D}\overline{\chi} \exp\left[\int \mathrm{d}^4x \mathrm{d}^4y \overline{\chi}(x) K(x,y) \chi(y)\right] = \det K \ , \tag{15.57}$$

Thus, we can evaluate the Faddeev–Popov determinant by introducing a set of scalar Grassmannian fields, conventionally denoted as c^a and $\overline{c^a}$. We then have

$$\Delta(A) = \det(K) = \int \mathcal{D}c \mathcal{D}\overline{c}\, e^{iS_{\text{ghost}}} \ , \tag{15.58}$$

where

$$S_{\text{ghost}} = -i \int \mathrm{d}^4x \mathrm{d}^4y \overline{c^a}(x) K^{ab}(x,y) c^b(y) \ . \tag{15.59}$$

Substituting for K we have

$$\mathcal{L}_{gh} = -\int d^4y\overline{c}^a(x)\partial^\mu D_\mu^{ab}\delta^4(x-y)c^b(y) \ , \tag{15.60}$$

$$= \partial^\mu\overline{c}^a D_\mu^{ab}c^b \ , \tag{15.61}$$

$$= \partial^\mu\overline{c}^a\partial_\mu c^a - gf^{abc}A_\mu^c(\partial^\mu\overline{c}^a)c^b \ . \tag{15.62}$$

As before, we are free to multiply the path integral by an arbitrary constant

$$Z_\sigma = \int \mathcal{D}\sigma \exp\left(-\frac{i}{2\xi}\int d^4x\sigma^a(x)\sigma^a(x)\right) \ , \tag{15.63}$$

which upon integrating in (15.46) will pick up a $\partial^\mu A_\mu^a\partial^\nu A_\nu^a$ term. Thus, putting everything together we have

$$Z(J) \propto \int \mathcal{D}A\mathcal{D}c\mathcal{D}\overline{c}\exp\left[iS_{\mathrm{YM}+\psi} + iS_{gh} + iS_{gf}\right] \ , \tag{15.64}$$

with $\mathcal{L}_{\mathrm{YM}+\psi}$ given as in the previous section and

$$\mathcal{L}_{gh} = \partial^\mu\overline{c}^a\partial_\mu c^a - gf^{abc}A_\mu^c(\partial^\mu\overline{c}^a)c^b \ , \tag{15.65}$$

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}\partial^\mu A_\mu^a\partial^\nu A_\nu^a \ , \tag{15.66}$$

The fields c^a and $\overline{c}^a$ are known as *ghosts*. Thus, to define a gauge boson propagator in the non–abelian case we are led to introduce new fields into our Lagrangian. These ghosts are scalar anti–commuting fields, which therefore violate the spin–statistics theorem. Thus these cannot correspond to physical propagating states, and this is indeed the case, as we will see when discussing BRST symmetry later on. On the other hand these fields do propagate internally and must in general be included in loop calculations.

16. QCD

16.1 Feynman Rules

The full QCD Lagrangian is given by $\mathcal{L}_{QCD} = \mathcal{L}_{\mathrm{YM}+\psi} + \mathcal{L}_{gh} + \mathcal{L}_{gf}$. For clarity we reproduce these all below:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}+\psi} &= i\overline{\psi}_i \partial\!\!\!/ \psi_i - m\overline{\psi}_i\psi_i + gT^a_{ij}\overline{\psi}_i A\!\!\!/^a\psi_j - \frac{1}{2}\partial^\mu A^{a\nu}\partial_\mu A^a_\nu + \frac{1}{2}\partial^\mu A^{a\nu}\partial_\nu A^a_\mu \\
&\quad - gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \frac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu \,. \qquad (16.1)\\
\mathcal{L}_{gh} &= \partial^\mu \overline{c}^a \partial_\mu c^a - gf^{abc}A^c_\mu(\partial^\mu \overline{c}^a)c^b \,, \qquad (16.2)\\
\mathcal{L}_{gf} &= -\frac{1}{2\xi}\partial^\mu A^a_\mu \partial^\nu A^a_\nu \,, \qquad (16.3)
\end{aligned}
$$

where as before we leave the quark flavour indices implicit for simplicity. The Feynman rules can be read off from this, and are summarised below:

- The gluon propagator follows in exactly the same way as the photon case, but with a δ^{ab} in the (adjoint) colour group indices. Thus we associate

$$
-i\frac{g^{\mu\nu} - (1-\xi)\,\frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon}\delta^{ab} \,. \qquad (16.4)
$$

  with each internal gluon line carrying momenta p.

- The fermion propagator follows in exactly the same way as for QED, but with a δ_{ij} in the (fundamental) colour group indices. Thus with each internal quark line we associate

$$
i\frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon}\delta_{ij} \,, \qquad (16.5)
$$

  where as usual p is the momentum pointing along the direction of the fermion arrow

- The $gq\overline{q}$ vertex has the same spinor structure as for QED, but now has a generator matrix T^a associated with it. We therefore have the vertex

$$
ig\gamma^\mu T^a_{ij} \,, \qquad (16.6)
$$

  where the adjoint index a is associated with the gluon, and the fundamental indices i (j) are associated with the fermion line pointing away from (towards) the vertex.

- There is a 3–gluon interaction vertex, given by

$$
gf^{abc}((p_1 - p_2)^\gamma g^{\alpha\beta} + (p_2 - p_3)^\alpha g^{\beta\gamma} + (p_3 - p_1)^\beta g^{\gamma\alpha}) \,, \qquad (16.7)
$$

for *incoming* gluon momenta p_1, p_2, p_3 associated with Lorentz indices α, β, γ and adjoint colour indices a, b, c, respectively.

- There is a 4–gluon interaction vertex, given by

$$-ig^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)+f^{ace}f^{bde}\left(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\sigma}g^{\nu\rho}\right)+f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\right] ,$$

  where the Lorentz indices μ, ν, ρ, σ labelling the four gluon lines are associated with the adjoint colour indices a, b, c, d, respectively.

- For every internal ghost line draw a fermion arrow over a dashed line and associate the propagator

$$\frac{i}{p^2 + i\epsilon}\delta^{ab} . \tag{16.8}$$

- The ghost–antighost–gluon vertex is given by

$$g f^{abc} p^\mu , \tag{16.9}$$

  where p is the momenta flowing along the ghost line pointing away from the vertex, b (c) are associate with the ghost lines flowing towards (away from) the vertex, and a is associated with the gluon. One arrow must point towards and one away from each vertex.

- For every internal ghost loop (as well as quark loop) we associate a factor of -1.

- Deal with external fermions in the usual way, but associating a fundamental colour index i with each one.

- For every incoming (outgoing) gluon, associate a polarization vector $\epsilon^\mu_{\lambda_i}$ ($\epsilon^{\mu *}_{\lambda_i}$).

16.2 Quark QED interactions

As well as pure quark–gluon interactions, the quarks/antiquarks carry electric charges $e_q = \pm\frac{1}{3}, \pm\frac{2}{3}$. These can therefore interact with photons via a photon–quark vertex, with Feynman rule:

- The $\gamma q\bar{q}$ vertex is given by

$$-iee_q\delta_{ij}\gamma^\mu , \tag{16.10}$$

  where i, j are the colour indices in the fundamental representation and e_q is the fractional quark charge.

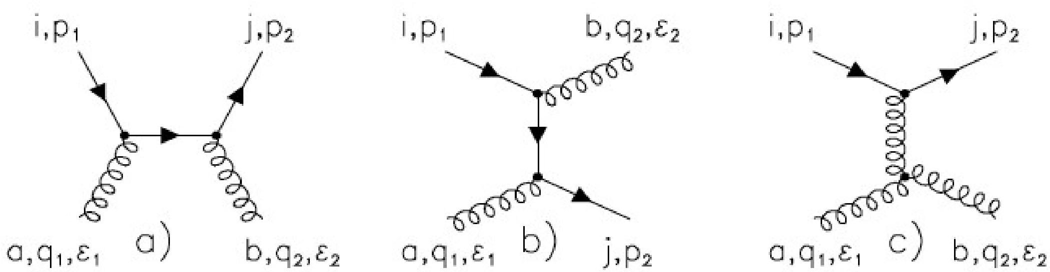

Figure 26: Leading–order Feynman diagrams for $qg \to qg$ scattering. The s, u and t channel diagrams are shown in a, b and c, respectively.

16.3 Example Process: $qg \to qg$ scattering

As an example we can consider the $q(p_1)g(q_1) \to q(p_2)g(q_2)$ scattering process, which at the LHC would contribute to inclusive jet production. The contributing Feynman diagrams at leading order are shown in Fig. 26, along with the corresponding colour labelling. Setting the quark masses to zero for simplicity we have

$$i\mathcal{M}_s = \frac{-ig^2}{s}\left(T^bT^a\right)_{ji}\overline{u}(p_2)\left[\not\epsilon_2^*(\not p_1 + \not q_1)\not\epsilon_1\right]u(p_1)\;, \tag{16.11}$$

$$i\mathcal{M}_u = \frac{-ig^2}{u}\left(T^aT^b\right)_{ji}\overline{u}(p_2)\left[\not\epsilon_1(\not p_1 - \not q_2)\not\epsilon_2^*\right]u(p_1)\;, \tag{16.12}$$

$$i\mathcal{M}_t = \frac{g^2}{t} f^{abc}T^c_{ji}\left[(q_1+q_2)^\sigma(\epsilon_1\cdot\epsilon_2^*) + (q_1-2q_2)\cdot\epsilon_1\epsilon_2^{\sigma *} + (q_2-2q_1)\cdot\epsilon_2\epsilon_1^\sigma\right]\overline{u}(p_2)\gamma_\sigma u(p_1)\;,$$

where we drop the spin labels on the spinor for brevity, but these are implied. We can then square the total amplitude and after averaging/summing over spins apply the same relations as in the QED case to calculate the factors associated with the kinematic terms above. However we now also have the factors associated with colour to deal with. Consider

$$|\mathcal{M}_s^{\rm colour}|^2 = \left(T^bT^a\right)_{ji}\left[\left(T^bT^a\right)_{ji}\right]^* = \left(T^bT^a\right)_{ji}\left(T^aT^b\right)_{ij}\;, \tag{16.13}$$

where we have used that the generator matrices are Hermitian in the last step, and repeated indices are not summed over (yet). Now, the important point here is that we never directly observe the individual colours of the quarks and gluons in any particular scattering process. Indeed only colour neutral hadrons are actually observed in Nature. To account for this, we must average over the colour indices of the quarks and gluons in the initial state, and sum over those in the final state. This gives

$$\langle|\mathcal{M}_s^{\rm colour}|^2\rangle = \frac{1}{N}\frac{1}{N^2-1}\left(T^bT^a\right)_{ji}\left(T^aT^b\right)_{ij} = \frac{1}{N}\frac{1}{N^2-1}{\rm Tr}[T^bT^aT^aT^b]\;, \tag{16.14}$$

$$= \frac{1}{N(N^2-1)}C(N)^2\delta_{ii} = \frac{N^2-1}{4N^2}\;, \tag{16.15}$$

$$= \frac{2}{9}\;, \tag{16.16}$$

where repeated indices are now summed over. In the first line we divide by the $N = 3$ and $N^2 - 1 = 8$ colours of the initial–state quark and gluon, as we are averaging over colour, and in the second line we have applied (15.32) and (15.33).

In fact these latter relations can always be used to express any combination of generator matrices and structure constants that appear in the colour averaged/summed amplitude squared. To give a couple more examples, we have

$$\langle|\mathcal{M}_t^{\rm colour}|^2\rangle = \frac{1}{N(N^2-1)}f^{abc}f^{abd}{\rm Tr}[T^cT^d]\;, \tag{16.17}$$

$$= \frac{1}{N(N^2-1)}T(N)f^{abc}f^{abc} = \frac{1}{N(N^2-1)}T(N)C(A)\delta_{aa}\;, \tag{16.18}$$

$$= \frac{1}{2}\;, \tag{16.19}$$

and finally for an interference term

$$\langle\mathcal{M}_t\mathcal{M}_s^*\rangle = \frac{1}{N(N^2-1)}f^{abc}{\rm Tr}[T^aT^bT^c]\;, \tag{16.20}$$

$$= \frac{1}{N(N^2-1)}\frac{1}{2}f_{abc}{\rm Tr}[T^a[T^b,T^c]]\;, \tag{16.21}$$

$$= \frac{1}{N(N^2-1)}\frac{i}{2}f^{abc}f^{bcd}{\rm Tr}[T^aT^d]\;, \tag{16.22}$$

$$= \frac{1}{N(N^2-1)}\frac{i}{2}C(A)T(N)\delta_{aa}\;, \tag{16.23}$$

$$= \frac{i}{4}\;. \tag{16.24}$$

Using similar results for the remaining terms, and evaluating the kinematic parts via the usual spin and polarization sum relations, we can then readily evaluate the cross section for spin and colour averaged/summed qg scattering.

16.4 Radiative Corrections

We can write the renormalized QCD Lagrangian as

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{2} Z_3 A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} - \frac{1}{2\xi}\partial_\mu A^{a\mu}\partial_\nu A^{av} - Z_{3g} g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu \\
& - \frac{1}{4} Z_{4g} g^2 f^{abc} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu + Z_{2'} \partial^\mu \overline{c}^a \partial_\mu c^a - Z_{1'} g f^{abc} A^c_\mu \partial^\mu \overline{c}^a c^b \\
& + i Z_2 \overline{\psi}_i \partial\!\!\!/ \psi_i - Z_m m \overline{\psi}_i \psi_i + Z_1 g A^a_\mu \overline{\psi}_i \gamma^\mu T^a_{ij} \psi_j \; .
\end{aligned}
\tag{16.25}
$$

Note that the gauge–fixing term does not require a renormalization factor; as with the photon propagator this term receives no corrections at higher order and hence does not require renormalization. Now, the coupling g appears in multiple places in the Lagrangian, but gauge invariance tells us that this should be universal, and therefore renormalize in the same way in each place, i.e. if the theory is to make any sense at all, higher–order corrections cannot break gauge invariance. Replacing the renormalized fields and couplings with their bare counterparts, $A^0_\mu = Z_3^{1/2} A_\mu$, we find that this requires

$$
g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{3g}^2}{Z_3^2} g^2 \tilde{\mu}^\epsilon = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^\epsilon \; . \tag{16.26}
$$

These relations do indeed hold, and are a result of the *Slavnov–Taylor* identities, which are the direct non–abelian analogue of the Takahashi–Ward identities we met in the case of QED. The proof of these relies on the fact that the above QCD Lagrangian, including the ghost terms, obeys a generalisation of gauge invariance known as *BRST* symmetry. We will discuss these further below.

In the following sections we will derive the β–function in the case of QCD. In a similar way to QED, we have

$$
g_0 = Z_1 Z_2^{-1} Z_3^{-1/2} \tilde{\mu}^{\frac{\epsilon}{2}} g \; , \tag{16.27}
$$

however we no longer have $Z_1 = Z_2$, which only holds in the abelian case. To motivate why, recall that the simplest argument for expecting this to hold in QED was that gauge invariance requires all derivatives in the Lagrangian to become covariant derivatives, $\partial_\mu \to \partial_\mu + ieA_\mu$. Then, demanding that this form is preserved after renormalization, in order to maintain gauge invariance of the Lagrangian, implied $Z_1 = Z_2$. While not watertight, this result does indeed follow when derived more rigourously. However, for QCD after gauge fixing, things are not quite so simple. In particular, we have introduced a ghost term

$$
\partial^\mu \overline{c}^a D^{ab}_\mu c^b \tag{16.28}
$$

which does not only feature a covariant derivative, and indeed we have introduced new ghost fields, for which their corresponding gauge transformation properties is far from clear. The symmetry that is preserved in this case is a more general BRST symmetry, and we find in this case that the implications are not as simple, with $Z_1 \neq Z_2$ in general. Thus we must calculate the renormalization factors Z_1, Z_2 and Z_3, which are associated with the $q\bar{q}g$ vertex, the quark propagator and the gluon propagator, respectively.

