# QCD Phenomenology – Final States

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Strong interaction physics is described using the quantum field theory QCD. Renormalization introduces arbitary scale $\mu^2$ into theory when regularizing loop diagrams. Coupling satisfies evolution equation

$$\frac{d\alpha_S}{d\ln\mu^2} = -\beta_0\alpha_S^2 + \cdots, \qquad \beta_0 = \frac{(11 - 2/3N_f)}{4\pi}$$

Negative $\beta$-function means strong at low scales but weaker at higher scales.

Ignoring the $\mathrm{O}(\alpha_s^3)$ corrections this may be solved

$$-\int_{\mu_0^2}^{\mu^2} d\,\ln\tilde{\mu}^2 = \frac{1}{\beta_0}\int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\,\tilde{\alpha}_s}{\tilde{\alpha}_s^2},$$

where $\mu_0$ is some fixed scale. Hence,

$$-\ln(\mu^2/\mu_0^2) = \frac{1}{\beta_0}\left[\frac{1}{\alpha_s(\mu_0^2)} - \frac{1}{\alpha_s(\mu^2)}\right].$$

This leads to

$$\alpha_s(\mu^2) = \frac{1}{\beta_0}\frac{1}{\ln(\mu^2/\mu_0^2) + \frac{1}{\beta_0\alpha_s(\mu_0^2)}}.$$

From this expression we can indeed see that $\alpha_s(\mu^2)$ decreases as $\mu^2$ increases, and that $\alpha_s(\mu^2) \to 0$ as $\mu^2 \to \infty$. However, the definition relies on an arbitrary boundary condition for the coupling at some fixed scale $\mu_0^2$.

It is simpler, and more illustrate to rewrite the solution for $\alpha_s(\mu^2)$ slightly. It may be expressed as

$$\alpha_s(\mu^2) = \frac{1}{\beta_0} \frac{1}{\ln(\mu^2) - (\ln(\mu_0^2) - \frac{1}{\beta_0 \alpha_s(\mu_0^2)})}.$$

Defining a scale $\Lambda_{QCD}$ by

$$\ln(\mu_0^2) - \frac{1}{\beta_0 \alpha_s(\mu_0^2)} = \ln(\Lambda_{QCD}^2),$$

$\Lambda_{QCD}$ is the value of $\mu_0^2$ for $\alpha_s(\mu_0^2) \to \infty$. Results in the solution.

$$\alpha_s(\mu^2) \approx \frac{4\pi}{(11 - 2/3N_f) \ln(\mu^2/\Lambda_{QCD}^2)}$$

Binds partons into hadrons at low scales, but can do perturbative calculations at higher scales.

Even in processes involving no incoming hadrons the final state is often dominated by hadrons, and thus a full understanding of the physics requires an understanding of this hadronic final state. This means we must use QCD and understand it as well as possible. The most obvious features in the hadronic final states are jets.

## General form of Perturbative Expansion

Suppose we calculate a total cross-section with one variable, e.g. centre of mass energy $\sqrt{s}$. Since the coupling depends on the renormalization scale $\mu$ the cross-section is scale-dependent. At LO in $\alpha_S$

$$\sigma(s) = A\alpha_S(\mu^2).$$

This automatically leads to

$$\frac{d\sigma(s)}{d\ln\mu^2} = -A\beta_0\alpha_S^2(\mu^2).$$

At NLO in $\alpha_S$ we have explicit scale depedence

$$\sigma(s) = A\alpha_S(\mu^2) + \alpha_S^2(\mu^2)(B + b\ln(\mu^2/s)).$$

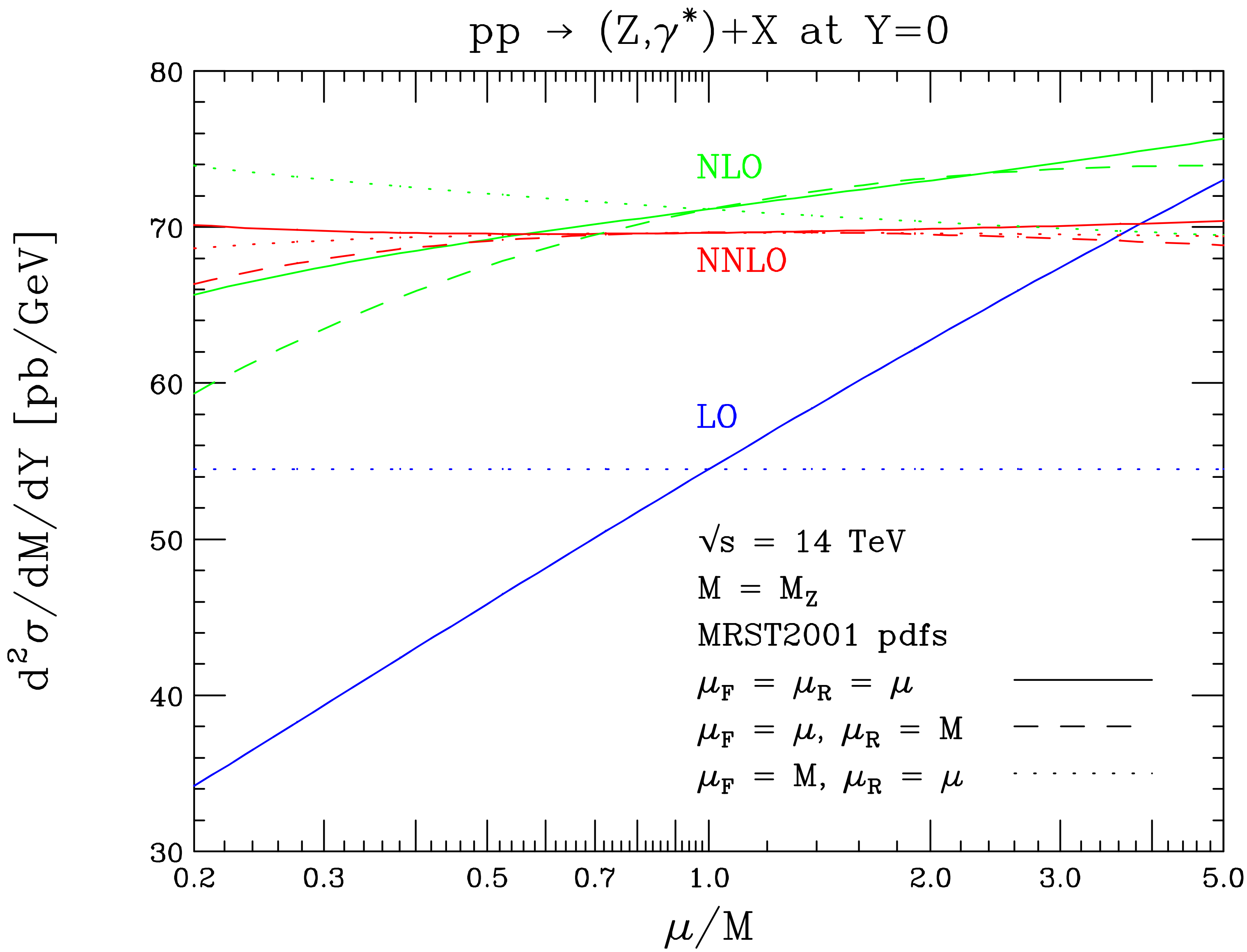
In general the scale dependence is

$$\frac{d\sigma(s)}{d\ln\mu^2} = -A\beta_0\alpha_S^2(\mu^2) + b\alpha_S^2(\mu^2) + \mathcal{O}(\alpha_S^3).$$

The scale dependence must decrease as we go to higher orders.

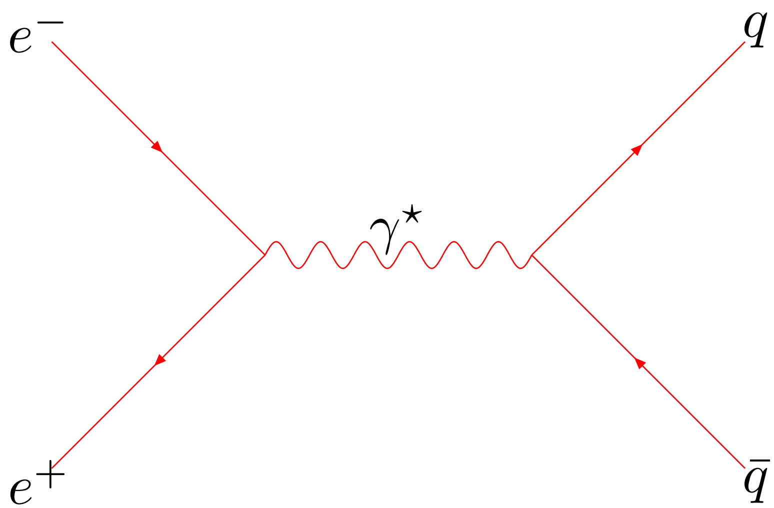
Achieved if $b = A\beta_0$, i.e. scale dependent part of NLO correction determined by lower orders and running of the coupling. Constant $B$ has to be calculated explicitly.

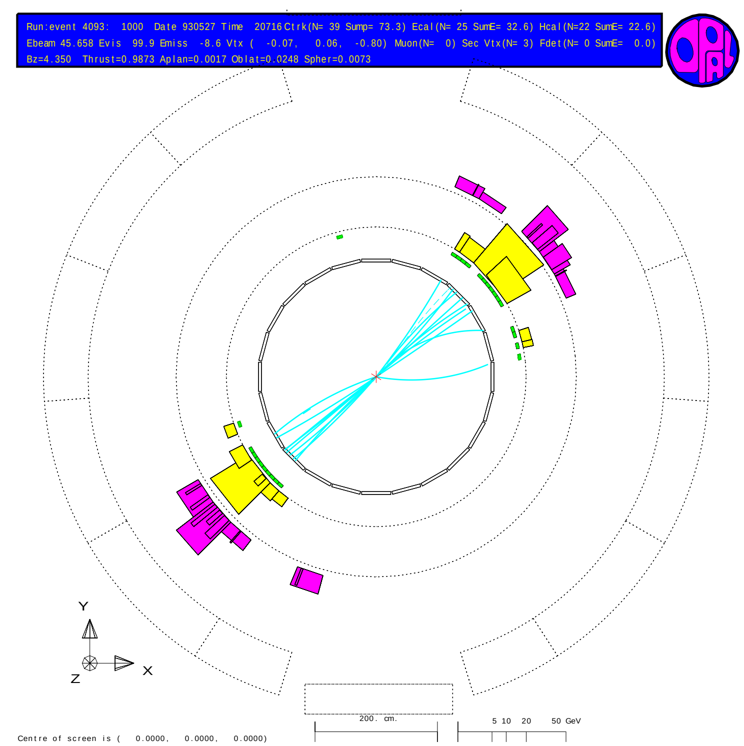
With a NNLO correction the scale dependence is postponed to $\mathcal{O}(\alpha_S^4)$.



