Answer FIVE questions

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

[Part marks]

1. (a) Give the definitions of arithmetic and geometric series. [4]

   (b) Assuming that the sum of squares of the natural numbers up to N can be written in the form [5]

$$S_N = \sum_{n=1}^{N} n^2 = aN^3 + bN^2 + cN \,,$$

   determine the constants a, b, and c, and explain whether this series can be classified as arithmetic or geometric.

   (c) Prove that [4]

$$\sum_{n=0}^{N-1} ar^n = \frac{a(1-r^N)}{1-r} \qquad\qquad (r \neq 1)\,.$$

   (d) A Fabry-Pérot interferometer consists of two parallel heavily silvered glass plates; light enters normally to the plates, and undergoes repeated reflections between them, with a small transmitted fraction emerging at each reflection. Find the intensity $|B|^2$ of the emerging wave, where [7]

$$B = A(1-R)\sum_{n=0}^{\infty} R^n \mathrm{e}^{\imath n\phi} \,,$$

   with A, R (the reflection coefficient < 1) and ϕ real constants.

2. (a) Write down the definition of the derivative of a function in terms of a limiting procedure. [4]

   (b) Starting from the definition of the derivative of a function, prove the product rule of differentiation. [6]

   (c) Explain the method of "integration by parts". [4]

   (d) Evaluate the following integral using two different methods: [6]

$$\int_{-\infty}^{+\infty} x^2 \mathrm{e}^{-ax^2}\, dx\,.$$

   You may use as given the following:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}\,.$$

3. (a) Find the real and imaginary parts of $\mathrm{e}^{\imath s}$, where $s = a + \imath b$. [3]

   (b) The damped oscillation of a vibrating block is given by $x = \mathrm{Re}(z)$, where [5]

$$z = \mathrm{e}^{(-0.2+0.5\imath)t}\,,$$

   in terms of the time t. Show that

$$x = \mathrm{e}^{-0.2t}\cos(0.5t)$$

   and determine the values of t where x is zero.

   (c) Find the velocity of the block (i) as dx/dt and (ii) as $\mathrm{Re}(dz/dt)$ and confirm that the answers are the same. [6]

   (d) For which values of t does the velocity get to a stationary value (either minimum or maximum) in each oscillation cycle? [6]

4. (a) The position vector of a particle moving on an ellipse in the $x-y$ plane is given as a function of time t by $\vec{r}=\hat{\imath}a\cos(\omega t)+\hat{\jmath}b\sin(\omega t)$, where a, b and ω are constants. Show that the particle's acceleration is always directed towards the origin of the coordinate system. [4]

(b) Show that the unit vectors in polar coordinates, $\hat{r}$ and $\hat{\theta}$, can be expressed in terms of the Cartesian unit vectors, $\hat{\imath}$ and $\hat{\jmath}$, as [4]

$$\hat{r}=\hat{\imath}\cos\theta+\hat{\jmath}\sin\theta \qquad \text{and} \qquad \hat{\theta}=-\hat{\imath}\sin\theta+\hat{\jmath}\cos\theta\,.$$

(c) Starting from the position vector in polar coordinates $\vec{r}=r\hat{r}$ show that the velocity in polar coordinates is given by [4]

$$\frac{d\vec{r}}{dt}=\frac{dr}{dt}\hat{r}+r\frac{d\theta}{dt}\hat{\theta}\,.$$

(d) The position vector of a moving particle is given in polar coordinates by $r=1/(\cos t)$, $\theta=t$, where t is time. Sketch the path of the particle for $0\leq t<\frac{1}{2}\pi$ and find its radial and angular components of acceleration as a function of time. [8]

5. (a) The equation $3y=z^3+3xz$ defines z implicitly as a function of x and y. Show that [7]

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{z^2-x}{(z^2+x)^3}\,.$$

(b) Determine the stationary point(s) of the function [7]

$$f(x,y)=xy(x^2+y^2-b)\,,$$

where b is a parameter.

(c) When $b>0$, one of the stationary points is $(x,y)=(\sqrt{b}/2,-\sqrt{b}/2)$. Determine whether this point is a minimum, maximum, or neither. [6]

6. (a) We would like to express an arbitrary function $f(x)$ in the form [5]

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n (x - x_0)^n \,.$$

Determine the constants a_n.

(b) Derive the MacLaurin series for e^x and $\cos x$, and, starting from the latter, show that [4]

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

(c) Prove that [4]

$$\mathrm{e}^{\imath\theta} = \cos\theta + \imath \sin\theta \,, \qquad (\cos\theta + \imath\sin\theta)^n = \cos(n\theta) + \imath\sin(n\theta) \,.$$

(d) Find the sum of the series [7]

$$1 + 2\cos\theta + \frac{2^2 \cos 2\theta}{2!} + \frac{2^3 \cos 3\theta}{3!} + \ldots$$

7. (a) Define the scalar product of two vectors $\vec{a}$ and $\vec{b}$ and from the definition obtain an expression for the scalar product in Cartesian coordinates. [5]

(b) Define the vector product of two vectors $\vec{a}$ and $\vec{b}$ and from the definition obtain an expression for the vector product in Cartesian coordinates. [5]

(c) Find the equation (in the form $ax + by + cz = d$) of the plane that is defined by the points $A : (-1, -2, 1)$, $B : (1, -2, 0)$ and $C : (-1, 0, 3)$. [5]

(d) Find the vector equation of a line that passes through the origin of the coordinate system and never crosses the above plane. [5]

END OF PAPER