16.5 1–loop Correction to Quark Propagator

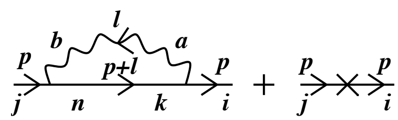

Figure 27: 1–loop corrections to quark propagator. Note here the wavy line represents an internal gluon.

The diagrams contributing at 1–loop to the fermion self–energy are shown in Fig. 27, and are exactly the same ones as those which contribute in the QED case, but with the photon simply replaced by a gluon. In Section 14.7 we found for the QED 1–loop self–energy:

$$\Sigma(\not{p})_{\rm QED} = -\frac{e^2}{8\pi^2}\int_0^1 {\rm d}x\,((\epsilon-2)(1-x)\not{p}+(4-\epsilon)m)\left[\frac{1}{\epsilon}-\frac{1}{2}\ln\left(\frac{-X}{\mu^2}\right)\right] \;, \\ +(Z_2-1)\not{p}-(Z_m-1)m+O(e^4)\;. \tag{16.29}$$

The QCD case will be exactly the same, except with $e \to -g$ (minus as the electric charge of the electron is negative and we define $e > 0$), $\delta_{2,m} \to \delta_{2,m}\delta_{ij}$, and an additional colour factor associated with the 1–loop diagram. This is

$$(T^a T^a)_{ij} = C(N)\delta_{ij}\;, \tag{16.30}$$

so that

$$Z_2^{\overline{\rm MS}} = 1 - C(N)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}\;, \tag{16.31}$$

and similarly for the mass renormalization. We note that here and in what follows we work in the Feynman gauge for the gluon propagator, i.e. with $\xi = 1$. As in QED, the individual renormalization constants do in general depend on ξ, but as usual the physical result does not. In particular, when we calculate the running of the QCD coupling at the end, these ξ dependent factors cancel, and so for simplicity we take this gauge in what follows.

16.6 1–loop Correction to QCD Vertex

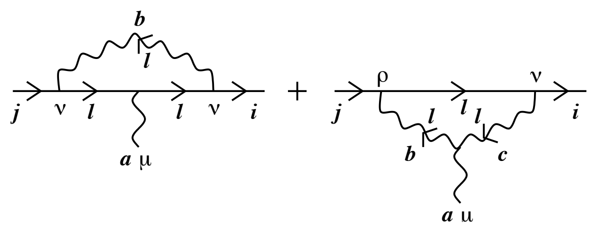

Figure 28: 1–loop corrections to $q\bar{q}g$ vertex. Note here the wavy line represents an internal gluon.

The 1–loop diagrams contributing to the QCD vertex are shown in Fig. 28. The 'abelian' left hand diagram is identical to the QED case, with the photon replaced by a gluon, whereas the right hand 'non–abelian' diagram, which contains a gluon self–coupling, is new. As with the quark propagator, we can simply read off the associated correction for the abelian diagram once we have worked out the corresponding colour factor. This is

$$
\begin{aligned}
T^b T^a T^b &= T^b \left[T^b T^a + i f^{abc} T^c\right] \;, && (16.32)\\
&= C(N) T^a + \frac{i}{2} f^{abc} \left[T^b, T^c\right] \;, && (16.33)\\
&= C(N) T^a + \frac{1}{2} (i f^{abc})(i f^{bcd}) T^d \;, && (16.34)\\
&= C(N) T^a - \frac{1}{2} (T^a_A)^{bc} (T^d_A)^{cb} T^d \;, && (16.35)\\
&= \left(C(N) - \frac{1}{2} T(A)\right) T^a \;. && (16.36)
\end{aligned}
$$

As discussed in the case of scalar QED, as we are only interested in the divergent piece we are free to set the external momenta to zero. Given we work in the $\overline{\text{MS}}$ scheme, where we are only interested in the ϵ poles, up to the usual $\tilde{\mu} \to \mu$ redefinition, this is indeed all we need. The contribution to the exact vertex $iV^{a\mu}_{ij}(p', p)$ from the non–abelian diagram is then

$$
(ig)^2 g f^{abc} (T^c T^b)_{ij} (-i)^2 i \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma_\nu (\not{l} + m) \gamma_\rho}{l^4 (l^2 - m^2)} \left[(0 + l)^\rho g^{\mu\nu} + (-l - l)^\mu g^{\nu\rho} + (l - 0)^\nu g^{\mu\rho}\right] \;, \tag{16.37}
$$

where here and in what follows we work in the Feynman gauge. We can simplify the colour factor with

$$f^{abc}(T^cT^b) = \frac{1}{2}f^{abc}[T^c,T^b] = \frac{i}{2}f^{abc}f^{cbd}T^d = -\frac{i}{2}T(A)T^a \ . \tag{16.38}$$

The numerator is

$$N^\mu = \gamma_\nu(\not{l}+m)\gamma_\rho(l^\rho g^{\mu\nu} + l^\nu g^{\mu\rho} - 2l^\mu g^{\nu\rho}) \ , \tag{16.39}$$

$$\stackrel{!}{=} \frac{l^2}{D}\gamma_\nu\gamma_\sigma\gamma_\rho(g^{\sigma\rho}g^{\mu\nu} + g^{\nu\sigma}g^{\mu\rho} - 2g^{\mu\sigma}g^{\nu\rho}) \ , \tag{16.40}$$

$$= \frac{l^2}{D}(\gamma^\mu\gamma_\sigma\gamma^\sigma + \gamma_\sigma\gamma^\sigma\gamma^\mu - 2\gamma^\sigma\gamma^\mu\gamma_\sigma) \ , \tag{16.41}$$

$$= \frac{l^2}{D}(2D - 2(2-D))\gamma^\mu \ , \tag{16.42}$$

$$= 3l^2\gamma^\mu \ , \tag{16.43}$$

where in the second line we have dropped the terms proportional to m, which are odd in l, and used $l^\mu l^\nu \to l^2 g^{\mu\nu}/D$. We have then made use of the standard results for gamma matrices, and finally set $D = 4$ for simplicity, as we are only interested in the divergent pieces, and any $O(\epsilon)$ contribution will only affect the finite part of the correction. Combining the above our result becomes

$$\frac{3}{2}T(A)g^3T^a_{ij}\gamma^\mu \int \frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(l^2-m^2)} \ . \tag{16.44}$$

From our master formula we have

$$\int \frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(l^2-m^2)} = \frac{i}{8\pi^2}\frac{1}{\epsilon} + O(\epsilon) \ . \tag{16.45}$$

Finally, recalling from (14.94) that we have

$$V^\mu_{\mathrm{QED}}(p',p) = -Z_1e\gamma^\mu - \frac{e^3}{8\pi^2}\frac{1}{\epsilon}\gamma^\mu + \text{finite} \ , \tag{16.46}$$

we can readily read off the divergent part associated with the abelian contribution. Combining the above we then get

$$V^{a\mu}_{ij}(p',p) = \left(Z_1 + \left[C(N) - \frac{1}{2}T(A)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + \frac{3}{2}T(A)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}\right)gT^a_{ij}\gamma^\mu + \text{finite} \ , \tag{16.47}$$

from which we have

$$Z_1^{\overline{\rm MS}} = 1 - [C(N) + T(A)]\, \frac{g^2}{8\pi^2}\frac{1}{\epsilon} \,. \qquad (16.48)$$

16.7 1–loop Correction to Gluon Propagator

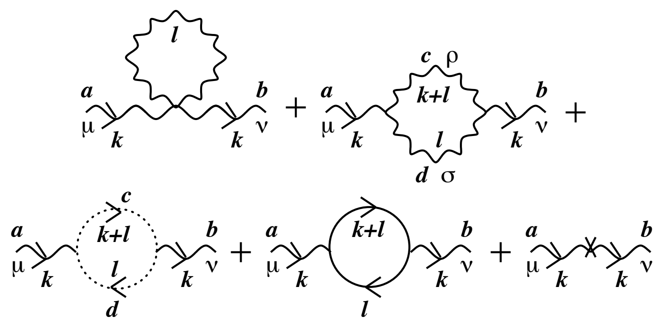

Figure 29: 1–loop corrections to gluon propagator. Note here the wavy line represents an internal gluon.

The 1–loop diagrams contributing to the gluon vacuum polarization are shown in Fig. 29. In addition to the abelian diagram we have three extra contributions, two due to the gluon self–coupling and one from a ghost loop. The first diagram, due to the 4–gluon coupling, will yield an integral

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{l^2} \,. \qquad (16.49)$$

We saw for the $\gamma\gamma\phi\phi$ correction to the scalar propagator in scalar QED that such an integral vanishes, see (8.97). Indeed it is no coincidence that the same integral occurs here, as the diagrams are of similar forms. Therefore it will not contribute.

The second diagram, due to the 3–gluon vertex, gives the contribution to the self–energy

$$i\Pi_{3g}^{\mu\nu,ab}(k) = \frac{1}{2} g^2 f^{acd} f^{bcd} (-i)^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{N^{\mu\nu}}{l^2 (l+k)^2} \,, \qquad (16.50)$$

where the $1/2$ is a symmetry factor, as the loop integration will map the gluons in the loop onto each other, and therefore we will double count without it.

The colour factor is simply $f^{acd} f^{bcd} = T(A)\delta^{ab}$, while the numerator is

$$\begin{aligned} N^{\mu\nu} = &\left[(2k+l)^\sigma g^{\mu\rho} - (2l+k)^\mu g^{\rho\sigma} + (l-k)^\rho g^{\mu\sigma}\right] \\ &\cdot \left[-(2k+l)_\sigma g_\rho{}^\nu + (2l+k)^\nu g_{\rho\sigma} + (k-l)_\rho g_\sigma{}^\nu\right] \,. \end{aligned} \qquad (16.51)$$

Introducing Feynman parameters as usual, and continuing to D dimensions we get

$$i\Pi_{3g}^{\mu\nu,ab}(k) = -\frac{1}{2}g^2 T(A)\delta^{ab}\tilde{\mu}^{\frac{\epsilon}{2}} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{N^{\mu\nu}}{(q^2+X)^2} \,, \tag{16.52}$$

where $X = x(1-x)k^2$ and $q = l + xk$. The numerator thus becomes

$$\begin{aligned} N^{\mu\nu} = &\left[(q+(2-x)k)^\sigma g^{\mu\rho} - (2q+(1-2x)k)^\mu g^{\rho\sigma} + (q-(1-x)k)^\rho g^{\mu\sigma}\right] \\ &\cdot \left[-(q+(2-x)k)_\sigma g_\rho{}^\nu + (2q+(1-2x)k)^\nu g_{\rho\sigma} - (q-(1-x)k)_\rho g_\sigma{}^\nu\right] \,. \end{aligned} \tag{16.53}$$

The terms linear in q will integrate to zero, and we end up with

$$\begin{aligned} N^{\mu\nu} \stackrel{!}{=} &-q^2 g^{\mu\nu} - (4D-6)q^\mu q^\nu - [(1+x)^2 + (2-x)^2]k^2 g^{\mu\nu} \\ &- [D(1-2x)^2 + 2(1-2x)(1+x) - 2(2-x)(1+x) - 2(2-x)(1-2x)]k^\mu k^\nu \,. \end{aligned} \tag{16.54}$$

Again we are free to put $D = 4$, as we are only interested in the divergent piece. Doing this, and replacing $q^\mu q^\nu \to q^2 g^{\mu\nu}/4$ we get

$$N^{\mu\nu} \stackrel{!}{=} -\frac{9}{2}q^2 g^{\mu\nu} - (5-2x+2x^2)k^2 g^{\mu\nu} + (2+10x-10x^2)k^\mu k^\nu \,. \tag{16.55}$$

Now, we apply (8.66) to replace

$$q^2 \to \frac{D}{2-D}X = -2x(1-x)k^2 \,, \tag{16.56}$$

and we finally get

$$N^{\mu\nu} \stackrel{!}{=} -(5-11x+11x^2)k^2 g^{\mu\nu} + (2+10x-10x^2)k^\mu k^\nu \,, \tag{16.57}$$

Again applying (16.45) we get

$$\begin{aligned} i\Pi_{3g}^{\mu\nu,ab}(k) &= -\frac{ig^2}{16\pi^2}\frac{1}{\epsilon}T(A)\delta^{ab}\int_0^1 \mathrm{d}x N^{\mu\nu} + \text{finite} \,, &(16.58)\\ &= -\frac{ig^2}{16\pi^2}\frac{1}{\epsilon}T(A)\delta^{ab}\left(-\frac{19}{6}k^2 g^{\mu\nu} + \frac{11}{3}k^\mu k^\nu\right) + \text{finite} \,, &(16.59) \end{aligned}$$

which is not transverse. However we are not in trouble yet, as we have not included the contribution from the ghost loop. This gives

$$i\Pi^{\mu\nu,ab}_{gh}(k) = (-1)g^2 f^{acd} f^{bdc} i^2 \int \frac{d^4 l}{(2\pi)^4} \frac{(l+k)^\mu l^\nu}{l^2(l+k)^2} \,, \tag{16.60}$$

where the (-1) sign comes from the ghost loop. The colour factor is $f^{acd} f^{bdc} = -T(A)\delta^{ab}$. The numerator is

$$(l+k)^\mu l^\nu = (q+(1-x)k)^\mu (q-xk)^\nu \,. \tag{16.61}$$

$$\stackrel{!}{=} \frac{1}{4} q^2 g^{\mu\nu} - x(1-x)k^\mu k^\nu \,, \tag{16.62}$$

$$\stackrel{!}{=} -\frac{1}{2} x(1-x)k^2 g^{\mu\nu} - x(1-x)k^\mu k^\nu \,, \tag{16.63}$$

where we have shifted variables after introducing Feynman parameters and applied the usual tricks (taking $D = 4$ when applying PV reduction, as we are only interested in the ϵ poles here). Using (16.45) and integrating over x we end up with

$$i\Pi^{\mu\nu,ab}_{gh}(k) = -\frac{ig^2}{8\pi^2}\frac{1}{\epsilon} T(A)\delta^{ab}\left(-\frac{1}{12}k^2 g^{\mu\nu} - \frac{1}{6}k^\mu k^\nu\right) + \text{finite} \,. \tag{16.64}$$

Now adding these contribution together we find

$$i\Pi^{\mu\nu,ab}_{3g}(k) + i\Pi^{\mu\nu,ab}_{gh}(k) = \frac{ig^2}{8\pi^2}\frac{1}{\epsilon}\frac{5}{3} T(A)\delta^{ab}\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) + \text{finite} \,, \tag{16.65}$$

which is transverse; although we have not shown it explicitly here, this also holds for the finite part. Thus we see the importance of including the ghost loops. It is only when this is done that the physical degrees of freedom of the gluon propagate and the corresponding self–energy remains transverse at higher orders, as required by gauge invariance.