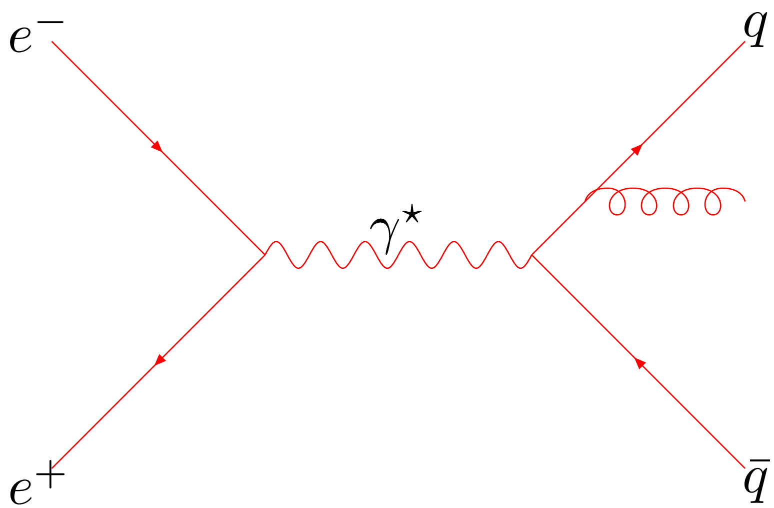
## Jet Events

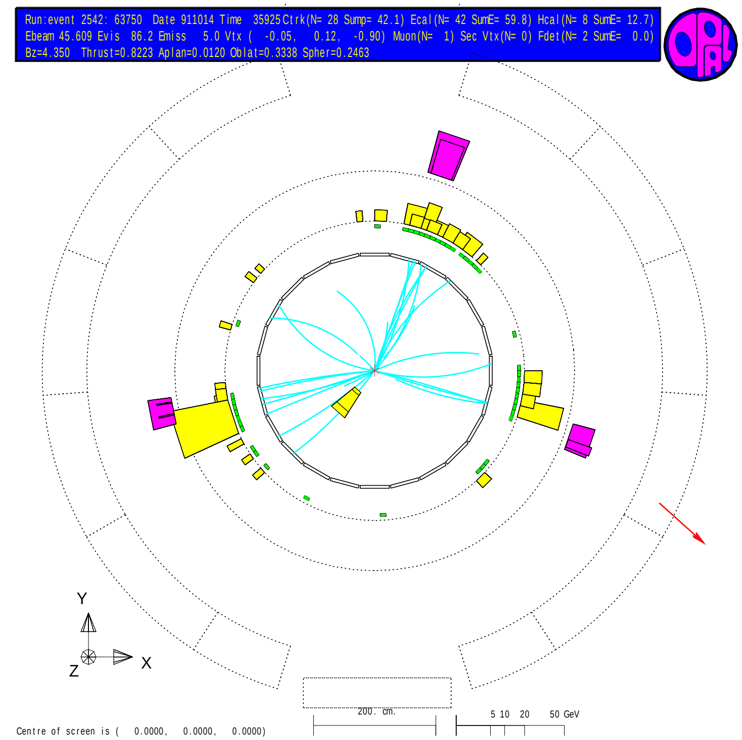
Consider the simplest case of $e^+e^-$ annihilation into a photon producing a hadronic final state. At zeroth order in $QCD$ this will simply be a quark-antiquark pair at parton level, and each quark will hadronize into at jet. Hence we have a two jet final state.



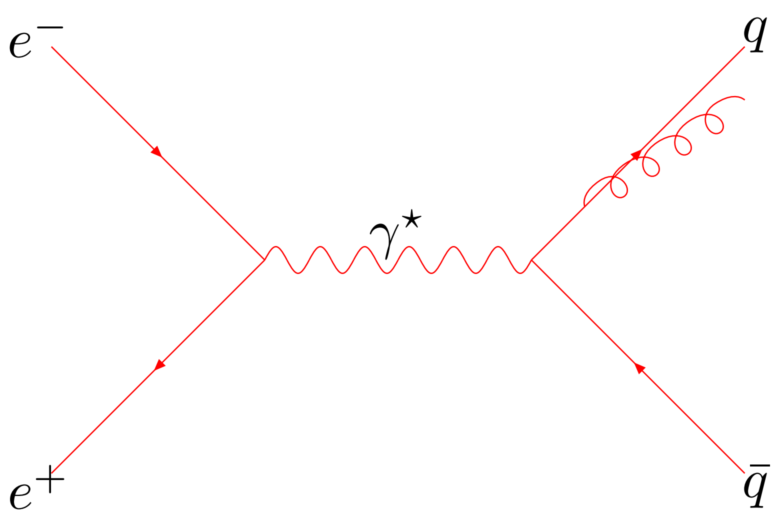
At first order in $\alpha_S$ we can also have the emission of a gluon from the quark or anti-quark, e.g.

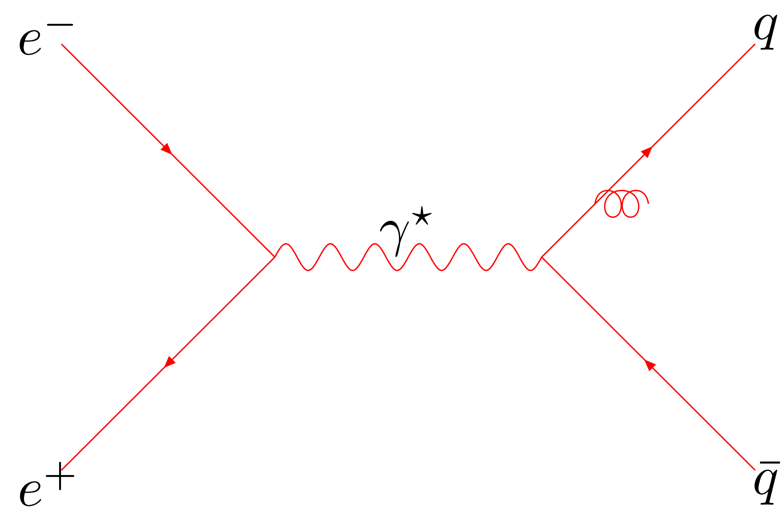



In general this will lead to a 3-jet final state.

At order $\alpha_S^2$ we could have two gluons, or the one gluon could fragment into a quark-antiquark pair, and we could obtain a 4-jet final state. This continues to higher orders, i.e. at $n_{th}$ order $\alpha_S$ we can have a $(n+2)$-jet state. However, if a final state parton is sufficiently collinear with another parton or is sufficiently unenergetic (soft) it will simply go into the jet of the initial parton. i.e.



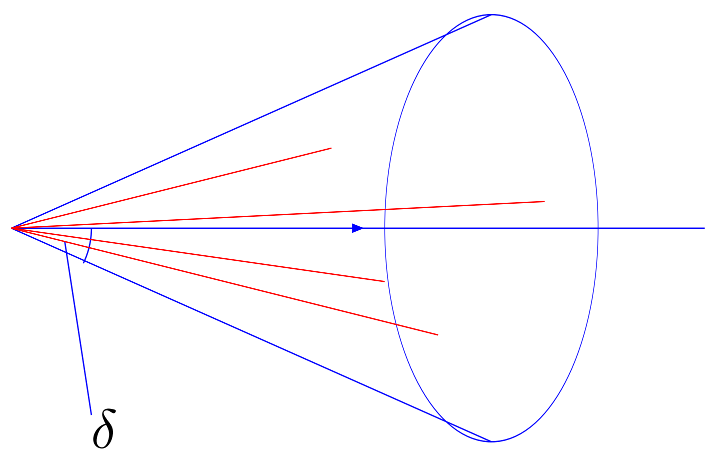




will contribute to the 2-jet rate rather than the 3-jet rate.

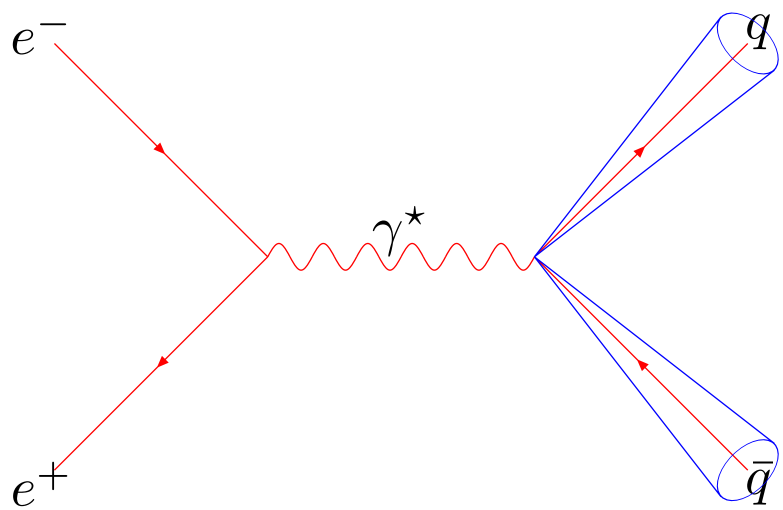
# Jet Definitions

In order to be quantitative we need a precise definition of what constitutes an $n$-jet event, i.e. we need a jet definition.

The simplest definition (in principle) is the cone algorithm. Define a cone with an opening angle $\delta$. The jet is made up of all partons within the cone, with the axis chosen such that the energy within the cone is maximized, and the momentum of the jet is the sum of the hadron momenta.

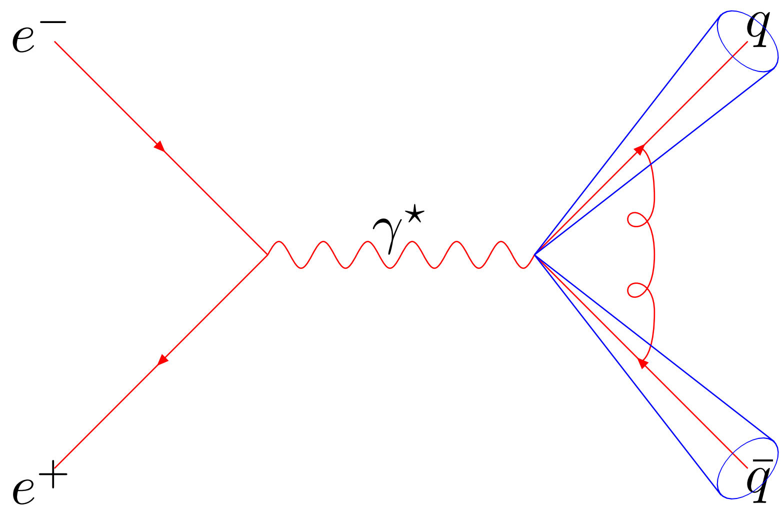


The same definition must be used at parton level in order to predict the theoretical jet rate. At leading order



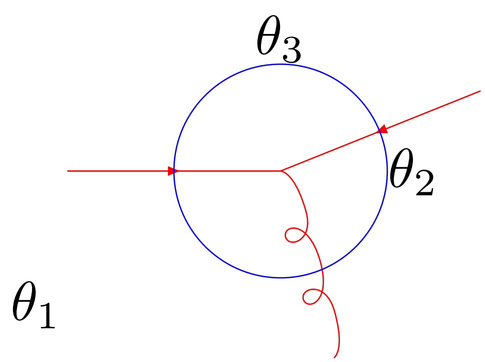

Hence at this order $\sigma_{2jet} = \sigma_{tot}$, and every event has two jets each of energy $1/2\sqrt{s}$.

When working to order $\alpha_S$ we must consider the virtual correction




which always leads to a two jet event. However, the amplitude for this diverges. We must also consider the emission of a gluon off the quark (antiquark) line.

Real emission of a gluon will lead to a 3-jet event if $\theta_1, \theta_2, \theta_3 > \delta$ (in the limit $\theta_i \to 0$ the 3-parton amplitude also diverges).



Defining $x_{1,2,3} = \frac{2E_{q,\bar{q},g}}{\sqrt{s}}$ the three parton cross section is calculated as

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

where

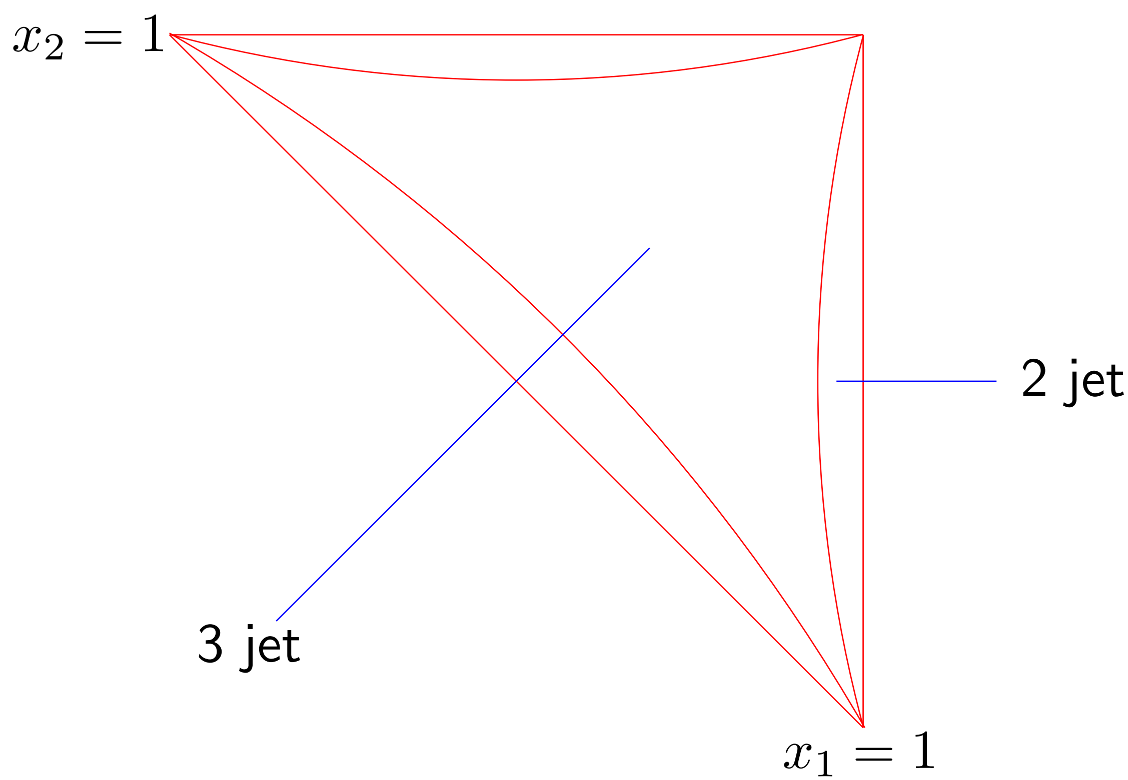
$$(1 - x_1) = \frac{1}{2} x_2 x_3 (1 - \cos\theta_{qg}) + \text{permutations.}$$

Hence divergences when $\theta_i \to 0$.

$$\theta_1 > \delta \to \frac{2(1-x_1)}{x_2(2 - x_1 - x_2)} > 1 - \cos(\delta)$$

So if $\delta << 1$ we have $x_1 > 1 - 0.25\delta^2 x_2(1 - x_2)$.

Using the kinematic constraint that $x_1, x_2 \leq 1$ and $x_1 + x_2 \leq 2$, the phase space can be pictured as below




However, this single constraint does not render results for the 2-jet and 3-jet cross-sections finite, i.e. there is not a complete cancellation of divergence between the real and virtual contributions. This is because we still have divergences coming from the situation when one of the 3 jets has vanishingly small momenta, i.e. when the corresponding parton is sufficiently soft.

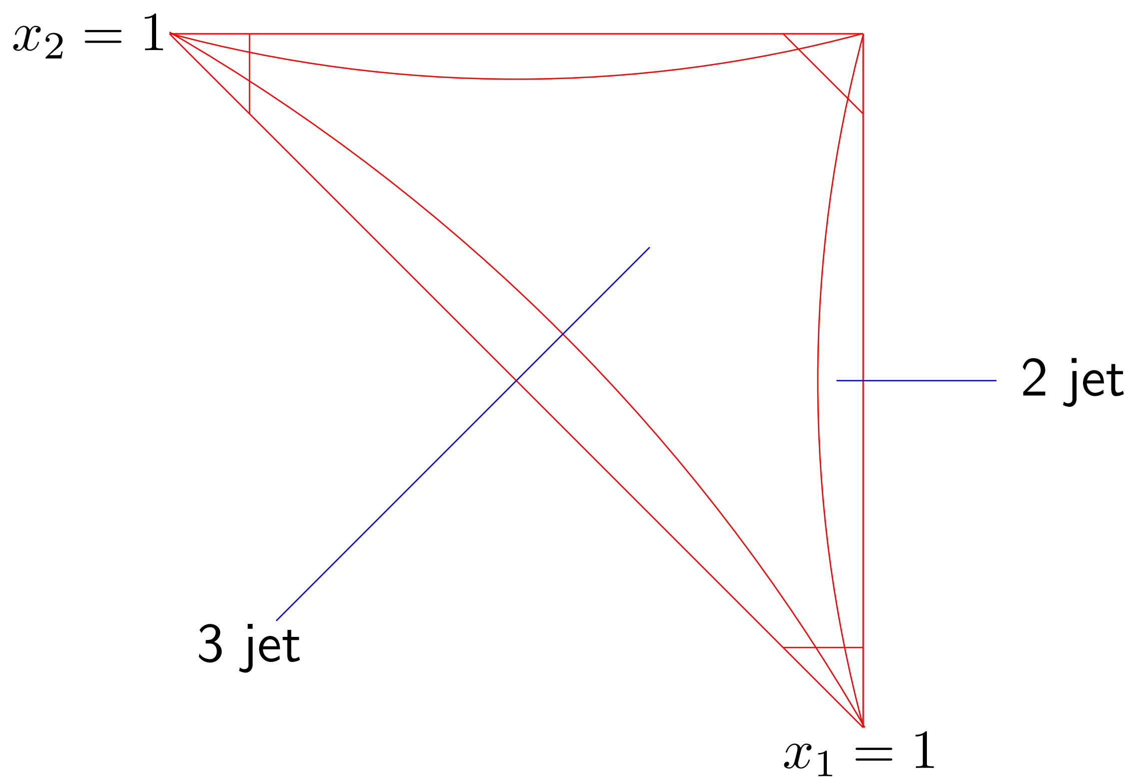
## Infrared Safety

To obtain a finite answer we must ensure that both soft and collinear real emission is treated the same way as virtual contributions.

$\rightarrow$ as well as finite angle $\delta$ we need an energy cut-off in the jet definition.

$$E_{jet} > \epsilon \times 1/2\sqrt{s}$$

This modifies our previous diagram.



The cancellation of divergences between real and virtual contributions is now complete and the fraction of 3-jet events $R_3$ at first order in $\alpha_S$ is now given by.

$$R_3 = C_F \frac{\alpha_S}{2\pi} \left[ \log(1/\delta) \log(1/\epsilon) + \log(1/\epsilon) + \log(1/\delta) + C \right]$$

$$R_2 = 1 - R_3$$

where the latter definition avoids complications in regularizing the divergent contributions.

Note: as $\epsilon \longrightarrow 0$ or $\delta \longrightarrow 0$, $R_3$ cab become very large, i.e. cancellation of divergences is ceasing to be effective. Possible for $R_3 > 1$ i.e. $R_2 < 0$ - clearly not sensible.

Intuitively simple jet definition. But suffers from some problems in details, e.g. need two completely separate cuts-off; possible for two cones to overlap. What happens in this case?

$\longrightarrow$ alternative definitions.