We now turn to the final diagram, due to an internal quark loop. This is of exactly the same form as in QED, but with a colour factor

$$\text{Tr}(T^a T^b) = T(N)\delta^{ab} \,. \tag{16.66}$$

Thus reading off from (14.66) we have

$$i\Pi^{\mu\nu,ab}_{q}(k) = -\frac{ig^2}{6\pi^2}\frac{1}{\epsilon} n_F T(N)\delta^{ab}\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) + \text{finite} \,, \tag{16.67}$$

where we have multiplied by a factor of the number of quark flavours, n_F, as each one will give a separate contribution to the loop[14]. Writing

$$i\Pi^{\mu\nu,ab}(k) = \Pi(k^2)(k^\mu k^\nu - k^2 g^{\mu\nu})\delta^{ab} \ , \tag{16.68}$$

and combining the above results, we find

$$\Pi(k^2) = \left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(N)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} - (Z_3 - 1) + \text{finite} \ , \tag{16.69}$$

at 1–loop, and thus

$$Z_3^{\overline{\text{MS}}} = 1 + \left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(N)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} \tag{16.70}$$

16.8 QCD β–Function

In summary, we have

$$Z_1^{\overline{\text{MS}}} = 1 - \left[C(N) + T(A)\right]\frac{\alpha_S}{2\pi}\frac{1}{\epsilon} \ , \tag{16.71}$$

$$Z_2^{\overline{\text{MS}}} = 1 - C(N)\frac{\alpha_S}{2\pi}\frac{1}{\epsilon} \ . \tag{16.72}$$

$$Z_3^{\overline{\text{MS}}} = 1 + \left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(N)\right]\frac{\alpha_S}{2\pi}\frac{1}{\epsilon} \ , \tag{16.73}$$

where we have defined the strong coupling $\alpha_S = g^2/4\pi$ in direct analogy to the fine–structure constant. We now have

$$\alpha_{S,0} = Z_1^2 Z_2^{-2} Z_3^{-1} \tilde{\mu}^\epsilon \alpha_S \ . \tag{16.74}$$

Taking the logarithm and differentiating we then get

$$\frac{\partial \alpha_S}{\partial \ln \mu} = \alpha_S \left[\frac{\partial \alpha_S}{\partial \ln \mu}\left(-2\frac{\partial \ln Z_1}{\partial \alpha_S} + 2\frac{\partial \ln Z_2}{\partial \alpha_S} + \frac{\partial \ln Z_3}{\partial \alpha_S}\right) - \epsilon\right] \ . \tag{16.75}$$

[14]More precisely, only those *active* quark flavours for which $|k^2| \gtrsim m_q^2$ contribute here, and thus $n_F \leq 6$. As we will see below, this does not effect the conclusions about the QCD beta function.

Rearranging we get

$$\frac{\partial \alpha_S}{\partial \ln \mu} = -\epsilon\alpha_S \left[1 - \alpha_S \left(-2\frac{\partial \ln Z_1}{\partial \alpha_S} + 2\frac{\partial \ln Z_2}{\partial \alpha_S} + \frac{\partial \ln Z_3}{\partial \alpha_S}\right)\right]^{-1} , \tag{16.76}$$

$$= -\epsilon\alpha_S \left[1 + \alpha_S \left(-2\frac{\partial \ln Z_1}{\partial \alpha_S} + 2\frac{\partial \ln Z_2}{\partial \alpha_S} + \frac{\partial \ln Z_3}{\partial \alpha_S}\right)\right] + O(\alpha_S^3) , \tag{16.77}$$

and thus to the order we are calculating in we have

$$\beta(\alpha_S) = -\alpha_S^2\epsilon \left(-2\frac{\partial \ln Z_1}{\partial \alpha_S} + 2\frac{\partial \ln Z_2}{\partial \alpha_S} + \frac{\partial \ln Z_3}{\partial \alpha_S}\right)\Bigg|_{\epsilon=0} , \tag{16.78}$$

$$= -\alpha_S^2\epsilon \left(-2\frac{\partial Z_1}{\partial \alpha_S} + 2\frac{\partial Z_2}{\partial \alpha_S} + \frac{\partial Z_3}{\partial \alpha_S}\right)\Bigg|_{\epsilon=0} + O(\alpha_S^3) , \tag{16.79}$$

$$= -\frac{\alpha_S^2}{2\pi}\left(2\left[C(N) + T(A)\right] - 2C(N) + \left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(N)\right]\right) , \tag{16.80}$$

which recalling (14.117) for

$$\alpha_S(\mu_f) = \frac{\alpha_S(\mu_i)}{1 - b_0\alpha_S(\mu_i)\ln\left(\frac{\mu_f}{\mu_i}\right)} . \tag{16.81}$$

gives

$$b_0 = -\frac{1}{2\pi}\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(N)\right] . \tag{16.82}$$

For QCD we have $T(A) = N = 3$ and $T(N) = 1/2$, and thus

$$b_0 = -\frac{1}{2\pi}\left[11 - \frac{2}{3}n_F\right] , \tag{16.83}$$

which is negative provided the number of quark flavours $n_f \leq 16$, which it certainly is for QCD.

This calculation tells us that the QCD coupling is *asymptotically free*, and has important physical consequences:

- As the scale μ decreases the coupling α_S increases, and thus at low energies QCD becomes strongly coupled, that is $\alpha_S \sim 1$ or higher and perturbation theory can no longer be reliably applied. This is completely consistent with what we see in Nature: we do not observe free quarks and gluons, but rather hadrons which are invariant under $SU(3)$ ('colour singlet') and correspond to tightly bound systems of quarks and gluons[15]

- As the scale μ increases the coupling α_S decreases to zero. Thus if we probe QCD at sufficiently high scales the coupling becomes small and hence amenable to perturbation theory. So, in for example the *Deep Inelastic Scattering* (DIS) of a lepton off a proton $lp \to l + X$, provided the virtuality of the exchanged photon is large enough, we can describe the reaction in terms of the scattering of the lepton off free quarks within the proton, and include a range of higher–order QCD corrections to this picture. More generally this fact is responsible for us being able to calculate essentially anything in high–energy hadron colliders; prior to this physicists had to resort to making general statements based on basic symmetry and unitarity requirements, via so–called S–Matrix and Regge theory.

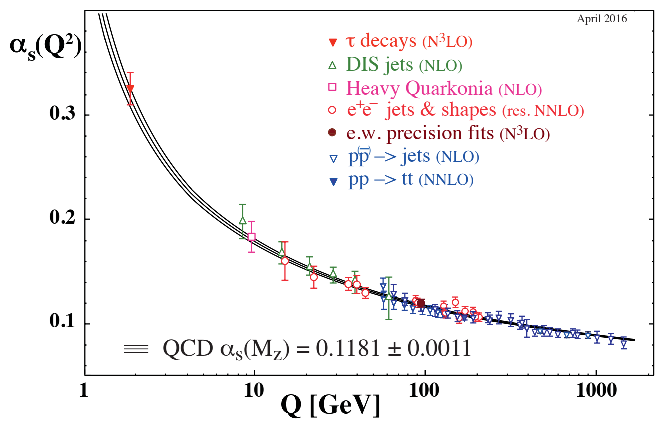

Figure 30: Summary of measurements of the strong coupling α_S, at different scales Q..

The fact that we can say anything at all about the strong force using perturbative methods is quite a remarkable one, and as such this calculation is arguably the most important one in the history of high–energy collider physics. Without it, there would be no LHC. It is no surprise then that the Nobel prize for physics in 2004 went to Gross, Politzer and Wilczek '*for the discovery of asymptotic freedom in the theory of the strong interaction*'. An up–to–date summary of the measurements of α_S is shown

[15]Note that some care in naively applying (16.81) is needed here, and in particular it is not the case that α_S becomes arbitrarily large at low energies, as this expression might suggest. Rather, the whole argument leading up to it relies on the perturbative approach of throwing away of higher–order terms in α_S. Thus while we know that α_S becomes large at low scale, we cannot predict its behaviour using the above methods. Indeed, the fact that quarks and gluons can only exist in colour–singlet hadrons, known as *confinement* has still yet to be proved; in fact, doing so is one of the millennium prize problems.

in Fig. 30, taken from the Particle Data Group (PDG). The decrease in the coupling as the scale increases is clear, and is found to be exactly in line with the expectations from QCD.

17. BRST

BRST (Becchi, Rouet, Stara, Tyutin) symmetry is an additional exact symmetry present in the QCD Lagrangian when ghost fields are added. It guarantees that additional gauge–violating terms, on top of the usual gauge–fixing term, do not appear at higher orders and e.g. spoil the universality of the gauge coupling g. In addition to this, it guarantees that the ghost fields themselves, as well as the unphysical longitudinal polarization states of the gluons, do not propagate as external physical modes.

17.1 Abelian Case

BRST symmetry can be demonstrated most simply by considering the Abelian limit of the Faddeev–Popov Lagrangian with complex scalar matter field ϕ

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi^*)(D^\mu\phi) - m^2\phi^*\phi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \overline{c}\partial^2 c\,. \tag{17.1}$$

Due to the gauge fixing term this is no longer invariant under a general gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\alpha\,, \tag{17.2}$$

but there is still a residual symmetry under this transformation if $\alpha(x)$ satisfies $\partial^2\alpha = 0$.

Instead of the gauge transformation parameter $\alpha(x)$, let us take $\alpha(x) = \theta c(x)$ for some Grassmann number θ. Then consider

$$\phi(x) \to e^{-ie\theta c(x)}\phi(x) = \phi(x) - ie\theta c(x)\phi(x)\,, \tag{17.3}$$

$$A_\mu(x) \to A_\mu(x) + \theta\partial_\mu c(x)\,. \tag{17.4}$$

As these are of the same form as a local gauge transformation, the first three terms in (17.1), which are manifestly gauge invariant, are also invariant under this. Now, the equations of motion for the ghost fields are

$$\partial^2 c = \partial^2\overline{c} = 0\,, \tag{17.5}$$

and thus provided we can use the equations of motion for the ghost fields, the gauge fixing term is also invariant, and hence the entire Lagrangian. If we do not use these, then for the gauge fixing term we have

$$(\partial_\mu A^\mu)^2 \to (\partial_\mu A^\mu)^2 + 2(\partial_\mu A^\mu)(\theta\partial^2 c) \ , \tag{17.6}$$

where we have dropped a term $\sim \theta^2 = 0$ due to the Grassmannian nature of θ. Clearly this is invariant if the equations of motion are satisfied, as we expect. However, even if they are not, comparing with the final term of (17.1) we can see that the Lagrangian will still be invariant if we require that the field $\overline{c}$ transforms as

$$\overline{c}(x) \to \overline{c}(x) - \frac{1}{\xi}\theta\partial_\mu A^\mu(x) \ . \tag{17.7}$$

This, combined with (17.3) and (17.4), are known as a *BRST transformation*, and the Lagrangian (17.1) is said to be BRST invariant. We can see that it is essentially a generalization of gauge invariance that still holds despite the presence of the gauge fixing term.

17.2 Non–Abelian Case

The Lagrangian is now

$$\mathcal{L} = \mathcal{L}[A^a_\mu, \psi_i] - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + (\partial_\mu \overline{c}^a)(D_\mu c^a) \ , \tag{17.8}$$

where

$$D_\mu c^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c \ . \tag{17.9}$$

We can then proceed in a similar way to the abelian case, with

$$\psi_i(x) \to \psi_i(x) + ig\theta c^a T^a_{ij}\psi_j \ , \tag{17.10}$$

$$A^a_\mu \to A^a_\mu + \theta D_\mu c^a \ , \tag{17.11}$$

$$\overline{c}^a \to \overline{c}^a - \frac{1}{\xi}\theta\partial^\mu A^a_\mu \ , \tag{17.12}$$

under which the first term is manifestly invariant, as this corresponds to a gauge transformation with $\theta^a(x) = \theta c^a(x)$. The transformation of $\overline{c}^a$ is then designed to exactly cancel the transformation of the gauge fixing term, as in the Abelian case. However, unlike in the Abelian case the $D_\mu c^a$ is not invariant by default, because of the presence of the A^a_μ field in the covariant derivative. We have

$$D_\mu c^a \to D_\mu c^a + g\theta f^{abc}(D_\mu c^b)c^c \ . \tag{17.13}$$

After a few lines of algebra it can be shown that the non–trivial BRST transformation of c^a via

$$c^a \to c^a - \frac{1}{2} g\theta f^{abc} c^b c^c \;, \tag{17.14}$$

will cancel this additional term and leave $D_\mu c^a$ invariant. To see this, we have

$$D_\mu c^a \to D_\mu c^a + g\theta f^{abc}(D_\mu c^b)c^c - \frac{1}{2} g\theta f^{abc}\left[(\partial_\mu c^b)c^c + c^b(\partial_\mu c^c) + g f^{cde} A^b_\mu c^d c^e\right] \;,$$

where the second term is already invariant, as shifting either c field leads to terms $\sim \theta^2 = 0$. The first two terms in the brackets are equal, as $(\partial_\mu c^b)c^c = -c^c(\partial_\mu c^b)$ and $f^{abc} = -f^{acb}$. For the third term, we have

$$\begin{aligned} f^{abc} f^{cde} A^b_\mu c^d c^e &= -f^{bdc} f^{cae} A^b_\mu c^d c^e - f^{dac} f^{cbe} A^b_\mu c^d c^e \;, \\ &= 2 f^{abc} f^{bed} A^e_\mu c^d c^c \;, \end{aligned} \tag{17.15}$$

where in the first line we have used the *Jacobi identity*

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \;, \tag{17.16}$$

which automatically follows from the complete antisymmetry of the structure constants. In the second we have used the antisymmetry of f^{abc} and some index relabelling. Thus

$$\begin{aligned} D_\mu c^a \to D_\mu c^a &+ g\theta f^{abc}(D_\mu c^b)c^c - g\theta f^{abc}\left[(\partial_\mu c^b)c^c + g f^{bed} A^e_\mu c^d c^c\right] \;, \\ &= D_\mu c^a + g\theta f^{abc}(D_\mu c^b)c^c - g\theta f^{abc}(D_\mu c^b)c^c = D_\mu c^a \;, \end{aligned} \tag{17.17}$$

as required.

Now, we may object that the transformation of c and $\overline{c}$ are inconsistent with these being related by complex conjugation, and indeed it is. However, there is in fact no requirement at all to identify the fields in this way. We simply needed a path integral over two Grasmmann valued fields in order to calculate the Faddeev–Popov determinant, and the logic follows through irrespective of whether these are related in this way or not. Indeed they do not even have to be complex. We are therefore free to give them completely distinct transformation properties.

17.3 BRST operator

n.b. This section is non–examinable

Writing the BRST transformations as $\psi_i \to \psi_i + \theta\delta_B\psi_i$, $A^a_\mu \to A^a_\mu + \theta\delta_B A^a_\mu$ etc, we have

$$\delta_B A_\mu^a(x) = D_\mu^{ab} c^b(x) = \partial_\mu c^a(x) - g f^{abc} A_\mu^c(x) c^b(x) \ , \tag{17.18}$$

$$\delta_B \psi_i(x) = i g c^a(x) (T_R^a)_{ij} \psi_j(x) \ , \tag{17.19}$$

$$\delta_B c^c(x) = -\frac{1}{2} g f^{abc} c^a(x) c^b(x) \ , \tag{17.20}$$

$$\delta_B \overline{c}^a(x) = -\frac{1}{\xi} \partial^\mu A_\mu^a \ . \tag{17.21}$$

More generally, one can consider the action of this symmetry by introducing a further field $B^a(x)$, which is scalar and not Grassmann valued. In this case the Lagrangian (17.8) becomes

$$\mathcal{L} = \mathcal{L}[A_\mu^a, \psi_i] + \frac{\xi}{2}(B^a)^2 + B^a \partial^\mu A_\mu^a + (\partial_\mu \overline{c}^a)(D_\mu c^a) \ . \tag{17.22}$$

This is required to be invariant under a BRST transformation, while the transformation of the anti–ghost field is defined in terms of this, with

$$\delta_B B^a(x) = 0 \ , \tag{17.23}$$

$$\delta_B \overline{c}^a(x) = B^a(x) \ . \tag{17.24}$$

As this has a quadratic term without derivatives, it is not interpreted as a standard propagating field. Indeed the functional integral over B^a can be performed by completing the square in (17.22), which is equivalent to substituting for B^a in terms of the classical equations of motion that follow from this Lagrangian:

$$B^a(x) = -\frac{1}{\xi} \partial^\mu A_\mu^a \ . \tag{17.25}$$

This then gives us back (17.8) as before.