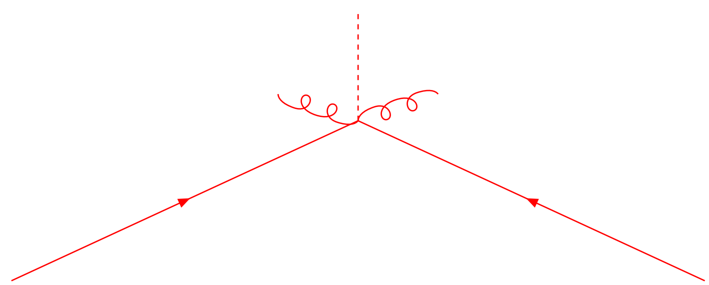
## Cluster Algorithms.

Consider a system of two hadrons in the experimental case, or two partons in the theoretical case. JADE algorithm will combine them into a single jet if the invariant mass $m^2$ satisfies.

$$m^2 = (p_1 + p_2)^2 < y_{cut} s.$$

$m^2 = 2E_1 E_2 (1 - \cos\theta_{12})$ and so $\rightarrow 0$ if $E_i \rightarrow 0$ (soft) or $\theta_{12} \rightarrow 0$ (collinear).

Works pretty well, but has some problems for small $y_{cut}$. Exhibited by below event.



The two gluons may be combined into a spurious jet indicated by the dashed line. This may be avoided by a simple modification.

## $K_T$ (Durham) Algorithm.

In this case the algorithm will combine into a single jet if

$$K_T^2 < y_{cut} s.$$

where $K_T^2 = 2min(E_1^2, E_2^2)(1 - \cos\theta_{12})$ and so $\rightarrow 0$ if $E_i \rightarrow 0$ (soft) or $\theta_{12} \rightarrow 0$ (collinear).

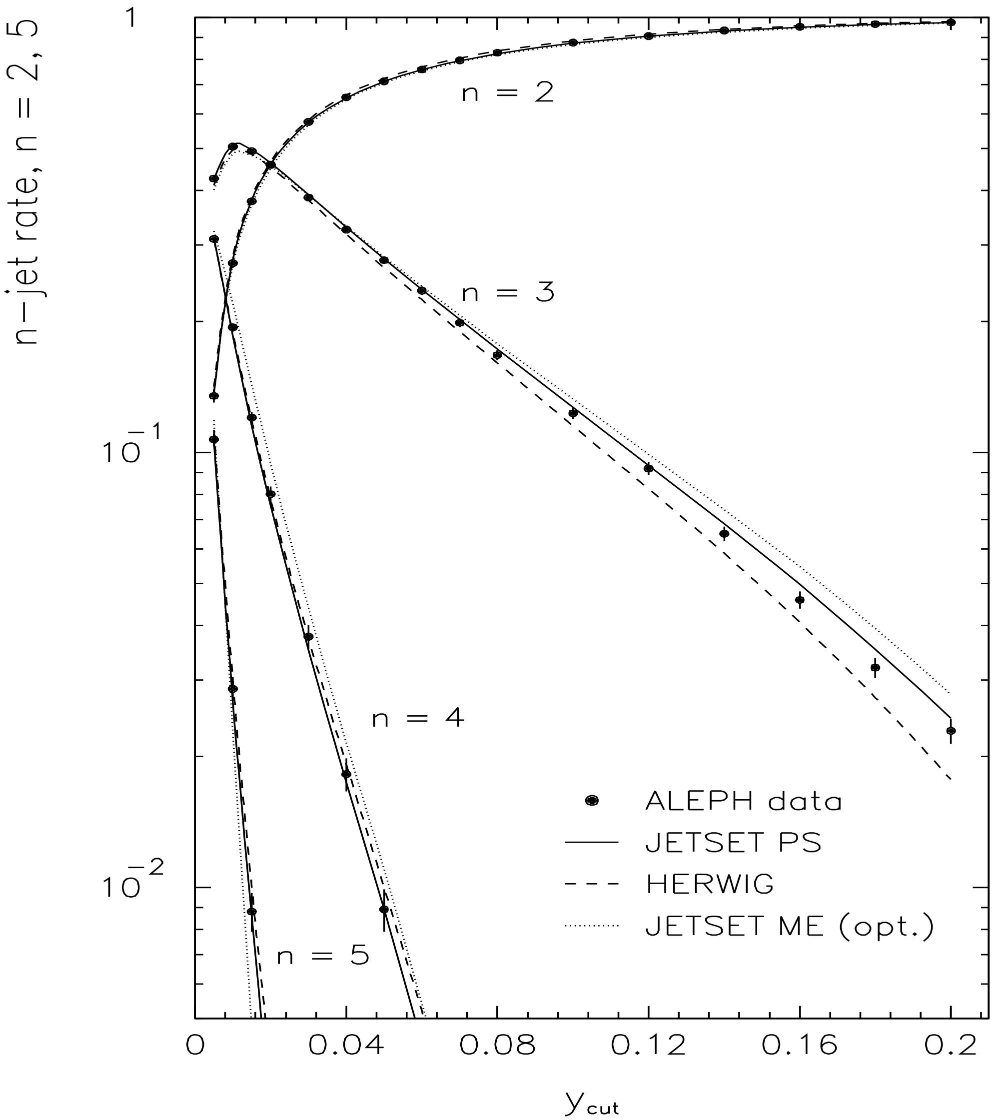
Use an iterative procedure

1. Find the pair with the smallest $y_{ij}$.
2. If $y_{ij,min} < y_{cut}$ combine $i$ and $j$, e.g. $p_{ij} = p_i + p_j$ and go back to 1.
3. If $y_{ij,min} > y_{cut}$ stop. All remaining momenta are called jets.

(Avoids previous problem because gluons combined with quarks before each other.)

In practice simpler than cone algorithm. Can be applied at any order. $\rightarrow$ powerful test of QCD.

Comparison with $n$-jet rates at ALEPH.

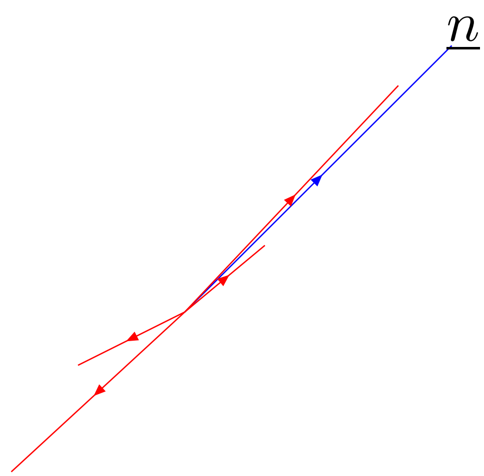


## Event Shapes.

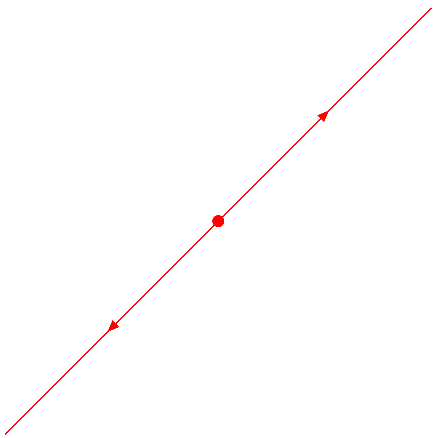
These are quantities which contain more detailed information about the final hadronic state.

e.g. Thrust $T = max_{\underline{\mathsf{n}}} \frac{\sum_i |\underline{\mathsf{p}}_i \cdot \underline{\mathsf{n}}|}{\sum_i |\underline{\mathsf{p}}_i|}$

this measures how unidirectional a set of jets are,

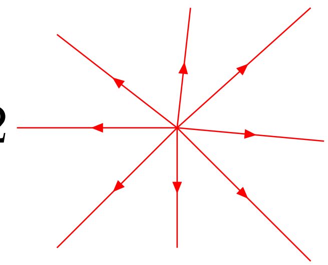


# Has extreme limits



$T = 1$

$T = 1/2$



isotropic

Other variable such as spherocity $S$

$$S = \left(\frac{4}{\pi}\right)^2 min_{\underline{\mathsf{n}}} \left(\frac{\sum_i |\underline{\mathsf{p}}_i \times \underline{\mathsf{n}}|}{\sum_i |\underline{\mathsf{p}}_i|}\right)^2$$

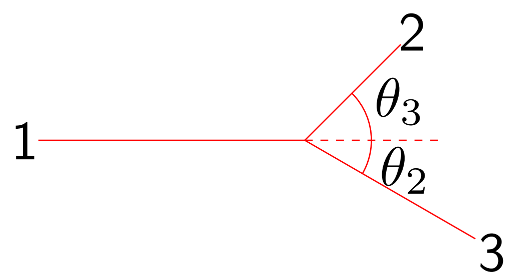
which has value $S = 1$ for an isotropic event and $S = 0$ for a linear event. Also $C - \mathrm{parameter}$

$$C - \mathrm{parameter} = \frac{3}{2} \frac{\sum_{i,j} |\underline{\mathsf{p}}_i||\underline{\mathsf{p}}_j| \sin^2 \theta_{ij}}{(\sum_i |\underline{\mathsf{p}}_i|)^2}$$

which is similar to spherocity.

Concentrate on Thrust as example.