One then finds that the BRST operator has the property that $\delta_B^2 = 0$, i.e. it is *nilpotent*. This can be explicitly shown by acting with δ_B on any of the above transformations, which will give 0; this happens automatically when written in terms of the auxillary field B^a, while taking (17.21) one has to assume the equations of motion for $\overline{c}$ are satisfied. One consequence of this is that we are free to add any term that is the BRST variation of some object O to the Lagrangian

$$\mathcal{L} = \mathcal{L}_{YM} + \delta_B O \ , \tag{17.26}$$

and this will still be BRST invariant (remember $\mathcal{L}_{YM}$ is automatically invariant, as in this case a BRST transformation simply corresponds to a gauge transformation). With a suitable choice of O this can in fact be used to recover the gauge fixing and ghost contributions to the Lagrangian we found before, via an alternative route (see Srednicki chapter 74 for details, using a slightly different notation).

The symmetries of the action are (1) Lorentz invariance; (2) CPT; (3) global gauge invariance; (4) BRST invariance; (5) ghost number conservation; (6) anti–ghost translation invariance. Ghost number conservation corresponds to assigning a ghost number $+1$ to c^a and -1 to $\overline{c}^a$ and demanding that every term in $\mathcal{L}$ has ghost number zero. Anti–ghost translation invariance corresponds to $\overline{c}^a(x) \to \overline{c}^a(x)+\chi$, where χ is a Grassmann constant. This follows because the Lagrangian (17.8) only contains derivatives in $\overline{c}^a$.

Now, this Lagrangian (17.8) in fact includes all terms consistent with these symmetries that have coefficients with positive or zero mass dimension. Now BRST symmetry requires that g renormalize in the same way at each of its appearances, and as loop corrections should respect the symmetry of the Lagrangian, this guarantees that the various renormalization parameters are related as we required above.

17.4 Physical states

n.b. This section is non–examinable

Regarding the BRST transformation as infinitesimal, we can construct a Noether current via the standard formula

$$j^\mu_B(x) = \sum_I \frac{\partial \mathcal{L}}{\partial(\partial\Phi_I(x))}\delta_B\Phi_I(x) \; , \tag{17.27}$$

where $\Phi_I(x)$ stands for all fields, including matter (scalar and/or spinor), gauge and ghost. We can then define a BRST charge

$$Q_B = \int \mathrm{d}^3x j^0_B(x) \; . \tag{17.28}$$

In a QFT this charge is an operator, as the fields themselves which enter (17.27) are operators. By considering the (anti) commutation relations between the canonically conjugate field variables, it is possible to show that

$$[Q_B, \Phi_I]_\pm = -i\delta_B\Phi_I \; , \tag{17.29}$$

where the $\pm$ indicates a commutator of anti–commutator, depending on whether the field is scalar or fermionic. We write this for the explicit case of BRST, but this holds for any general charge. Thus Q_B generates a BRST transformation, with

$$i[Q_B, A_\mu(x)] = D_\mu^{ab} c^b(x) \ , \tag{17.30}$$
$$i\{Q_B, c^a(x)\} = -\frac{1}{2} g f^{abc} c^b(x) c^c(x) \ , \tag{17.31}$$
$$i\{Q_B, \overline{c}^a\} = -\frac{1}{\xi} \partial^\mu A_\mu^a \ , \tag{17.32}$$
$$i[Q_B, \phi_x(x)]_\pm = igc^a(x)(T_R^a)_{ij}\phi_j(x) \ , \tag{17.33}$$

where in the last line ϕ is a scalar or fermion transforming in the representation R. As the BRST transformation of a BRST transformation is zero, Q_B is nilpotent

$$Q_B^2 = 0 \ . \tag{17.34}$$

Now, consider the action of Q_B on some state $|\psi\rangle$, then if

$$|\psi\rangle = Q_B|\phi\rangle \tag{17.35}$$

for some state $|\phi\rangle$, we automatically have

$$Q_B|\psi\rangle = 0 \ . \tag{17.36}$$

In this case, the state $|\psi\rangle$ is said to be in the *image* of Q_B. If (17.36) is satisfied, but

$$|\psi\rangle \neq Q_B|\phi\rangle \tag{17.37}$$

then this state is said to belong to the *cohomology* of Q_B. In both cases these are annihilated by Q_B and are said to be in the *kernel* of Q_B. A third option is that the state is not in the kernal, and is not annihilated by Q_B. Note that any state that is in the image of Q_B has zero norm, as

$$\langle\psi|\psi\rangle = \langle\phi|Q_B^2|\phi\rangle = 0 \ . \tag{17.38}$$

and so we are led to identify such states as unphysical. Moreover, as the Lagrangian is BRST invariant, the Hamiltonian H that we derive from it must commute with the BRST charge

$$[H, Q_B] = 0 \ , \tag{17.39}$$

and thus a state which is annihilated by Q_B at earlier times will be annihilated at later times, as $Q_B e^{-iHt}|\psi\rangle = e^{-iHt}Q_B|\psi\rangle$. Furthermore, as unitary time evolution does not change the norm of a state, any state in the cohomology will remain so at later times. We therefore claim that all physical states of a theory correspond to the cohomology of Q_B.

Consider an initial state of widely separated wave packets of incoming particles, with the usual expansions

$$A^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3k}{(2\pi)^3 k^0} \left[\epsilon_\lambda^{\mu *}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \epsilon_\lambda^\mu(\mathbf{k}) a_\lambda^\dagger(\mathbf{k}) e^{-ikx}\right] , \tag{17.40}$$

$$c(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 k^0} \left[c(\mathbf{k}) e^{ikx} + c^\dagger(\mathbf{k}) e^{-ikx}\right] , \tag{17.41}$$

$$\overline{c}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 k^0} \left[b(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}\right] , \tag{17.42}$$

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 k^0} \left[a_\phi(\mathbf{k}) e^{ikx} + a_\phi^\dagger(\mathbf{k}) e^{-ikx}\right] , \tag{17.43}$$

where for the gauge field we sum over all four polarization states. For $k = \omega(1,0,0,1)$ these are given by

$$\epsilon_>^\mu = \frac{1}{\sqrt{2}}(1,0,0,1) , \tag{17.44}$$

$$\epsilon_<^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1) , \tag{17.45}$$

$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0) , \tag{17.46}$$

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,i,0) . \tag{17.47}$$

The first two correspond to the unphysical longitudinal polarization states, while the last two are the usual physical transverse ones.

Setting $g = 0$, as we are interested in the asymptotic states, and substituting (17.40)–(17.43) into (17.30)–(17.33) and matching coefficients of e^{-ikx}, we find

$$[Q_B, a_\lambda^\dagger(\mathbf{k})] = \sqrt{2}\omega\delta_{\lambda >} c^\dagger(\mathbf{k}) , \tag{17.48}$$

$$\{Q_B, c^\dagger(\mathbf{k})\} = 0 , \tag{17.49}$$

$$\{Q_B, b^\dagger(\mathbf{k})\} = \frac{1}{\xi}\sqrt{2}\omega a_<^\dagger(\mathbf{k}) , \tag{17.50}$$

$$[Q_B, a_\phi^\dagger(\mathbf{k})] = 0 . \tag{17.51}$$

Now if we start with some normalized state $|\psi\rangle$ that is in the cohomology:

$$\langle\psi|\psi\rangle = 1 \;, \qquad Q_B|\psi\rangle = 0 \;, \tag{17.52}$$

then (17.48) gives

$$Q_B a^\dagger_>(\mathbf{k})|\psi\rangle \propto c^\dagger(\mathbf{k})|\psi\rangle \;, \tag{17.53}$$

and thus if we create a photon with polarization state $>$ by acting with $a^\dagger_>$ then this state is not annihilated by Q_B and is therefore not in the cohomology. In addition, if we create a ghost field by acting with $c^\dagger$, then this is proportional to Q_B acting on a state, and is also not in the cohomology. In a similar way (17.50) gives

$$Q_B b^\dagger(\mathbf{k})|\psi\rangle \propto a^\dagger_<(\mathbf{k})|\psi\rangle \;, \tag{17.54}$$

demonstrating that the $<$ polarization state and the anti–ghost fields are also not in the cohomology.

On the other hand, if we create physical polarization states by acting with $a^\dagger_\pm$, then (17.48) gives that

$$Q_B a^\dagger_\pm(\mathbf{k})|\psi\rangle = 0 \tag{17.55}$$

and so these are annihilated by Q_B. As these cannot be written as Q_B acting on some other state, these are in the cohomology. A similar result holds for the matter fields.

Thus we can build an initial state of widely separated particles in the cohomology only if we exclude (massless) gauge boson fields with unphysical polarization states, and any ghost fields. As a state in the cohomology cannot evolve to a state not in the cohomology, these can also not be produced in the scattering process.

18. Spontaneous Symmetry Breaking

In this section we discuss the concept of spontaneous symmetry breaking. In a nutshell, this is the idea that while the physics describing a system may possess some symmetry, the ground state of this system may not itself be invariant under this symmetry. In QFT language, while the Lagrangian itself may possess some symmetry, the vacuum state may not. Though we will focus on particle physics here, this possibility is widespread in other areas, for example Bose–Einstein condensates, superfluids, superconductors, crystals and ferromagnetic materials all exhibit this phenomenon. In the latter case, within the so–called Ising model the atoms interact through a spin–spin interaction, with corresponding Hamiltonian (in the absence of any external magnetic field):

$$H = -\sum_{i,j} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j \;, \tag{18.1}$$

where $J_{ij} > 0$. This is itself invariant under rotations, as we might expect, i.e. there is no preferred direction with respect to which the atomic spins might align. However we can clearly see that the minimum energy state for the above expression is achieved by having all spins aligned with each other, maximising the dot product of spin vectors. Yet we have just stated that within our theory there is no preferred direction along which these might align. So what happens in reality? Well, if one takes such a material and lowers the temperature, then at some point the spins will all align into exactly such a ground state, with the overall direction of this being random. This ground state is no longer rotationally invariant, and we talk of the original symmetry being spontaneously broken.

The reason this is a particularly attractive idea from the point of view of particle physics is because it has been known for a long time that the force carrying particles of the weak nuclear force, the W and Z bosons, are massive. The latest experimental values are:

$$M_Z = (91.1876 \pm 0.0021)\,\text{GeV} \;, \qquad M_W = (80.385 \pm 0.015)\,\text{GeV} \;. \tag{18.2}$$

Given the success in describing QED and QCD as gauge theories, it is tempting to try the same for the weak interaction. However we immediately run into a problem, as we recall from (5.34) that a photon mass term in our Lagrangian for the case of e.g. QED:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} m_A^2 A_\mu A^\mu \;, \tag{18.3}$$

is definitely not gauge invariant. For more complicated non-abelian symmetries the result is the same, namely we cannot explicitly write down a mass term for the gauge bosons of our theory that respects gauge invariance. One might still be tempted to disregard the requirement of gauge symmetry, and put in some sort of mass term by hand. However it turns out that if we try this the resulting theory exhibits all sorts of bad behaviour: particular scattering cross sections become arbitrarily large at high energies, breaking the conservation of probability, and moreover the theory itself is not renormalizable.

Given this, it is natural to ask whether SSB might come to the rescue, i.e. might it be possible to start off with a gauge theory of the weak interaction, but for which the ground state does not preserve the underlying gauge symmetry, and hence it might be permitted to have massive gauge bosons. We can then hope that the underlying (albeit broken) gauge symmetry preserves the nice features we see in QED and QCD, i.e. well behaved scattering cross sections and renormalizability, while still allowing for massive gauge bosons. As we will see, this is indeed possible, and is (as far as we know today) precisely what is realised in our gauge theory for the electroweak interaction, as it appears in the Standard Model.

18.1 Global Symmetries – Abelian Case

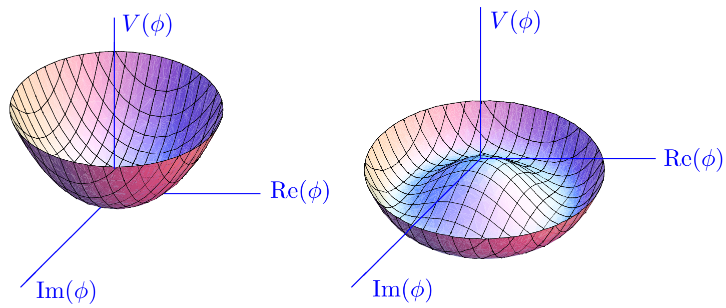

Figure 31: Potential for complex scalar field with positive (negative) mass term shown in the left (right) figure.

To introduce the basic concept, we first consider a theory which is invariant under a global symmetry, i.e. not a gauge theory. Consider in particular the Lagrangian for one complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{1}{4}\lambda(\phi^*\phi)^2 \, , \tag{18.4}$$

which is invariant under the global $U(1)$ transformation

$$\phi(x) \to e^{-i\alpha}\phi(x) \, , \tag{18.5}$$

where we absorb the charge labelling q into α for simplicity. For $m^2 > 0$ the second piece in (18.4) will enter the equations of motion for ϕ and ϕ^* in the usual way, and we interpret m as the mass of the particles ϕ (and antiparticle ϕ^*) in the quantum theory. Now, what happens if we instead take $m^2 < 0$? We can certainly no longer interpret this as a mass term in any straightforward sense. This corresponds to a so–called tachyonic field configuration, for which the usual relativistic relation $\gamma^2 = E^2/m^2$ tells us that a negative m^2 corresponds to $v > c$, i.e. faster than light propagation. This is certainly not something we would like to be present in our theory.

Fortunately we do not have deal with such a field configuration, as in this case the potential term in the Lagrangian has the form shown in Fig. 31, with

$$V(\phi) = m^2\phi^*\phi + \frac{1}{4}\lambda(\phi^*\phi)^2 \,, \tag{18.6}$$

which has a stable minimum for the non–zero constant field configuration

$$\phi(x) = \frac{1}{\sqrt{2}}ve^{-i\theta} \,. \tag{18.7}$$

where

$$v = \left(\frac{4|m^2|}{\lambda}\right)^{1/2} \,, \tag{18.8}$$

and the phase θ is arbitrary. This corresponds to the minimum energy *ground state* configuration, i.e. which minimises the Hamiltonian corresponding to (18.4). Thus in this ground state configuration, the field has a constant and non–zero value. This is in contrast to the positive m^2 case, where the stable minimum simply occurs at $\phi = 0$, i.e. with the field turned off, as one might more naturally expect. In the $m^2 < 0$ case there is also a minimum here, which would correspond to the tachyonic field configuration discussed above, however as this is unstable this will not correspond to a (stable) ground state.

Under $U(1)$ transformations θ changes to $\theta + \alpha$, and thus different minimum energy configurations are related by this symmetry. In the quantum theory, there is a continuous family of ground states, labelled by θ, with

$$\langle\theta|\phi(x)|\theta\rangle = \frac{1}{\sqrt{2}}ve^{-i\theta} \,, \tag{18.9}$$

and $\langle\theta|\theta'\rangle = 0$. That is, the non–zero value of the ground–state field configuration in the classical case corresponds to a non–zero vacuum expectation value, with magnitude v, in the quantum case.

In the classical language, there is a flat direction in field space along which we can move without changing energy. The physical consequence of this is the existence of a massless particle known as a *Goldstone boson* in the spontaneously broken theory. To see this, we first choose our phase $\theta = 0$, which we are completely free to do due to the global $U(1)$ symmetry of the Lagrangian; this corresponds to a coordinate choice, if you like. We then parameterise our field as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \,, \tag{18.10}$$

where χ and ρ are real scalar fields. Physically, this corresponds to expanding around the vacuum, i.e.

$$\langle 0|\phi(x)|0\rangle = \frac{1}{\sqrt{2}}v \;, \tag{18.11}$$

implies that

$$\langle 0|\rho(x)|0\rangle = \langle 0|\chi(x)|0\rangle = 0 \;. \tag{18.12}$$

Substituting this into (18.4) and dropping an overall constant $\sim v^4\lambda/16$, which doesn't contribute to the physics, we find

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(1+\frac{\rho}{v}\right)^2 \partial_\mu\chi\partial^\mu\chi \\ &+\frac{1}{2}\partial_\mu\rho\partial^\mu\rho - |m^2|\rho^2 - \frac{1}{4}\lambda v\rho^3 - \frac{1}{16}\lambda\rho^4 \;,\end{aligned} \tag{18.13}$$

and thus by expanding around the minimum in this way we can see that the two degrees of freedom of our original complex scalar field in the unbroken theory behave quite differently. We in particular have one real ρ field with mass $m_\rho^2 = 2|m^2|$ and zero vacuum expectation value (vev), and a massless field χ. This second field does not appear in the potential at all, and thus parameterises the flat direction. The $U(1)$ transformation of the field ϕ simply corresponds to $\chi(x) \to \chi(x) + v\alpha$. However, we can see that the fields χ and ρ which now describe our Lagrangian no longer have a $U(1)$ symmetry. We therefore say that this symmetry is *spontaneously broken* by the choice of ground state; the theory has undergone spontaneous symmetry breaking (SSB).