Order $\alpha_S$ - three partons




It is not too difficult to see that $\underline{n}$ must lie along the axis of a particle.

$$T = max\left(\frac{x_1 + x_2\cos\theta_2 + x_3\cos\theta_3}{x_1 + x_2 + x_3}, \cdots\right)$$

$$T = max\{x_1, x_2, x_3\}$$

Leads to a constraint $\rightarrow T > 2/3$ at this order

From the previous calculation of the three parton rate

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F\frac{\alpha_S}{2\pi}\int dx_1\, dx_2\, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\delta(T - max\{x_1, x_2, x_3\})$$

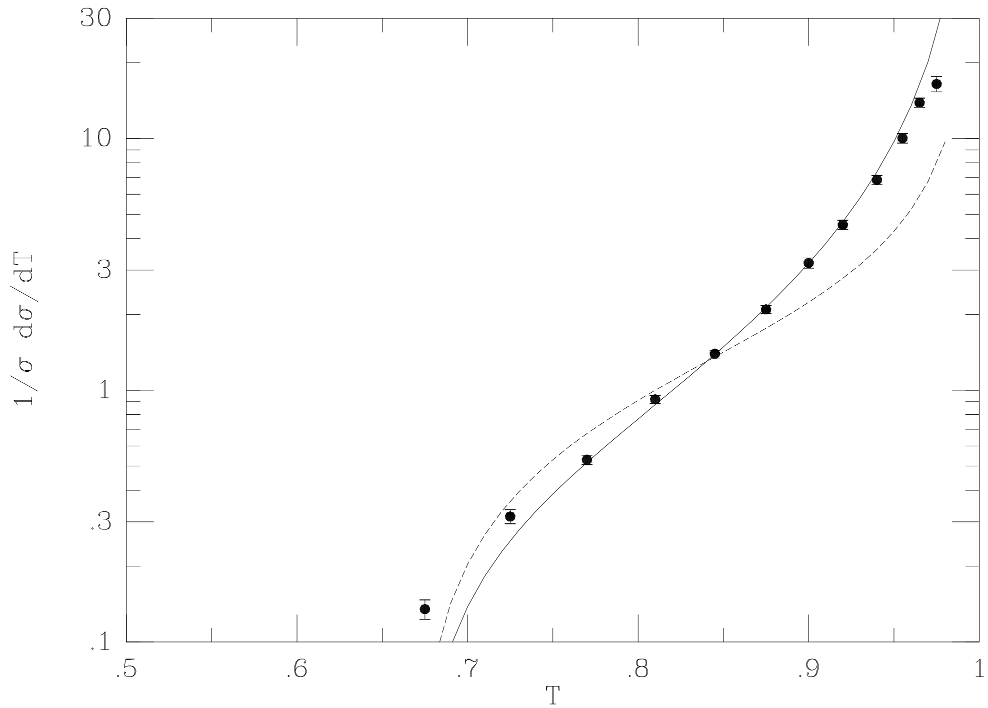
We obtain a cross-section which diverges in the limit $T \to 1$ tending to

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F\frac{\alpha_S}{2\pi}\left[\frac{4}{(1-T)}\log\left(\frac{1}{1-T}\right) - \frac{3}{1-T}\right]$$

This is due to soft and collinear singularities. The $\mathcal{O}(\alpha_S)$ virtual correction $\propto \delta(1-T)$ such that a finite total cross-section is obtained. Of course in the limit $T \to 1$ we should use a proper jet definition.

The figure shows the thrust distribution measured at DELPHI compared with the theory (solid line) and for scalar gluon (dashed line).

Deficiency at small $T$ due to kinematic bound for 3-parton process. Just starting to fail as $T \to 1$.



The $\mathcal{O}(\alpha_S^2)$ corrections are also known.

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{\alpha_S(\mu^2)}{2\pi}A_1(T) + \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2 [2\pi A_1(T)\beta_0 \log(\mu^2/s) + A_2(T)] + \mathcal{O}(\alpha_S^3)$$

where the first NLO term is determined from the renormalization group, i.e. from the independence of the total on $\mu^2$ and the known running of the coupling, i.e.

$$\frac{d}{d\ln\mu^2}\frac{1}{\sigma}\frac{d\sigma}{dT} = -\beta_0\left(\frac{\alpha_S^2(\mu^2)}{2\pi}\right)A_1(T) + \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2 [2\pi A_1(T)\beta_0] + \mathcal{O}(\alpha_S^3) = \mathcal{O}(\alpha_S^3),$$

and we see the independence of the cross-section on $\mu$ is higher order.

Comparison of experiment to theory good test of both QCD matrix elements and of value of $\alpha_S(M_Z^2)$.

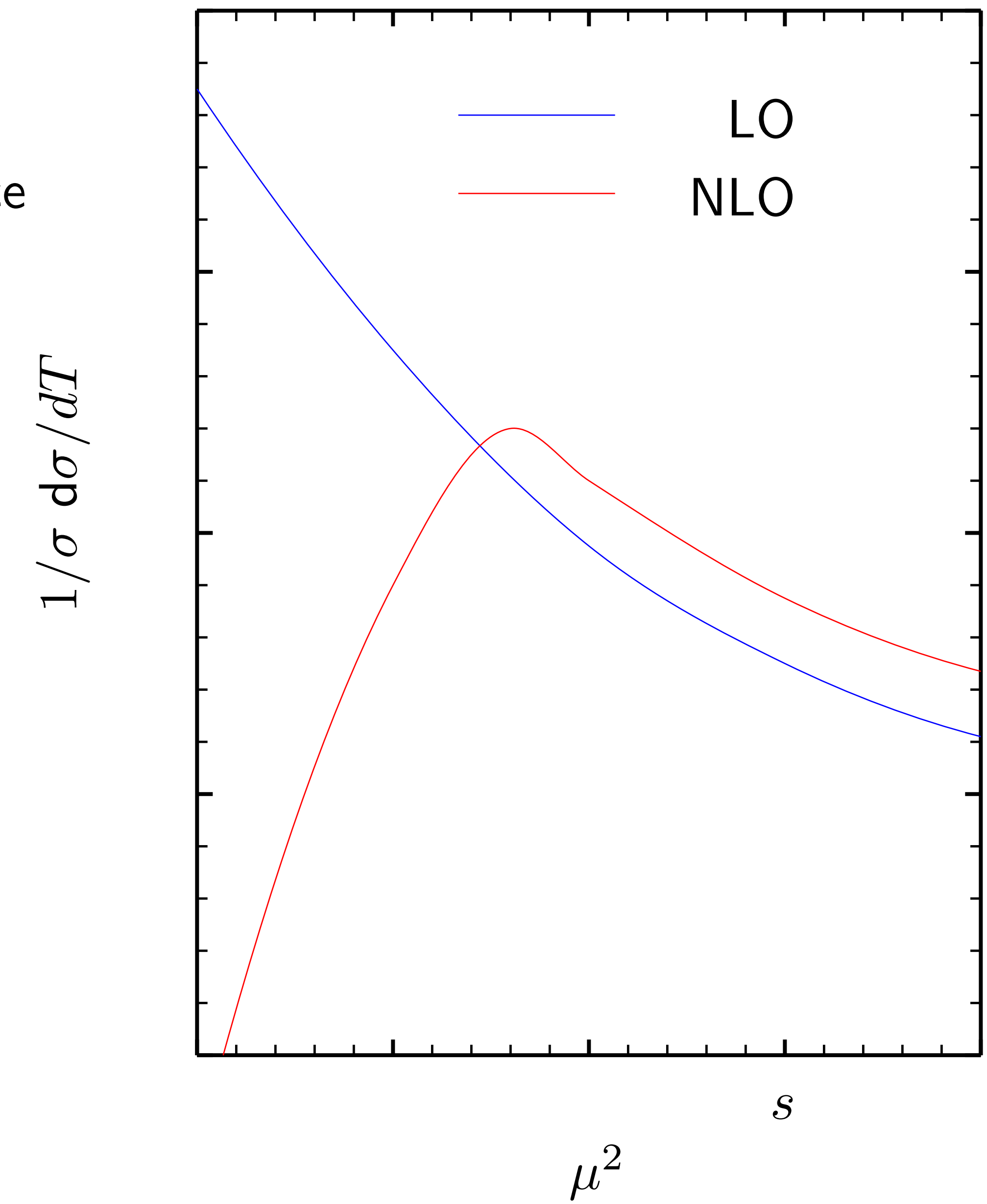
However, some problems.

$$T \to 1, \quad A_2(T) \to \frac{\log^3(1-T)}{(1-T)}$$

NLO works well except at lowest and highest $T$ (can become negative), **provided** $\mu^2$ chosen appropriately.

Large renormalization scale dependence at high (and lowest) $T$.

Due to increasing order of divergence in $T$ with increasing order in $\alpha_S$.



Useful to define an effective **two jet rate** using the thrust distribution.

$$R(\tau) \equiv \int_{1-\tau}^{1} dT \, \frac{1}{\sigma}\frac{d\sigma}{dT}$$

where $\sigma$ is now summed over all $n$-parton final states.

This is like the previous two-jet fraction, but with $\tau = 1 - T$ being the jet resolution $y$. As $\tau \to 0$

$$R(\tau) \to 1 - C_F \frac{\alpha_S}{2\pi} 2\log^2\tau + \left(C_F \frac{\alpha_S}{2\pi}\right)^2 2\log^4\tau$$

Extra $\log^2\tau$ at every order of $\alpha_S$.

Although $\alpha_S$ small $\alpha_S \log^2\tau$ large. Implies breakdown of perturbation theory.

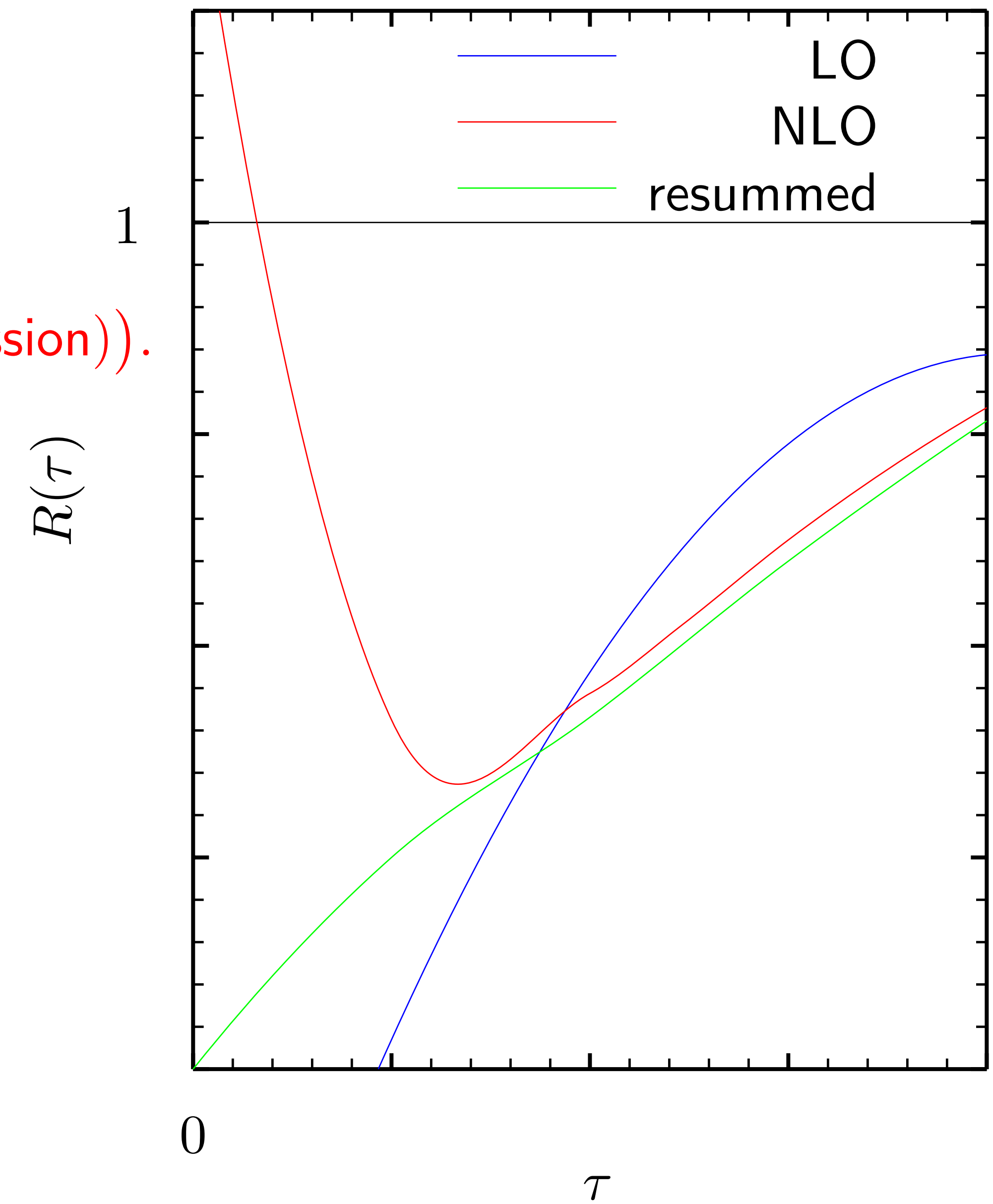
Fortunately terms in $\log^2\tau$ can be very neatly summed to all orders (resummed) $\to$ Sudakov form factor

$$R(\tau) \to \exp\left[-C_F \frac{\alpha_S}{2\pi} 2\log^2\tau\right]$$

which is finite (zero) as $\tau \to 0$.

Hence the probability to produce a $q\bar{q}$ pair with no accompanying gluon $= 0$. (Similar to QED.)

$P$(no emission) $= \exp\big(-P_{simple}$(emission)$\big)$.



## Higher order resummations.

Exponentiation removes unphysical behaviour, but still problems with perturbative convergence due to large $\log\tau$ terms and corresponding $\mu^2$-dependence.

By iterative means able to even resum exponent.

$$R(\tau) \to \exp\Big[-C_F\frac{\alpha_S}{2\pi}2\log^2\tau + a_2\alpha_S^2\log^3\tau + a_3\alpha_S^3\log^4\tau + \cdots$$

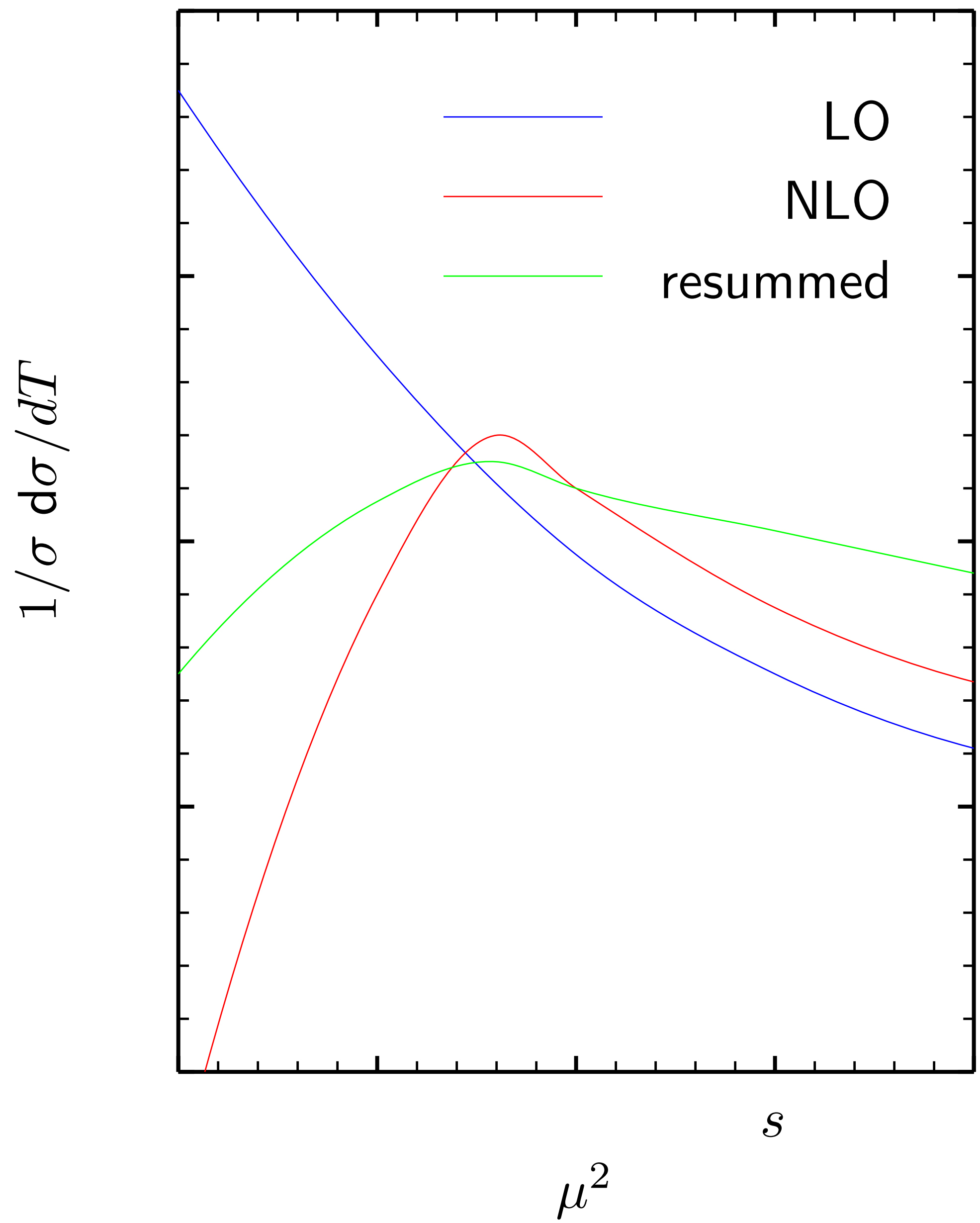
$$+C_F\frac{\alpha_S}{2\pi}3\log\tau + b_2\alpha_S^2\log^2\tau + b_3\alpha_S^2\log^4\tau + \cdots\Big]$$

where we have the notation

$\alpha_S^n\log^{n+1}\tau$ are leading logs

$\alpha_S^n\log^n\tau$ are next-to-leading logs

NLL resummation $\longrightarrow$ much improved scale dependence (and convergence).



**Warning**, order by order expansion in $\alpha_S$ not always enough in perturbative QCD. Exponentiation sometimes necessary for correct probabilistic interpretation. Possible for various processes.

Also - if scale dependence large usually a good reason.

In this case not clear if scale is $s$ or $\tau s$ or some other combination.

Resummation of large $\log \tau$ terms removes ambiguity, e.g.

$$\alpha_S(s\tau) = \alpha_S(s) - \beta_0 \alpha_S^2(s) \log \tau + \cdots .$$

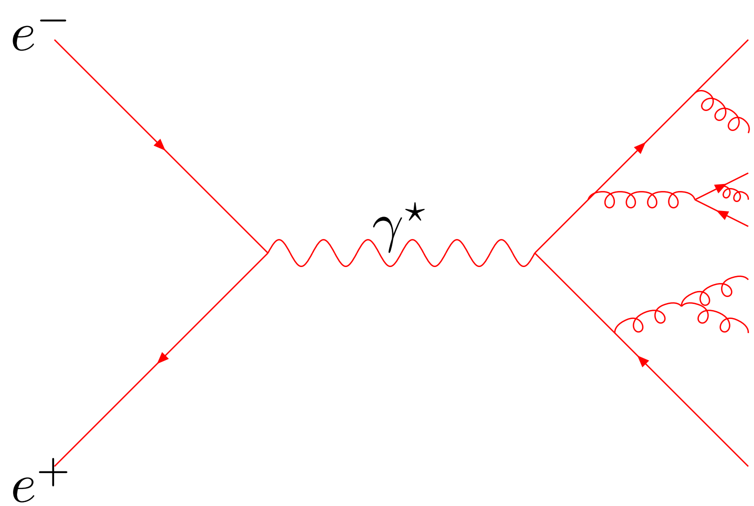
Similar procedure for other processes - C-parameter, jet rates ....

Renormalization scale variation ($s/4 \leq \mu^2 \leq 4s$) is definitely **NOT** a good estimate of theory error unless details of higher orders are understood and if necessary included.

## Parton Showers.

Very often it is extremely difficult to account for the enhancements in the perturbative expansion in such a neat analytic manner. However, we have seen that in a partonic final state Sudakov form factor $\longrightarrow$ probability of each parton being unaccompanied $= 0$.

Phase space fills with partons



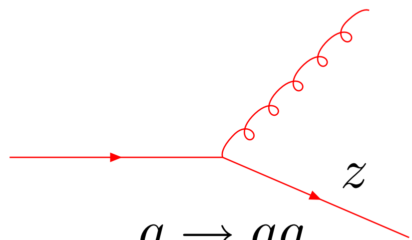

Dominated by soft and collinear partons.

Must calculate parton shower.

**Collinear limit.** In the limit that branching angle $\to 0$

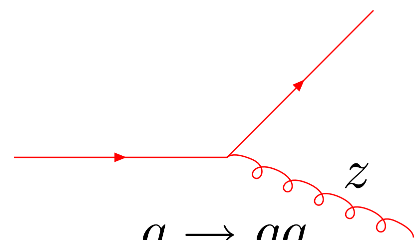
$$d\sigma = \sigma_0 \frac{\alpha_S}{2\pi} \frac{d\theta^2}{\theta^2} dz\, P(z)$$

$P(z)$ = splitting function.