Note also that both of these physical fields are perfectly well–behaved, i.e. the apparently tachyonic behaviour implied by the original Lagrangian is absent once we expand about the correct ground state. We can also see that a quartic and a new cubic interaction term between the ρ fields are present in this physical Lagrangian. That is, as well as changing the mass spectrum of our theory, SSB introduces new interactions between the fields. Both of these properties will be present in the more general case of gauge symmetry breaking we will consider below, albeit in a slightly different form.

Now this may seem like mathematical trickery, as all we have done is rewrite our original Lagrangian via a simple substitution (18.10), which defines ρ and χ in terms of ϕ. How can this have introduced any genuine physical results? The crucial point is (18.12), i.e. that the vacuum expectation values of these new fields is 0. We expect our physical degrees of freedom to be turned off in the vacuum, and when we scatter particles we excite this vacuum. Thus it is indeed the new fields ρ and χ, and *not* the field ϕ that represent the physical states in the system.

Does χ remain massless in the full quantum theory, i.e. after including quantum corrections? If this is the case, its exact propagator $\tilde{\Delta}_\chi^{\rm exact}(k^2)$ should have a pole at $k^2 = 0$, that is the self–energy should satisfy $\Pi_\chi(k^2 = 0) = 0$. The diagrams that generate these corrections will come from summing all IPI diagrams with two external χ lines. The $\rho\rho\chi\chi$ and $\rho\chi\chi$ vertices which will generate these are, due to the derivative acting on the χ fields in these terms in the Lagrangian, proportional to $k_1 \cdot k_2$, where $k_{1,2}$ are the momenta of the two χ lines at the vertex. Thus as the external lines have zero momenta, the attached vertices vanish, and we do indeed have $\Pi_\chi(k^2 = 0) = 0$, as required.

18.2 Global Symmetries – Non–Abelian Case

The above results can be generalised straightforwardly to the non–abelian case. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i - \frac{1}{16}\lambda(\phi_i\phi_i)^2 \ , \tag{18.14}$$

for a set of N real scalar fields, and where repeated indices are summed over as usual. This is invariant under the $SO(N)$ transformation

$$\phi_i \to O_{ij}\phi_j \ , \tag{18.15}$$

where the real transformation matrices satisfy $O^T = O^{-1}$ and $\det(O) = 1$. As usual, we can write the transformation matrix in the infinitesimal form

$$O_{ij} = \delta_{ij} + i\theta^a(T^a)_{ij} \ , \tag{18.16}$$

in terms of a basis of generator matrices T^a and real infinitesimal parameters θ_a. The orthogonality condition for the real matrices O implies that the T^a must be imaginary and anti–symmetric, for which there are $\frac{1}{2}N(N-1)$ independent matrices. A convenient choice for these generators is to take for each T^a that it has one single non–zero entry $-i$ above the main diagonal, and a corresponding $+i$ below, e.g.

$$T^1 = \begin{pmatrix} 0 & -i & 0 & 0 & \\ i & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ & & & & \ldots \end{pmatrix} \ , \tag{18.17}$$

and so on. Taking as before $m^2 < 0$ in the Lagrangian, the minimum of the potential is now achieved for

$$\frac{\partial V}{\partial \phi_j} = 0 \Rightarrow \phi_i \phi_i = \frac{4|m^2|}{\lambda} \,, \tag{18.18}$$

which is satisfied for $\phi_i = v_i$ with

$$v^2 = v_i v_i = \frac{4|m^2|}{\lambda} \,, \tag{18.19}$$

and the direction in which this N–component vector $\vec{v}$ points is arbitrary. This v_i is the vev of the fields ϕ_i. We are free to choose our coordinate system so that

$$v_i = v\delta_{iN} \,, \tag{18.20}$$

and the vev lies entirely in the last component. This will simplify the considerations below, but the physics will not depend on this choice. Now, consider the impact on the vev of making an infinitesimal $SO(N)$ transformation. This gives

$$v_i \to v_i + i\theta^a (T^a)_{ij} v_j = v\delta_{iN} + i\theta^a (T^a)_{iN} v \,. \tag{18.21}$$

Crucially for many values of a the second term on the right hand side of this expression vanishes. The only cases where this is non–vanishing is where the corresponding T^a has a non–zero entry in the last column. There are $N-1$ of these, corresponding to the case of an $-i$ in the first row and last column, second row and last column etc up to the $(N-1)$th row. An $SO(N)$ transformation involving these changes the vev of the fields, but not the energy. Thus this corresponds to a flat direction in field space, in a direct generalization of the $U(1)$ case discussed before. These are known as *broken* generators, while those for which the contribution from the T^a vanishes are known as *unbroken generators*, as these act on a subspace that is orthogonal to the vev direction, and are thus unaffected by the SSB. Considering for concreteness the case of $SO(3)$, with a vev chosen to lie along the z direction, the unbroken case corresponds to a rotation in the $x-y$ plane, with the remaining two independent rotations being broken.

In the above case we therefore have

$$(T^a)_{ij} v_j = 0 \Rightarrow \text{Unbroken} \,, \tag{18.22}$$

$$(T^a)_{ij} v_j \neq 0 \Rightarrow \text{Broken} \,, \tag{18.23}$$

although some care is needed in interpreting this for a general coordinate choice; we will comment on this below.

As the unbroken generators do not change the vev of the field, after rewriting the Lagrangian in terms of the shifted fields (each with zero vev), there should still be a manifest symmetry corresponding to the number of unbroken generators. In the present case the number of these is

$$\frac{1}{2}N(N-1) - (N-1) = \frac{1}{2}(N-1)(N-2) \ , \tag{18.24}$$

which is the number of generators for a $SO(N-1)$ symmetry. Thus, we expect our Lagrangian, when it is rewritten in terms of shifted fields, to still have a $SO(N-1)$ symmetry.

To see that this is indeed the case, it simplifies things if we note that

$$V(\phi) = \frac{1}{2}m^2\phi_i\phi_i + \frac{1}{16}\lambda(\phi_i\phi_i)^2 = \frac{\lambda}{16}(\phi_i\phi_i - v^2)^2 - \frac{\lambda v^4}{16} \ , \tag{18.25}$$

and then drop the last constant term, which plays no role in the physics. We can thus rewrite our Lagrangian as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{16}\lambda(\phi_i\phi_i - v^2)^2 \ , \tag{18.26}$$

where the repeated index i sums from 1 to N. Now, we take our coordinate choice (18.20), which allows us to simply expand

$$\phi_N(x) = v + \rho(x) \ , \tag{18.27}$$

with the other field components unchanged. Plugging this into (18.26) we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - V(\rho,\phi) \ , \tag{18.28}$$

where i now sums from 1 to $N-1$, and

$$\begin{aligned} V(\rho,\phi) &= \frac{1}{16}\lambda[(v+\rho)^2 + \phi_i\phi_i - v^2]^2 \ , \\ &= \frac{1}{16}\lambda(2v\rho + \rho^2 + \phi_i\phi_i)^2 \ , \\ &= \frac{1}{4}\lambda v^2\rho^2 + \frac{1}{4}\lambda v\rho(\rho^2 + \phi_i\phi_i) + \frac{1}{16}\lambda(\rho^2 + \phi_i\phi_i)^2 \ , \end{aligned} \tag{18.29}$$

which indeed has a manifest $SO(N-1)$ symmetry in the fields $\phi_{i=1\ldots N-1}$.

The first term in the last line above corresponds to the ρ mass, with $m_\rho^2 = \lambda v^2/2$.

The remaining terms correspond to new interactions between the ρ and ϕ fields: we have in particular cubic $\rho\rho\rho$, $\rho\phi\phi$ as well as the usual quartic ρ^4, $\rho\rho\phi\phi$ and ϕ^4 interactions. Crucially, we can see that there is no mass term for the fields ϕ_i, and thus we now have $N-1$ massless Goldstone bosons, which corresponds to the number of broken generators. This is no accident, but is rather a specific example of the more general *Goldstone's theorem*, which states that for each spontaneously broken generator there will exist a massless particle. We now consider this theorem in a little more detail.

18.3 Goldstone's Theorem

This states that whenever a continuous symmetry of the Lagrangian is spontaneously broken, massless 'Goldstone bosons' emerge, with one present for each broken generator of the symmetry. To prove this, we note that the potential before spontaneous symmetry breaking will be invariant under the relevant global symmetry transformation. Considering the infinitesimal case

$$\phi_i \to \phi_i + \delta\phi_i = \phi_i + i(\theta^a)T^a_{ik}\phi_k \; . \tag{18.30}$$

where $\delta\phi_i$ denotes an arbitrary infinitesimal group transformation, which we can also expand in terms of the generators T^a. Keeping it general for now, we required that

$$V(\phi_i + \delta\phi_i) = V(\phi_i) \; . \tag{18.31}$$

This implies that

$$\frac{\partial V}{\partial \phi_j}\delta\phi_j = 0 \; . \tag{18.32}$$

Differentiating with respect to ϕ_i we have

$$\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\delta\phi_j + \frac{\partial V}{\partial\phi_j}\frac{\partial(\delta\phi_j)}{\partial\phi_i} = 0 \; , \tag{18.33}$$

where we can see from (18.30) that the $\delta\phi_j$ is a function of the fields and hence the second term is in general non–zero. Now, after SSB we expand in terms of

$$\phi_i(x) = v_i + \chi_i(x) \; . \tag{18.34}$$

Evaluating (18.33) at the minimum $\phi_i = v_i$ the second term by definition vanishes and we have

$$\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\phi_i=v_i}\delta\phi_j \equiv (M^2_\chi)_{ij}\delta\phi_j = 0 \; , \tag{18.35}$$

where the symmetric matrix $(M^2)_{ij}$ has eigenvalues corresponding to the squared masses of the fields. Now, if the transformation (18.30) leaves the vev v_i invariant, we will have $\delta\phi_j = 0$ and hence this is satisfied automatically. This corresponds precisely to the subspace that is orthogonal to the vev direction discussed in the previous section, that is the unbroken $SO(N-1)$ symmetry transformations that leave the vev untouched. On the other hand if the transformation does change the vev direction (i.e. corresponds to the broken symmetry direction), then $\delta\phi_j \neq 0$, and hence this corresponds to an eigenvector of this mass matrix with zero eigenvalue, i.e. precisely a massless Goldstone boson, with one occurring for each broken symmetry direction.

To be concrete, we can return to the choice of vev (18.20) and basis of generators (18.17) considered in the previous section. Writing our transformation in terms of the generators (18.35) becomes

$$(M^2_\chi)_{ij}(T^a v)_j = 0 \; . \tag{18.36}$$

We can therefore immediately see that the $N-1$ generators for which $T^a v \neq 0$, and which we identified as the broken generators, are precisely those which require a zero eigenvector of the mass matrix, and hence a massless Goldstone boson for each of these broken generators. On the other hand, some care is needed in making use of such a result for a general choice of vev and/or representation of the generator matrices. In particular, Goldstone's theorem tell us that we should identify the remaining symmetry that occurs for the Lagrangian, once it has has been rewritten in term of the shifted fields, i.e. after SSB. In the current case, irrespective of the choice of vev direction (and generator representation), this is $SO(N-1)$. We therefore identify from this that there are

$$\frac{1}{2}(N-1)(N-2) \tag{18.37}$$

unbroken generators. To identify the number of broken generators, and hence massless Goldstone bosons, we should identify the overall symmetry breaking pattern

$$SO(N) \to SO(N-1) \; , \tag{18.38}$$

where the LHS corresponds to the original symmetry of the Lagrangian, and the RHS to the symmetry after SSB. Then the number of broken generators is given by

$$\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1 \; , \tag{18.39}$$

as found in the explicit example.

In this example, we for demonstration instead identified the number of broken generators by directly confirming whether $T^a v \neq 0$ was satisfied, in order to motivate the underlying result. However, one cannot in general proceed in this way. In particular, in general we can have in (18.36) that $T^a v \neq 0$ is satisfied for more than $N-1$ generators. While this might suggest there are more than $N-1$ Goldstone bosons, this will not be the case if some of these are not linearly independent. To be concrete, we can consider SO(3), for which our choice of generators is

$$T^1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad T^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \qquad T^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} . \tag{18.40}$$

Now, if we instead of (18.20) take

$$v = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \tag{18.41}$$

then clearly $T^a v \neq 0$ is satisfied for all 3 (and not $N-1=2$) generators.

What is going on? Well, we can see that

$$T^3 v = T^2 v - T^1 v , \tag{18.42}$$

i.e. only 2 of the 3 apparent requirements (18.36) are in fact independent. Hence, for this choice of vev the mass matrix continues to have 2 and not 3 zero eigenvalues. To be concrete, by explicitly writing down $(M^2_\chi)_{ij}$ in terms of v we find that this has two linearly independent eigenvectors with zero eigenvalue, e.g.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} . \tag{18.43}$$

Therefore, one cannot immediately identify the number of broken generators by simply considering whether $T^a v \neq 0$ is in general true. This procedure in particular worked in the previous example precisely because we chose our vev direction (or equivalently our basis of generators) such that for the $SO(3)$ case the T^1 generator corresponds to a rotation that is orthogonal to the direction of the vev, allowing us to simply read this off as the unbroken generator of the remaining $SO(2)$ symmetry. For the choice above, on the other hand, none of the three generators act in a way that is purely orthogonal to the vev direction and so the result cannot be read off so simply. However, clearly there is an unbroken rotation in $SO(3)$ space that acts orthogonally to v (and indeed any particular choice of it), i.e. that leaves this vev untouched, but which corresponds to some linear combination of the three generators in this basis. That is, the physics and in particular the number of Goldstone bosons remains the same.

In summary, to identify the number of Goldstone bosons one should instead expand around the new vev and determine the symmetry breaking pattern, in order to identify the remaining symmetry of the Lagrangian after SSB. From this, one can determine the number of broken generators and hence Goldstone bosons (see (18.37) to (18.39) and the discussion there).

To clarify this, and the role of M_{ij} further, we note if we write the quadratic term in the potential after SSB as

$$V(\phi_i) = \frac{1}{2}(M_{ij}^2)\chi_i\chi_j + \cdots \tag{18.44}$$

then this precisely corresponds to

$$\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\phi_i=v_i} = \left.\frac{\partial^2 V}{\partial\chi_i\partial\chi_j}\right|_{\chi_i=0} = (M^2)_{ij} \ , \tag{18.45}$$

as claimed. One can in particular define mass eigenstates $\tilde{\rho}$ such that

$$V(\phi_i) = \sum_{i=1}^{N_{\text{eig}}} \frac{1}{2} m_i^2 \tilde{\rho}_i^2 \ , \tag{18.46}$$

where m_i^2 corresponds to eigenvalue i of M_{ij}^2. This corresponds precisely to a mass term for each field $\tilde{\rho}$.