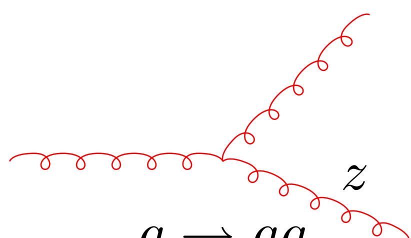

$q \longrightarrow qg$

$C_F \frac{1+z^2}{1-z}$



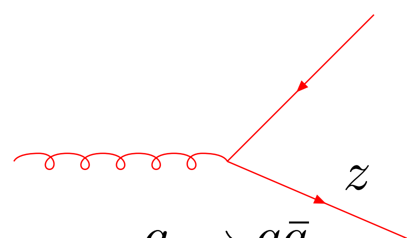

$q \longrightarrow gq$

$C_F \frac{1+(1-z)^2}{z}$




$g \longrightarrow gg$

$C_A \frac{z^4+1+(1-z)^4}{z(1-z)}$




$g \longrightarrow q\bar{q}$

$T_R(z^2 + (1-z)^2)$

Account for running coupling constant. Higher orders suggest that

$$\alpha_S \to \alpha_S(k_T^2).$$

$k_T^2$ is parton transverse momentum $k_T^2 = z(1-z)Q^2$ where $Q^2$ is initial parton virtuality. $\to$ enhancement for low $k_T$ partons.

However, need emitted parton to be resolvable, i.e. collinear parton pair indistinguishable from single parton.

Introduce resolution criterion $k_T > Q_0$ where $Q_0 \leq 1\mathrm{GeV}$.

$$\to z,\ (1-z) > Q_0^2/Q^2$$

Virtual corrections combined with unresolvable emissions $\to$ cancellation of divergences.

Unitarity: resolved $+$ unresolved $= 1$.

## Sudakov Form Factor.

Probability of emission between $q^2 + dq^2$ and $q^2$

$$d\mathcal{P} = \frac{\alpha_S(k_T^2)}{2\pi}\frac{dq^2}{q^2}\int_{Q_0^2/q^2}^{1-Q_0^2/q^2} dz P(z) \equiv \frac{dq^2}{q^2}\bar{P}(q^2).$$

Define probability of no emission between $Q^2$ and $q^2$ to be $\Delta(Q^2, q^2)$. Satisfies equation

$$\frac{d\Delta(Q^2,q^2)}{dq^2} = -\Delta(Q^2,q^2)\frac{d\mathcal{P}}{dq^2}$$

$$\Delta(Q^2,q^2) = \exp\left(-\int_{q^2}^{Q^2}\frac{dk^2}{k^2}\bar{P}(k^2)\right).$$

(Similar to radioactive decay. If decay constant $= \lambda$ probability of no decay before $t$ $= \exp - \int^t \lambda dt$.

$\Delta(Q^2, q^2) \equiv \Delta(Q^2)$ is the Sudakov form factor, i.e. probability for emitting no resolvable partons.

## Monte Carlo implementation.

Given parton with virtuality $Q^2 = t_1$ and momentum fraction $x_1$ what is $(t_2, x_2)$ after next branching?

Probability of evolving from $t_1$ down to $t_2$ without resolvable branching is $\Delta(t_1)/\Delta(t_2)$. If we have random number $\rho_1$ in the interval $[0,1]$ then $t_2$ determined by solving

$$\frac{\Delta(t_1)}{\Delta(t_2)} = \rho_1.$$

If $t_2 \leq t_0$ then no further branching takes place, else we find scale of next branching and can repeat.

Can reverse in order to evolve up (e.g. DIS).

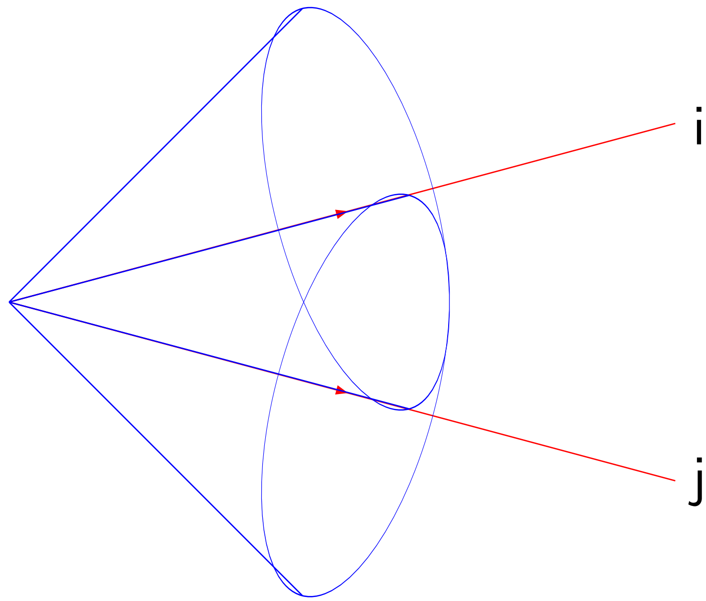
At $t_2$ also want momentum fraction $z = x_2/x_1$ for branching. Find this by using weight given by splitting function and solving

$$\int_{t_0/t}^{x_2/x_1} dz \frac{\alpha_S(k_T^2)}{2\pi} P(z) = \rho_2 \int_{t_0/t}^{1-t_0/t} dz \frac{\alpha_S(k_T^2)}{2\pi} P(z)$$

where $\rho_2$ is another random number.

Depends on various choices, particularly $Q_0^2$.

This accounts for collinear contributions, but must also take into account soft contributions. Turns out that including these leads to concept of angular ordering.




So parton i will only emit within a cone extending to parton j and *vice versa*.

Angular ordering gives basis for coherent parton branching which includes soft gluon enhancements to all orders.

Instead of virtuality $t$ use angular variable

$$\zeta = \frac{p_b \cdot p_c}{E_b E_c} \approx 1 - \cos\theta$$

as evolution variable for branching $a \to bc$ such that $\zeta_2 < \zeta_1$. Use angular cut-off $\zeta_0$.

Rapidity distribution of third jet in CDF events

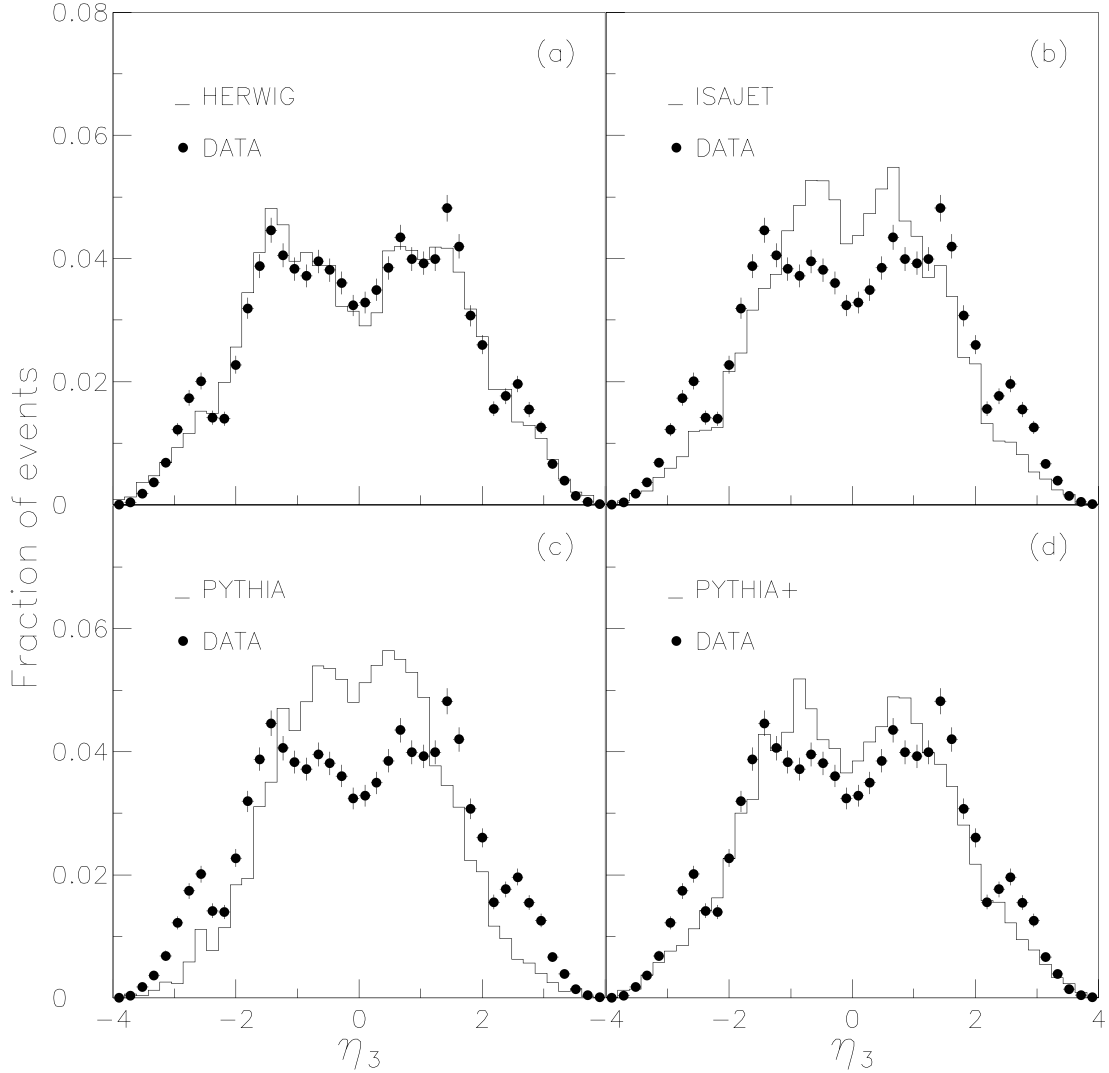
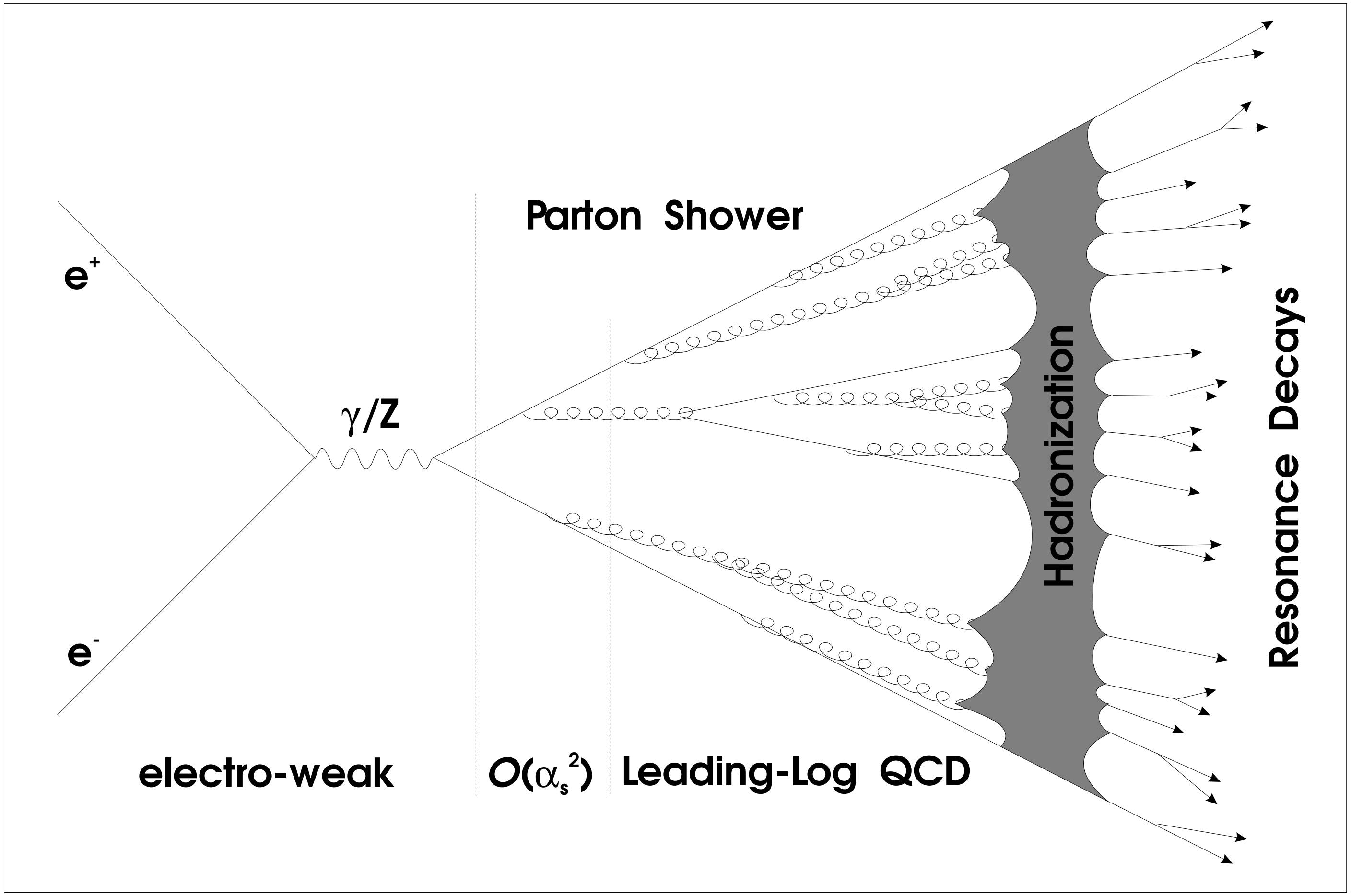