In the example of Section 18.2 we have simply

$$(M^2)_{ij} = \frac{\lambda}{2} v_i v_j \ . \tag{18.47}$$

Which if we consider a basis of vectors formed by v_j/v^2 and $N-1$ vectors n_j^I (with $I = 1, \cdots, N-1$) orthogonal to this, will give

$$(M^2)_{ij}\left(\frac{v_j}{v^2}\right) = \frac{\lambda}{2}v^2\left(\frac{v_i}{v^2}\right) \ , \qquad (M^2)_{ij}n_j^I = 0 \ . \tag{18.48}$$

Thus, this has precisely $N-1$ eigenvectors with zero eigenvalue. The massive state is then given by

$$\tilde{\rho}_i = \frac{\rho_j v_j}{v^2} v_i \ , \tag{18.49}$$

which gives

$$\frac{1}{2}(M^2)_{ij}\tilde{\rho}_i\tilde{\rho}_j = \frac{\lambda}{4}v^2\tilde{\rho}_i\tilde{\rho}_i \ , \tag{18.50}$$

consistent with our result when we took $v_i = \delta_{iN}v$.

18.4 $SU(N)$

In this case the Lagrangian takes the form

$$\mathcal{L} = \partial_\mu\phi_i^*\partial^\mu\phi_i - m^2\phi_i^*\phi_i - \frac{1}{4}\lambda(\phi_i^*\phi_i)^2 \ , \tag{18.51}$$

and minimising the potential gives a ground state for

$$\phi_i^*\phi_i = \frac{2|m^2|}{\lambda} \ . \tag{18.52}$$

Similarly to $SO(N)$, we are free to choose a coordinate system so that

$$v_i = ve^{-i\theta}\delta_{iN} \ , \tag{18.53}$$

where the v_i is now in principle complex, hence the introduction of the phase θ as in the original $U(1)$ example. We can then take a convenient basis for the generators of $SU(N)$, in this case a set of matrices with a factor of $-i$ above the main diagonal and a corresponding i below (as in $SO(N)$), and a set of matrices with a factor of 1 above the main diagonal and a corresponding 1 below. Finally, we in general have $N-1$ linearly independent traceless matrices with elements purely along the diagonal. We can choose these such that we take $N-2$ matrices with a zero in the final, Nth row, Nth column entry (i.e. traceless with respect to the $N-1$ other diagonal elements) and a final generator that has a non-zero value in this entry. In the first two cases there are $N(N-1)/2$ matrices, while in the latter there are $N-1$, giving N^2-1 in total as expected.

In such a case we will have $N-1$ generators for the first two sets of generators for which $(T^a)_{iN}$ is non–zero, and one from the diagonal set of matrices, giving in total $2N-1$ broken generators. Now, when we expand our fields around the vacuum (18.53), we should still have a $SU(N-1)$ symmetry in the first $N-1$ components of ϕ_i. In other words, we have a $SU(N) \to SU(N-1)$ breaking pattern, and counting generators before and after this corresponds to precisely

$$(N^2-1) - ((N-1)^2-1) = 2N-1 \ , \tag{18.54}$$

broken generators, as anticipated by our choice of basis above. We note that, as discussed above this particular choice of generators and vev direction allows us to read off the number of broken and unbroken generators directly, but for other choices this will not be manifest, although the physics (i.e. number of Goldstone bosons) will remain the same.

In fact (18.51) has a larger symmetry than $SU(N)$. Writing the fields as $\phi_i = (\phi_i^R + i\phi_i^I)/\sqrt{2}$ we find

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i^R\partial^\mu\phi_i^R + \frac{1}{2}\partial_\mu\phi_i^I\partial^\mu\phi_i^I - \frac{1}{2}m^2(\phi_i^R\phi_i^R + \phi_i^I\phi_i^I) - \frac{1}{16}\lambda(\phi_i^R\phi_i^R + \phi_i^I\phi_i^I)^2 \ , \quad (18.55)$$

which we can see by comparing to (18.14) corresponds to a $SO(2N)$ symmetry. According to the above logic, after SSB one might expect the Lagrangian to retain a $SO(2N-2)$ symmetry, however in this basis, we can write the potential in a similar form to the $SO(N)$ case, with

$$V(\phi) = \frac{1}{16}\lambda(\phi_i^R\phi_i^R + \phi_i^I\phi_i^I - v^2)^2 \ . \quad (18.56)$$

Now, the vev can be chosen to be aligned with one of the ϕ_R (say), and so in fact the broken potential retains a larger $SO(2N-1)$ symmetry, i.e.

$$SO(2N) \to SO(2N-1) \ . \quad (18.57)$$

Counting generators this gives

$$N(2N-1) - (2N-1)(N-1) = 2N-1 \ , \quad (18.58)$$

broken generators, consistent with the above. We note that, having written things in terms of the $SO(2N)$ symmetry, the same logic as described at the end of Section 18.3 immediately follows. That is, irrespective of the particular choice of vev direction, there are indeed $2N-1$ Goldstone bosons.

18.5 Gauge symmetry breaking – Abelian Case

So far, we have considered the case of spontaneously broken global symmetries. What happens if we consider a local symmetry, and gauge this in the usual way? Consider the Lagrangian for a complex scalar field invariant under a $U(1)$ symmetry

$$\mathcal{L} = (D^\mu\phi)^*(D_\mu\phi) - V(\phi) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \quad (18.59)$$

where $D_\mu = \partial_\mu - igA_\mu$ and

$$V(\phi) = m^2\phi^*\phi + \frac{1}{4}\lambda(\phi^*\phi)^2 \ . \tag{18.60}$$

Minimising the potential as before, we have that classically this has a non–zero vev given by

$$\langle 0|\phi(x)|0\rangle = \frac{1}{\sqrt{2}}v \ , \tag{18.61}$$

where we have made a global $U(1)$ transformation to set the phase of the vev to zero, and

$$v^2 = \left(\frac{4|m^2|}{\lambda}\right) \ . \tag{18.62}$$

We can therefore write, as before,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \ . \tag{18.63}$$

Substituting this into the potential, we find (dropping a constant term as before)

$$V(\phi) = \frac{1}{4}\lambda v^2\rho^2 + \frac{1}{4}\lambda v\rho^3 + \frac{1}{16}\lambda\rho^4 \ . \tag{18.64}$$

As χ does not appear in the potential, it is massless, and it corresponds to the Goldstone boson of the spontaneously broken $U(1)$ symmetry, while as before we have a massive state ρ, with $m_\rho^2 = \frac{1}{2}\lambda v^2$. At this stage, everything very closely resembles the case of a global symmetry, however here we have the big difference that for a gauge symmetry we can make a gauge transformation that shifts the phase of $\phi(x)$ to an arbitrary space time function

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \ . \tag{18.65}$$

We are then free to choose $\alpha(x) = \chi(x)/v$ in order to shift the χ dependence away, setting $\chi(x) = 0$ in (18.63). In other words, as we have

$$\begin{aligned} D_\mu\phi &= (\partial_\mu - igA_\mu)\frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \ , \\ &= \frac{1}{\sqrt{2}}e^{-i\chi(x)/v}\left[\partial_\mu\rho - ig(v+\rho)\left(A_\mu + \frac{1}{g}\frac{\partial_\mu\chi}{v}\right)\right] \ , \end{aligned} \tag{18.66}$$

we can simply transform

$$\phi'(x) = e^{i\chi(x)/v}\phi(x) \ , \qquad A'_\mu = A_\mu + \frac{1}{g}\frac{\partial_\mu \chi}{v} \ , \tag{18.67}$$

This choice is known as the *unitary gauge*. As the Goldstone field χ is no longer present in the gauge transformed Lagrangian, it cannot play the role of a physical field in the theory (in contrast to the global SSB case). Where has this degree of freedom gone? To see this, we expand the kinetic term in this gauge to give

$$\begin{aligned}(D^\mu\phi)^*(D_\mu\phi) &= \frac{1}{2}(\partial^\mu\rho + ig(v+\rho)A^\mu)(\partial_\mu\rho - ig(v+\rho)A_\mu) \ , \\ &= \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}g^2\rho^2 A_\mu A^\mu + g^2 v\rho A_\mu A^\mu + \frac{1}{2}g^2 v^2 A^\mu A_\mu \ .\end{aligned}$$

While the second and third terms corresponds to new interactions between the field ρ and the $U(1)$ gauge boson, this last contribution corresponds to precisely a mass term, with

$$M = gv \ . \tag{18.68}$$

This is the *Higgs Mechanism*. When a gauge symmetry undergoes SSB the Goldstone boson disappears from the theory entirely, and the gauge boson instead acquires a mass. We say that the gauge boson has 'eaten' the Goldstone boson. Note that the number of degrees of freedom is preserved, as the impact of the Goldstone boson is felt through the gauge transformed field in (18.67). In particular, while a massless gauge boson has only two transverse spin states and hence two degrees of freedom, a massive gauge boson has an additional longitudinal spin state, and therefore has three.

18.6 Massive Gauge Bosons

What does the solution for a massive gauge boson look like? In the unitary gauge, the terms in $\mathcal{L}$ that are quadratic in A_μ are

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2 A^\mu A_\mu \ , \tag{18.69}$$

with the gauge boson mass given as in (18.68). The equations of motion that follow from this are

$$\left[(\partial^2 + M^2)g^{\mu\nu} - \partial^\mu\partial^\nu\right] A_\nu = 0 \ . \tag{18.70}$$

Now, at this point in the consideration of the equations of motion for the massless photon field, we had additional gauge symmetries that we could use to further restrict the degrees of freedom of the solution. However, here we have already entirely fixed

A_μ through our choice of unitary gauge fixing (18.67), and so this option is no longer open to us. On the other hand, if we act again with ∂_μ on the above expression we find

$$M^2\partial^\mu A_\mu = 0 \;, \tag{18.71}$$

which does restrict the 4 degrees of freedom of A_μ to 3. Substituting this back into the equations of motion we have

$$(\partial^2 + M^2)A_\nu = 0 \;. \tag{18.72}$$

The general solution to this is

$$A^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left[\epsilon_\lambda^{\mu *}(\mathbf{k})a_\lambda(\mathbf{k})e^{ikx} + \epsilon_\lambda^\mu(\mathbf{k})a_\lambda^*(\mathbf{k})e^{-ikx}\right] \;, \tag{18.73}$$

where $k_0^2 = \mathbf{k}^2 + M^2$ and the sum is over the three independent polarization states which from (18.71) must satisfy

$$k^\mu \epsilon_\mu(k) = 0 \;. \tag{18.74}$$

These correpond to the usual transverse components we met in the case of the photon, and an additional longitudinal component, which is new. In the particle rest frame, with $k = (M, \mathbf{0})$, we can choose the polarization vectors to correspond to a definite spin along the z axis. This gives

$$\begin{aligned}
\epsilon_+(0) &= -\frac{1}{\sqrt{2}}(0,1,i,0) \;, \\
\epsilon_-(0) &= \frac{1}{\sqrt{2}}(0,1,-i,0) \;, \\
\epsilon_0(0) &= (0,0,0,1) \;.
\end{aligned} \tag{18.75}$$

These can be related to other frames by appropriate boosts and/or rotations. For example, for a particle moving along the z axis with momentum $\mathbf{k}$, the transverse vectors are unchanged, and we have

$$\epsilon_0(k) = \frac{1}{M}(|\mathbf{k}|, 0, 0, k_0) \;. \tag{18.76}$$

More generally, the three polarization vectors along with the timelike vector k^μ/M form an orthonormal and complete set, satisfying

$$\begin{aligned} k \cdot \epsilon_\lambda(k) &= 0 \ , \\ \epsilon_{\lambda'} \cdot \epsilon^*_\lambda &= -\delta_{\lambda'\lambda} \ , \\ \sum_\lambda \epsilon^{\mu *}_\lambda(k) \epsilon^\nu_\lambda(k) &= -g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \ . \end{aligned} \tag{18.77}$$

What about the propagator? To calculate this, we follow the logic of Section 6.3. The Lagrangian including the source term now becomes

$$\mathcal{L}_{\rm QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \to \frac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu + M^2 g^{\mu\nu}) A_\nu - J_\mu A^\mu \ . \tag{18.78}$$

Translating to momentum space as before, we have

$$S = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \tilde{A}_\mu(p) \left[p^2 P^{\mu\nu} \right] \tilde{A}_\nu(-p) + \tilde{J}_\mu(p) \tilde{A}^\mu(-p) + \tilde{J}_\mu(-p) \tilde{A}^\mu(p) \ , \tag{18.79}$$

but where the projection matrix is now given by

$$P^{\mu\nu}(p) = \frac{p^2 - M^2}{p^2} g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \ . \tag{18.80}$$

In contrast to the massless photon case, the presence of the mass term means that this is invertible, with

$$(P^{\mu\nu})^{-1} = \frac{p^2}{p^2 - M^2} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{M^2} \right) \ . \tag{18.81}$$

This corresponds to a propagator given by

$$\tilde{\Delta}^{\mu\nu}(p) = \frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{M^2}}{p^2 - M^2} \ . \tag{18.82}$$

18.7 Unitary Gauge Fixing

While the above formulation is sufficient at tree–level, when going to higher orders we have to be a bit more careful. In particular, all of this has been defined in the specific unitary gauge, and we know that in general care must be taken to properly gauge fix the Lagrangian when considering the full quantum path integral. In the unitary gauge we have imposed the gauge condition $\chi(x) = 0$, which corresponds to taking a functional delta function $\Pi_x \delta(\chi(x))$ in the path integral. In order to integrate over χ and impose this condition, we must make a change of variables from Reϕ and Imϕ (or equivalently

ϕ and ϕ^*) to ρ and χ. We have

$$\mathrm{dRe}\phi\, \mathrm{dIm}\phi = \frac{1}{2}\left(1+\frac{\rho}{v}\right) \mathrm{d}\rho\, \mathrm{d}\chi\ , \tag{18.83}$$

and hence

$$\prod_i \mathrm{dRe}\phi_i\, \mathrm{dIm}\phi_i = \prod_i \frac{1}{2}\left(1+\frac{\rho_i}{v}\right) \mathrm{d}\rho_i\, \mathrm{d}\chi_i\ , \tag{18.84}$$

where the product is over $i = 1 \cdots n$ discrete spacetime points, which we will in a moment send to the $n \to \infty$ continuum limit to recover our path integral. For now though we note that

$$\prod_i \left(1+\frac{\rho_i}{v}\right) = \det\left(\left[1+\frac{\rho_i}{v}\right]\delta_{ij}\right) = \int \mathrm{d}^n\chi \mathrm{d}^n\overline{\chi} \exp\left(\overline{\chi}_i\left[1+\frac{\rho_i}{v}\right]\chi_i\right)\ , \tag{18.85}$$

where χ and $\overline{\chi}$ are Grassmann valued, as usual. Then, taking the continuum limit and associating these with ghost fields c and $\overline{c}$ we have

$$\int \mathcal{D}\mathrm{Re}\phi\, \mathcal{D}\mathrm{Im}\phi\, \delta(\chi) \propto \int \mathcal{D}\rho \int \mathcal{D}\overline{c}\mathcal{D}c\, e^{im_{gh}^2 \int \mathrm{d}^4x \overline{c}(1+\rho/v)c} \tag{18.86}$$

where in the last line we have introduced an arbitrary mass parameter m_{gh}, corresponding to a choice of normalization. Thus as in the case of non–Abelian gauge theories, to correctly define the spontaneously broken Abelian theory we are lead to introduce additional unphysical ghost fields. Indeed, these have no kinetic term, and so the propagator is simply given by

$$\tilde{\Delta}(k^2) = \frac{1}{m_{gh}^2}\ . \tag{18.87}$$

In addition, the ghost term has introduced a $\rho\bar{c}c$ vertex, with an associated factor of $-im_{gh}^2/v$; as the number of ghost propagators and vertices will be equal in any Feynman diagram, this means that the dependence on the arbitrary factor m_{gh} will cancel in the end.

This all seems quite convenient for calculating loop diagrams. We simply apply the propagator (18.82) and take care to include ghosts in the loops where appropriate, which have particularly simple, momentum-independent, propagators. However, this property of the ghost propagator is in fact highly problematic, as its independence from the loop momenta in general leads to poorly convergent loop integrals. This same property can be seen in the gauge boson propagator, for which as $k \to \infty$, it again scales as a constant $\sim 1/M^2$. This property means that it is, for example, quite difficult to establish renormalizability in this gauge. In the following section we therefore take a more general approach.

18.8 R_ξ gauge

To overcome the issues discussed above with the unitary gauge, we can instead consider a generalization of the R_ξ gauge considered in the case of unbroken gauge theories before. To do this, we introduce a Cartesian basis for ϕ

$$\phi = \frac{1}{\sqrt{2}}(v + h + ib) \ , \tag{18.88}$$

where h and b are real scalar fields. In terms of these, the potential is (again dropping the constant term)

$$V(\phi) = \frac{1}{4}\lambda v^2 h^2 + \frac{1}{4}\lambda v h(h^2 + b^2) + \frac{1}{16}\lambda(h^2 + b^2)^2 \ , \tag{18.89}$$

and the covariant derivative is

$$D_\mu\phi = \frac{1}{\sqrt{2}}\left[(\partial_\mu h + gbA_\mu) + i(\partial_\mu b - g(v + h)A_\mu)\right] \ . \tag{18.90}$$

The kinetic term in ϕ can then be expanded out and written as

$$\begin{aligned}(D_\mu\phi)^* D^\mu\phi = &\frac{1}{2}\partial_\mu b\partial^\mu b + \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2 v^2 A_\mu A^\mu - gvA^\mu\partial_\mu b \\ &- gA^\mu(h\partial_\mu b - b\partial_\mu h) \\ &+ g^2 vhA^\mu A_\mu + \frac{1}{2}g^2(h^2 + b^2)A^\mu A_\mu \ . \end{aligned} \tag{18.91}$$

The first line contains all terms quadratic in the fields. The first two are the kinetic terms for the b and h fields, the third is the mass term for the gauge field, and the last term is difficult to identify, and is somewhat undesirable–looking, but seems to corresponds to a transition between the A and b fields. The remaining terms in the second and third lines correspond to interaction vertices between the h, b and A fields.

To implement the R_ξ gauge we proceed via Fadeev–Popov gauge fixing. For our gauge fixing function we make the choice

$$f(A) = \partial_\mu A^\mu + \xi gvb - \sigma(x) \ , \tag{18.92}$$

the reason for which will become clear below. This defines our generalisation of the R_ξ gauge. Here $\sigma(x)$ is an arbitrary function, as in the unbroken abelian case, and for $v = 0$ we go back to the choice (6.48). As in Section 6.4 we can add an arbitrary constant

Z_σ term to the path integral, however with the additional term, the gauge–fixing part of the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{gf} &= -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi g v b)^2 \;, \\ &= -\frac{1}{2\xi}(\partial^\mu A_\mu)^2 - gvb\partial^\mu A_\mu - \frac{1}{2}\xi g^2 v^2 b^2 \;, \qquad (18.93)\\ &= -\frac{1}{2\xi}(\partial^\mu A_\mu)^2 + gvA_\mu \partial^\mu b - \frac{1}{2}\xi g^2 v^2 b^2 \;, \qquad (18.94)\end{aligned}$$

where in the third line we have integrated by parts and dropped the surface term. Thus we have two additional contributions in comparison to the unbroken abelian case. Crucially, the second term on the third line cancels the unwanted piece on the first line of (18.91). The last term gives a mass $\xi^{1/2}M$ to the b field.

What is the ghost term corresponding to the choice (18.92)? We are interested, as before, in the functional determinant $\Delta(A)$, given by

$$[\Delta(A)]^{-1} = \int \mathcal{D}\theta\, \delta(f(A_\theta)) \;. \tag{18.95}$$

Under a gauge transformation with infinitesimal parameter θ we have

$$A_\mu \to A_\mu + \partial_\mu \theta \;, \qquad \phi \to \phi + ig\theta\phi \;. \tag{18.96}$$

In terms of the Cartesian fields, the latter corresponds to

$$h \to h - g\theta b \;, \qquad b \to b + g\theta(v+h) \;, \tag{18.97}$$

and thus we have

$$f(A_\theta) = f(A) + (\partial^2 + \xi g^2 v(v+h))\theta \;. \tag{18.98}$$

As usual, the $f(A)$ term can be dropped, and we are left with the second piece to evaluate. Note in the unbroken abelian case, this would simply give an overall factor of $\partial^2\theta$, which was dropped. Here, due to the second term we cannot do this, and instead we must follow the same approach as in Section 15.5 for the treatment of non–abelian gauge theories. That is, we introduce a ghost term to the action

$$S_{gh} = \int \mathrm{d}^4x \mathrm{d}^4y \bar{c}(x) K(x,y) c(y) \;, \tag{18.99}$$

where $\Delta(A) = \det(K)$ and we have dropped the gauge indices, as we are considering the abelian case. Explicitly we have

$$K(x,y) = -\delta^4(x-y)(\partial^2 + \xi g^2 v(v+h)) \tag{18.100}$$

where the overall sign corresponds to a choice of convention that we take for consistency with the case of QCD, and so

$$\begin{aligned} \mathcal{L}_{gh} &= -\bar{c}\left[\partial^2 + \xi g^2 v(v+h)\right] c \, , \\ &= \partial_\mu \bar{c} \partial^\mu c - \xi g^2 v^2 \bar{c} c - \xi g^2 v h \bar{c} c \, . \end{aligned} \tag{18.101}$$

From the second line we can see that the ghost has acquired the same mass $\xi^{1/2} M$ as the b field.

Turning to the vector field, including the gauge fixing piece the terms quadratic in the field can be written as

$$\mathcal{L}_0 = \frac{1}{2} A_\mu \left[g^{\mu\nu}(\partial^2 + M^2) - (1 - \xi^{-1})\partial^\mu \partial^\nu \right] A_\nu \, , \tag{18.102}$$

which in momentum space corresponds to

$$\tilde{\mathcal{L}}_0 = -\frac{1}{2} \tilde{A}_\mu(-k) \left[g^{\mu\nu}(k^2 - M^2) - (1 - \xi^{-1}) k^\mu k^\nu \right] \tilde{A}_\nu(k) \, . \tag{18.103}$$

We can rewrite the kinematic matrix as

$$\begin{aligned} [\cdots] &= (k^2 - M^2) g^{\mu\nu} - (1 - \xi^{-1}) k^\mu k^\nu \, , \\ &= (k^2 - M^2) \left(P^{\mu\nu}(k) + \frac{k^\mu k^\nu}{k^2} \right) - \left(1 - \frac{1}{\xi} \right) k^\mu k^\nu \, , \end{aligned} \tag{18.104}$$

$$= (k^2 - M^2) P^{\mu\nu}(k) + \frac{1}{\xi} \frac{k^2 - \xi M^2}{k^2} k^\mu k^\nu \, , \tag{18.105}$$

where $P^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu / k^2$ projects onto the subspace transverse to k; thus this and $k^\mu k^\nu$ are orthogonal projection matrices. It is therefore straightforward to invert the above formula, and we are left with the gauge boson propagator

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{P^{\mu\nu}(k)}{k^2 - M^2 + i\epsilon} + \frac{\xi k^\mu k^\nu / k^2}{k^2 - \xi M^2 + i\epsilon} \, , \tag{18.106}$$

where have included the $i\epsilon$ term as usual. We can see that the transverse components of the vector field propagate with the mass M, while the longitudinal component propagates with the same mass as the b and ghost fields, $\xi^{1/2} M$.

If we take $\xi = 1$, the propagator greatly simplifies to

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - M^2 + i\epsilon} , \tag{18.107}$$

although it can also be useful to leave ξ general, as the final result should not depend on this.

In summary, in the R_ξ gauge we are left with unphysical ghost fields c and $\bar{c}$, and a b field, with propagators

$$\tilde{\Delta}(k) = \frac{1}{k^2 - \xi M^2} . \tag{18.108}$$

These must be included in loops, but not as physical external states. We can also see from (18.89) that the scalar field that we have suggestively labelled as h is a physical propagating state with mass $m_h^2 = \frac{1}{2}\lambda v^2$, and propagator

$$\tilde{\Delta}(k) = \frac{1}{k^2 - m_h^2} . \tag{18.109}$$

An equivalent h scalar boson will also be present when we break the gauge symmetry of the Standard Model, in which case it corresponds to the well known Higgs boson. Finally, the interactions are given by

$$\begin{aligned}\mathcal{L}_I = &-\frac{1}{4}\lambda v h(h^2 + b^2) - \frac{1}{16}\lambda(h^2 + b^2)^2 \\ &- gA^\mu(h\partial_\mu b - b\partial_\mu h) + g^2 v h A^\mu A_\mu + \frac{1}{2}g^2(h^2 + b^2)A^\mu A_\mu \\ &- \xi g^2 v h \bar{c} c ,\end{aligned} \tag{18.110}$$

It is interesting to consider the $\xi \to \infty$ limit. In this case we have

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} - k^\mu k^\nu / M^2}{k^2 - M^2 + i\epsilon} , \tag{18.111}$$

which corresponds exactly to the propagator (18.82) in the unitary gauge. The b field becomes infinitely heavy, and so can be dropped. For the ghost fields we have to be a bit more careful, as the $hc\bar{c}$ vertex contains a ξ, and so these cannot necessarily be dropped. However we can omit the $k^2 \ll m_{gh}^2 = \xi M^2$ term in the propagator, and so we are indeed back to the case of a $1/m_{gh}^2$ propagator; for every ghost loop we will have two powers of ξ in the numerator from the $hc\bar{c}$ vertices at each end, and two in the denominator from the ghost propagators, which will then cancel. Thus the $\xi \to \infty$ limit does indeed correspond to the unitary gauge.

18.9 Non–Abelian Gauge Symmetry Breaking

The basic ingredients, and implications, are the same as above but with the added complication that one has to keep track of the non–abelian nature of the gauge symmetry when gauge fixing. In particular, in the end one can define a mass squared matrix analogous to Section 18.3 and which can be divided into two orthogonal subspaces, one of which describes the physical massive scalar states and one the unphysical goldstone bosons that result from the symmetry breaking. A ghost Lagrangian must also be included, again suitably generalised. Further details are given in e.g. Srednicki chapter 86, where this is dealt with in full generality. However, to get a more concrete idea of how this works we will simply jump straight to the explicit case of the Standard Model.

19. The Standard Model

19.1 Electroweak Symmetry Breaking

We know experimentally that the weak force has three massive gauge bosons associated with it, the electrically neutral Z boson, and the electrically charged W^+ and W^- bosons. The most up–to–date values for the masses of these gauge bosons are

$$M_Z = (91.1876 \pm 0.0021)\,\text{GeV} \;, \qquad M_W = (80.385 \pm 0.015)\,\text{GeV} \;, \tag{19.1}$$

Now, we wish to describe this interaction via a gauge theory, and a natural (and it turns out, the correct) choice is to consider $SU(2)$, for which we have precisely $N^2 - 1 = 3$ gauge bosons. To do this we deal with introduce the *electroweak* symmetry

$$SU(2) \times U(1)_Y \;, \tag{19.2}$$

which provides a unified description of the weak and EM interactions. After SSB we will arrive at three massive fields, which we associate with the W and Z bosons and one massless field, which corresponds to a particular combination of the unbroken generators, and which we associate with the photon. The breaking pattern is

$$SU(2) \times U(1)_Y \to U(1)_{\text{EM}} \;. \tag{19.3}$$

Note that it is only *after* SSB that the remaining $U(1)$ symmetry is associated with QED. The $U(1)$ symmetry prior to SSB is completely distinct, and we give it the label *hypercharge*. The $SU(2)$ symmetry we label *weak isospin*.

To see how this works in practice, we first consider a few general points about the breaking of non–abelian gauge symmetry. Consider some scalar field that transforms according to an arbitrary non–abelian gauge symmetry as

$$\phi(x) \to \exp\left[ig\Gamma^a(x)T^a\right]\phi(x) \ , \tag{19.4}$$

where the T^a are as usual the generators of the group. We have

$$(D_\mu\phi)_i = \partial_\mu\phi_i - igA^a_\mu T^a_{ij}\phi_j \ , \tag{19.5}$$

$$(D_\mu\phi)^*_i = \partial_\mu\phi^*_i + igA^a_\mu(T^a_{ij}\phi_j)^* = \partial_\mu\phi^*_i + igA^a_\mu\phi^*_j T^a_{ji} \ , \tag{19.6}$$

where we have used the fact that the generators are Hermitian in the last step. In our Lagrangian we will then have

$$(D_\mu\phi)^* D^\mu\phi = \partial_\mu\phi^*_i\partial^\mu\phi_i + igA^a_\mu(\phi^*_j T^a_{ji}[\partial^\mu\phi_i] - [\partial^\mu\phi^*_i]T^a_{ij}\phi_j) + g^2A^a_\mu A^{\mu b}\phi^*_j(T^aT^b)_{jk}\phi_k \ . \tag{19.7}$$

It is the final term which will generate gauge boson mass terms after SSB, with

$$g^2A^a_\mu A^{\mu b}\phi^*_j(T^aT^b)_{jk}\phi_k \to \frac{1}{2}g^2A^a_\mu A^{\mu b}v_j(T^aT^b)_{jk}v_k \equiv \frac{1}{2}A^a_\mu A^{\mu b}(M^2)_{ab} \ , \tag{19.8}$$

where $\langle 0|\phi_i|0\rangle = v_i/\sqrt{2}$, and we assume for simplicity that v_i is real. The mass–squared matrix M^2 has eigenvalues corresponding to the masses of the gauge bosons, and we have precisely one such state for each broken generator, $T^a_{ij}v_j \neq 0$. The zero eigenvalues correspond to the unbroken generators, and in these cases the gauge boson will remain massless.

We now return to the specific case of the SM. In general we can associate different couplings g and g' to the $SU(2)$ and $U(1)_Y$ symmetries we are interested in, in which case our covariant derivative can be written as

$$D_\mu = \partial_\mu - igW^a_\mu\tau^a - ig'YB_\mu \ , \tag{19.9}$$

where W^a_μ and B_μ are the $SU(2)$ and $U(1)_Y$ gauge bosons, Y is the hypercharge of the field acted on, and the generators $\tau^a = \sigma^a/2$ are given in terms of the usual Pauli spin matrices, with $a = 1, 2, 3$.

To our gauge theory we now add a completely new complex scalar field doublet $\phi = (\phi_1, \phi_2)$, which we assume to have hypercharge $Y = 1/2$. In this case the field transforms as

$$\phi \to e^{i\alpha^a\tau^a}e^{i\beta/2}\phi \ , \tag{19.10}$$

for arbitrary parameters α^a and β. The action of the covariant derivative is

$$D_\mu \phi = (\partial_\mu - igW^a_\mu \tau^a - i\frac{1}{2}g' B_\mu)\phi \ , \tag{19.11}$$

We then introduce a scalar potential $V(\phi)$ to the Lagrangian

$$V(\phi) = -\mu^2(\phi^\dagger \phi) + \lambda(\phi^\dagger \phi)^2 \ , \tag{19.12}$$

where $\mu, \lambda > 0$. The minimum of the potential corresponds to

$$|\phi| = \sqrt{\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \ , \tag{19.13}$$

and so our field has a non–zero vev, that for simplicity we can take to be

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \ , \tag{19.14}$$

although the physics does not depend on this choice.