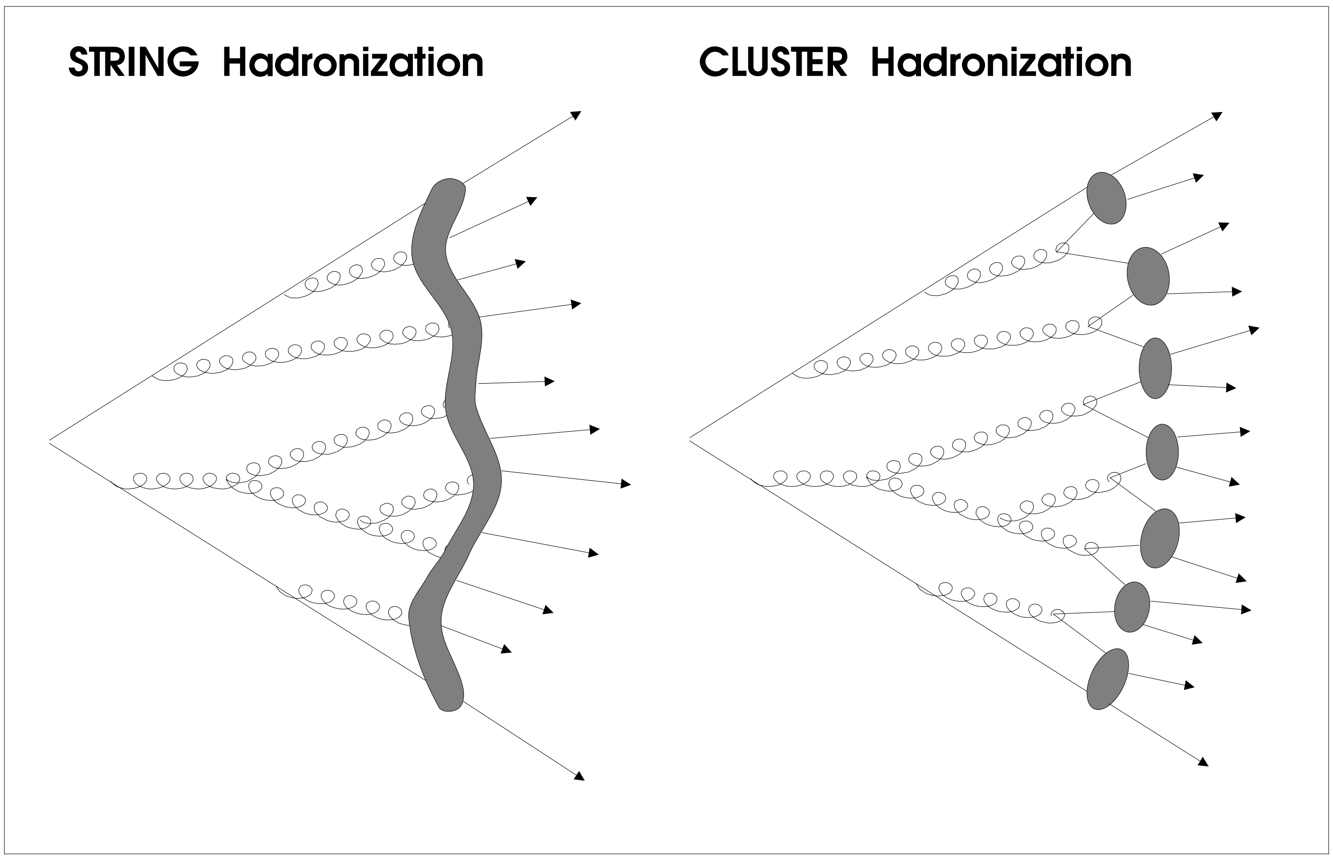

Figure 10: Observed $\eta_3$ distribution compared to the predictions of: (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

HERWIG has complete treatment of angular ordering, PHYTHIA+ partial.

Final step to simulate final state is hadronization.

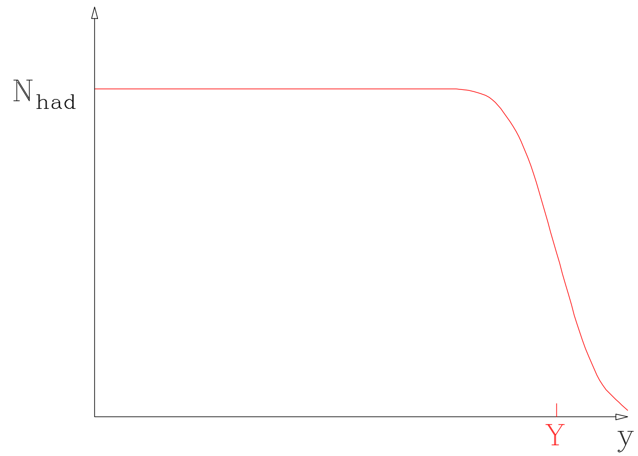


Main approaches (models) string hadronization and cluster hadronization. Not discussed here.

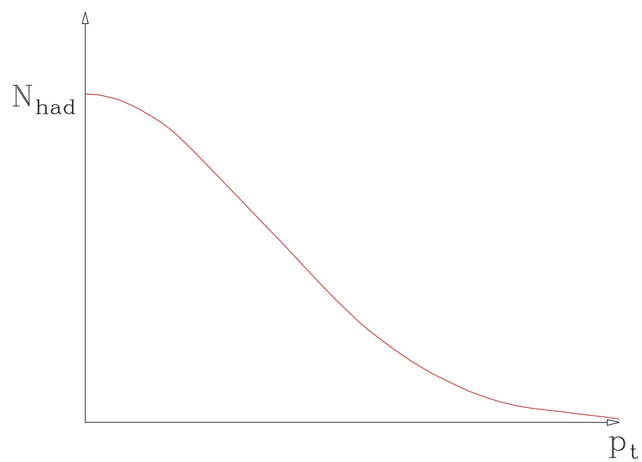


General properties. Defining rapidity $y$ by $y = \frac{1}{2}\log\frac{E+p_z}{E-p_z}$

two jets events have flat $y$ distribution




and $p_t$ distribution $\rho(p_t^2) \sim \exp(-p_t^2/2p_0^2)$




Can use this to estimate hadronization corrections.

Average transverse momentum

$$\lambda = \int d^2p_t \rho(p_t^2) p_t \sim 1\mathrm{GeV}.$$

Calculate jet energy and momentum

$$E = \int_0^Y dy \int d^2p_t \rho(p_t^2) p_t \cosh(y) = \lambda \sinh(Y).$$

$$p = \int_0^Y dy \int d^2p_t \rho(p_t^2) p_t \sinh(y) = \lambda(\cosh(Y) - 1).$$

$Y$ relatively large $\to p \sim E - \lambda$.

Consider thrust distribution for three jets.

$$T = \frac{p_1}{\sqrt{s}} = 1/2x_1 \quad \text{at parton level}$$

$$= \frac{E_1 - \lambda}{\sqrt{s}} = 1/2x_1 - \frac{\lambda}{\sqrt{s}}.$$

So hadronization correction of $\Delta T \sim 2\lambda/\sqrt{s}$, i.e. thrust shifted.

Typical corrections $\mathcal{O}(\lambda/\sqrt{s})$ (less inclusive) and $\mathcal{O}((\lambda^2/s))$ (more inclusive, e.g jet rates). Accounted for well by hadronization models in event generators.

Full machinery of final state QCD calculations works well, but requires many contributions.

Still refinements going on.