Now, it turns out that this results in precisely the pattern of SSB of our original $SU(2) \times U(1)_Y$ that is required experimentally to reproduce the electroweak sector of the SM. To see how this works, we can consider the corresponding term as in (19.8) which will generate the gauge boson masses. Taking into account the differing couplings we can write the mass–squared matrix as

$$M^2_{ab} = g_a g_b v_j (T^a T^b)_{jk} v_k \stackrel{!}{=} \frac{1}{2} g_a g_b v_j (\{T^a, T^b\})_{jk} v_k \ , \tag{19.15}$$

where we can make the replacement due to the contraction with the A fields, which are symmetric in $a \leftrightarrow b$; this simplifies the calculation below. Here we define $T^a = \sigma^a/2$ and $g_a = g$ for $a = 1, 2, 3$ and $T^a = I/2$ and $g_a = g'$ for $a = 4$. Making use of the usual relation $\{\sigma^a, \sigma^b\} = 2\delta_{ab}$ we get

$$M^2_{ab} = \frac{g^2 v^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -g'/g \\ 0 & 0 & -g'/g & g'^2/g^2 \end{pmatrix} \ , \tag{19.16}$$

which has determinant zero and hence at least one massless gauge boson remains. In fact, we can read off immediately from the first two diagonal components that we have two identical non–zero eigenvalues, given by

$$M_W^2 = \frac{g^2 v^2}{4} \ , \tag{19.17}$$

and thus we have one remaining non–zero eigenvalue, with

$$M_Z^2 = \frac{(g'^2 + g^2)v^2}{4} \ . \tag{19.18}$$

Thus this does indeed have exactly the pattern of symmetry breaking we require to produce the observed electroweak sector of the SM, i.e. with three massive gauge bosons for the three broken generators of $SU(2)$ weak isospin. We associate the above masses with those of the W and Z bosons, while we associate the final massless gauge boson with the photon. The mass eigenstates which diagonalise M^2, corresponding to the Z and the photon are

$$\begin{aligned} Z_\mu &\equiv \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \ , \\ A_\mu &\equiv \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \ , \end{aligned} \tag{19.19}$$

where we have introduced the *weak mixing angle* θ_W via

$$\tan\theta_W \equiv \frac{g'}{g} \Rightarrow \qquad \cos\theta_W = \frac{g}{\sqrt{g'^2 + g^2}} \ , \qquad \sin\theta_W = \frac{g'}{\sqrt{g'^2 + g^2}} \ . \tag{19.20}$$

Thus, indeed as claimed the photon field A_μ is a mixture of the original W^3 and B gauge fields from the combined $SU(2) \times U(1)_Y$ symmetry of our unbroken theory, and QED and the weak force are embedded, or unified, within a larger electroweak sector. We can also define the fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2) \ , \tag{19.21}$$

which we will see when we consider interactions with quarks and leptons correspond to the positively and negatively electrically charged W bosons. Inverting the above relations, we can write the covariant derivative in terms of these new fields, giving

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - iZ_\mu(g\cos\theta_W T^3 - g'\sin\theta_W Y) - ig\sin\theta_W A_\mu(T^3 + Y) \ , \tag{19.22}$$

where $T^\pm = T^1 \pm iT^2$. We can then read off from this that we should associate the unbroken generator with the electric charge

$$Q = T_3 + Y \tag{19.23}$$

and identify the coefficient of the EM interaction with the (positive) electric charge e

$$e = g \sin\theta_W \ . \tag{19.24}$$

The covariant derivative then becomes

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_W} Z_\mu (T^3 - \sin^2\theta_W Q) - ieA_\mu Q \ . \tag{19.25}$$

Finally, we can see that the W and Z boson masses are not independent, but rather are related by

$$M_W = M_Z \cos\theta_W \ , \tag{19.26}$$

where the weak mixing angle can be calculated from the measured couplings e and g. This is a remarkable constraint, which relates the seemingly unrelated masses of the W and Z bosons with the electroweak and EM couplings. By measuring all of these quantities we can then place tight consistency checks on the SM, which would not necessarily hold if there were some physics beyond it hiding in e.g. loop corrections[16]. So far, it has passed these with essentially flying colours.

19.2 The Higgs Potential

This section is non–examinable

We now return to the kinetic and potential terms for our scalar field ϕ. Working in the unitary gauge for simplicity, we expand around our vev with

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \ , \tag{19.27}$$

where h is a real scalar field. This is the famous Higgs field, and the corresponding particle is the Higgs boson. Substituting this into our expression for the potential (19.12) we find

$$V(\phi) = \frac{1}{4}\lambda h^4 + v\lambda h^3 + \lambda v^2 h^2 \ , \tag{19.28}$$

[16]Indeed, even within the SM these exact results only hold at tree–level and will receive loop corrections. These however can be calculated to a high degree of precision.

and therefore the Higgs particle has a mass

$$m_h = \sqrt{2\lambda} v \ , \tag{19.29}$$

which is directly proportional to the vev of the scalar field undergoing SSB. We can also see that this has introduced cubic and quartic self–interaction term for the Higgs. While the Higgs itself has indeed been seen, with roughly

$$m_h = (125.1 \pm 0.2)\,\text{GeV} \ , \tag{19.30}$$

it is currently an ongoing effort to measure these cubic and quartic self–couplings, and see if they really conform to the expectations from the SM Higgs, or have some different form, which would indicate a more complicated and non–standard form of SSB had taken place.This is a very difficult thing to measure, and will no doubt take many years, and perhaps even a new collider to do well.

19.3 Gauge boson Interactions

This section is non–examinable

As we are dealing with a non–abelian gauge theory, we will expect there to be couplings between the gauge bosons. These are contained as usual within the kinetic term

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F^{a,\mu\nu} F_{a\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \ , \tag{19.31}$$

where $a = 1, 2, 3$ are the $SU(2)$ group indices and

$$F^c_{\mu\nu} = \partial_\mu W^c_\nu - \partial_\nu W^c_\mu + g\epsilon_{abc} W^a_\mu W^b_\nu \ , \tag{19.32}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ , \tag{19.33}$$

We can then simply substitute our known expressions (19.19) and (19.21) for our basis of $W^\pm$, Z and A gauge fields to work out the corresponding interactions between these. Instead of going through all of this in detail (see any good textbook and the following Standard Model course for more), we will simply concentrate on one example, namely the $\gamma W^+ W^-$ vertex. With a little algebra it is possible to show that

$$-\frac{1}{4}(F^{1\mu\nu} F_{1\mu\nu} + F^{2\mu\nu} F_{2\mu\nu}) = (\partial_\mu + igW^3_\mu) W^-_\nu \left[(\partial^\nu - igW^{3\nu}) W^{+\mu} - (\partial^\mu - igW^{3\mu}) W^{+\nu} \right] \ . \tag{19.34}$$

This gives contributions of the form

$$igW^3_\mu W^-_\nu \partial^\nu W^{+\mu} = ig(\sin\theta_W A_\mu + \cos\theta_W Z_\mu) W^-_\nu \partial^\nu W^{+\mu} \ , \tag{19.35}$$

and thus a γW^+W^- vertex

$$ig\sin\theta_W A_\mu W_\nu^- \partial^\nu W^{+\mu} = ieA_\mu W_\nu^- \partial^\nu W^{+\mu} \;, \tag{19.36}$$

thus justifying our association of the $W^\pm$ fields with unit charge.

19.4 The Weak Interaction

This section is non–examinable

The covariant derivative (19.25) uniquely determines the couplings of the W and Z bosons to fermions, once we have specified the quantum numbers of these under the $SU(2) \times U(1)_Y$ group. To do this, we use the experimental observation that the weak interaction is known to violate parity, and in particular the W boson only couples to the left–handed chiral states of quarks and leptons. Our theory will therefore be chiral, that is it will distinguish between the left and right handed fermion components.

To reproduce the known form of the weak interaction, we assign our left–handed fermion fields to doublets of $SU(2)$, that is for the leptons we write our theory in terms of

$$L^i = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \;. \tag{19.37}$$

where the i labels the three generations of leptons in the SM. We then assign by hand hypercharge quantum numbers so that these reproduce the correct electric charges according to (19.23). Thus

$$e_L : \qquad Y = Q - T_3 = -1 + \frac{1}{2} = -\frac{1}{2} \;, \tag{19.38}$$

$$\nu_{eL} : \qquad Y = Q - T_3 = 0 - \frac{1}{2} = -\frac{1}{2} \;. \tag{19.39}$$

We then take that the right–handed field components are uncharged under weak isospin, i.e. are singlets, in order to remove interactions via the weak force. We have

$$e_R^i = \{e_R, \mu_R, \tau_R\} \;, \tag{19.40}$$

which following the logic above have hypercharge $Y = Q = -1$ (as $T_3 = 0$). As neutrinos have zero electric charge, this implies they cannot be produced at all in the SM, and so we do not include a right–handed neutrino component. The interactions between the leptons and gauge bosons are then determined from the Lagrangian term

$$-i\overline{e}_R(\partial\!\!\!/ - ig'Y_R B\!\!\!/)e_R + i\overline{L}(\partial\!\!\!/ - igW\!\!\!\!/ \cdot \tau - ig'Y_L B\!\!\!/)L \;, \tag{19.41}$$

where we omit the family index for simplicity, and the $\overline{e}$ has the usual meaning when we interpret these as 4–component Dirac spinors. For the Z boson interactions, we read off from (19.25) that the relevant term is

$$-i\frac{g}{\cos\theta_W}\left[-\sin^2\theta_W Q\overline{f}_R \not{Z} f_R + (T^3 - \sin^2\theta_W Q)\overline{f}_L \not{Z} f_L\right] \ , \tag{19.42}$$

where $f = e, \mu, u, d\cdots$ corresponds to the lepton or quark flavour, and we have used that $T_3 = 0$ for the right–handed fields. To avoid working in terms of Weyl spinors, we can then use the chirality projection as in (10.35), i.e.

$$\psi_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\psi \ , \tag{19.43}$$

where again we deal with 4–component Dirac spinors here (thus $\overline{\psi}_{L,R} = (\frac{1}{2}(1\mp\gamma_5)\psi)^\dagger\gamma_0$). Then, from the usual properties of the γ matrices, we find

$$\overline{\psi}_L\gamma^\mu\psi_L = \frac{1}{2}\overline{\psi}\gamma^\mu(1-\gamma_5)\psi \ , \qquad \overline{\psi}_R\gamma^\mu\psi_R = \frac{1}{2}\overline{\psi}\gamma^\mu(1+\gamma_5)\psi \ , \tag{19.44}$$

allowing us to write (19.42) as

$$-i\frac{g_W}{2\cos\theta_W}\overline{f}\gamma^\mu(g_V^f - g_A^f\gamma_5)f \ , \tag{19.45}$$

where θ_W is the weak mixing angle, with $e = g_W \sin\theta_W$, and the vector and axial–vector couplings are given by

$$g_V^f = T_f^3 - 2Q_l \sin^2\theta_W \ , \tag{19.46}$$

$$g_A^f = T_f^3 \ . \tag{19.47}$$

Here T_f^3 is the weak isospin of the left–handed fermion field. Thus for example for an electron we have the interaction vertex

$$-i\frac{g}{2\cos\theta_W}\gamma^\mu\left[\left(-\frac{1}{2} + 2\sin^2\theta_W\right) + \frac{1}{2}\gamma_5\right] \ . \tag{19.48}$$

The interaction between a $W^\pm$ boson and a lepton $SU(2)$ doublet can be read off from (19.25), i.e. we are interested in

$$-i\frac{g}{\sqrt{2}}\overline{L}(\not{W}^+T^+ + \not{W}^-T^-)L \ . \tag{19.49}$$

The $T^{\pm}$ are given explicitly by

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{19.50}$$

which we can see pick out a single component of the lepton doublet; these are the weak isospin raising and lowering operators. Applying (19.44) as before we get, for e.g. the interaction between an electron and neutrino we have the terms in the Lagrangian

$$-i\frac{g}{2\sqrt{2}} \left(W_\mu^- \overline{e}\gamma^\mu(1-\gamma_5)\nu + W_\mu^+ \overline{\nu}\gamma^\mu(1-\gamma_5)e \right) , \tag{19.51}$$

from which the interactions can be read off. A similar term can be written down for quark transitions, with e.g.

$$-i\frac{g}{2\sqrt{2}} \left(V_{ud} W_\mu^- \overline{u}\gamma^\mu(1-\gamma_5)d + V_{ud}^* W_\mu^+ \overline{d}\gamma^\mu(1-\gamma_5)u \right) , \tag{19.52}$$

where V is the CKM matrix, which describes the fact that the quark mass and weak eigenstates are not aligned, and allows for further, e.g. $u \to s$ transitions. This will be discussed in detail in the Standard Model course, and so we will not consider this further here.

19.5 Fermion masses

This section is non–examinable

The observed masses of the W and Z bosons are in fact not the only issue that arises if we attempt to describe the weak interaction with an unbroken gauge theory. In particular, we saw in Section 10.3 that a fermion mass term must mix left and right–handed chiral components in order to be Lorentz invariant. However, we know experimentally that the weak interaction treats such left and right–handed components quite differently, and in order to describe this we have had to introduce additional quantum numbers into the game. In particular, the combination $e_L e_R$ carries non–zero weak isospin and hence is not $SU(2)$ invariant. It is therefore impossible to write down a gauge–invariant fermion mass term in our unbroken chiral weak gauge theory.

Here again, the Higgs field saves the day. In particular, we saw before that the Higgs is charged under $SU(2)$, and therefore we can write down gauge–invariant interactions terms of the type

$$-\lambda_f \overline{L}\phi e_R + \text{h.c.} , \tag{19.53}$$

where 'h.c.' stands for the Hermitian conjugate expression. This is indeed $SU(2)$ invariant as our choice of Higgs field (19.27) has $T_3 = -1/2$, while the $\overline{L}$ has $T_3 = 1/2$.

Here λ_f is the *Yukawa* coupling between the Higgs and fermion. Expanding our Higgs fields as in (19.27) this becomes

$$-\frac{\lambda_f}{\sqrt{2}}(v+h)\left(\overline{f}_l f_R + \overline{f}_R f_l\right) = -\frac{\lambda_L}{\sqrt{2}}(v+h)\overline{f}f \ , \tag{19.54}$$

$$= -m_f\overline{f}f - \frac{m_f}{v}h\overline{f}f \ . \tag{19.55}$$

That is, the fermions acquire a mass given by

$$m_f = \frac{\lambda_f v}{\sqrt{2}} \ . \tag{19.56}$$

19.6 The Higgs Interactions

This section is non–examinable

The new h field we have introduced via (19.27) will interact with the SM particles in a number of ways. We can immediately read off from (19.55) that there will be a $hf\overline{f}$ interaction due to the term

$$-\frac{m_f}{v}h\overline{f}f \ . \tag{19.57}$$

The gauge boson interactions come from the action of the covariant derivative on the Higgs field. From (19.25) we have

$$D_\mu h = \frac{1}{\sqrt{2}}\left(\partial_\mu - i\frac{g}{\sqrt{2}}W_\mu^+ + i\frac{g}{2\cos\theta_W}Z_\mu\right)(v+h) \ , \tag{19.58}$$

where we have used that $T^-\phi = 0$, $T^+\phi = (v+h)/\sqrt{2}$, for our choice of vacuum (19.27) and $T_3 = -1/2$, $Q = 0$ for the Higgs. After a little rearranging, this gives

$$(D_\mu h)^* D_\mu h = \frac{1}{2}\partial_\mu h \partial^\mu h + \left[m_W^2 W_\mu^+ W^{\mu -} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu\right]\left(1+\frac{h}{v}\right)^2 \ . \tag{19.59}$$

We can read off the usual mass terms for the W and Z from this, but in addition we can see that hW^+W^- and hZZ interactions are present, with vertices

$$2i\frac{M_V^2}{v}g_{\mu\nu} \ , \tag{19.60}$$

and hhW^+W^- and $hhZZ$ interactions, with vertices

$$2i\frac{M_V^2}{v^2}g_{\mu\nu} \ , \tag{19.61}$$

where $M_V = M_{W,Z}$. In all cases we can see that the strength of the interaction grows with the mass of the fermion or gauge boson. Thus the Higgs boson interacts more strongly with the heavier SM particles, such as the top quark and W, Z bosons, and only quite weakly with the lighter particles such as the electron and u, d, s quarks. This has a wide range of phenomenological consequences when it comes to measuring this object in the collider.

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