

Projectile problem

A particle of mass m is fired with an initial speed v_0 at an initial angle θ to the horizontal Earth's surface. Neglecting any drag forces the equations of motion are well known:

$$m \frac{d^2 x}{dt^2} = 0; \quad m \frac{d^2 y}{dt^2} = -mg,$$

where x, y are the horizontal and vertical positions relative to the origin at time t . These differential equations have solutions for x and y as functions of t as

$$x(t) = v_0 t \cos \mathbf{q}; \quad y(t) = v_0 t \sin \mathbf{q} - \frac{1}{2} g t^2$$

giving a trajectory

$$y(x) = x \tan \mathbf{q} - \frac{g x^2}{2 v_0^2 \cos^2 \mathbf{q}}.$$

The range R is given by $R = \frac{2 v_0^2 \sin \mathbf{q} \cos \mathbf{q}}{g} = \frac{v_0^2 \sin 2\mathbf{q}}{g}$ and the time, T , to the highest point is $T = \frac{v_0 \sin \mathbf{q}}{g}$.

The present task is to simulate this motion in a spreadsheet. To make it a little more like the “real world” you will include in the simulation a drag force ($-\gamma m v$) that is proportional to the instantaneous speed v . The equations of motion then become

$$m \frac{d^2 x}{dt^2} = -mg v \cos \mathbf{q} = -mg \frac{dx}{dt},$$

$$m \frac{d^2 y}{dt^2} = -mg - mg v \sin \mathbf{q} = -mg - mg \frac{dy}{dt},$$

where γ is a drag coefficient per unit mass.

The earlier exercise on derivatives showed how to get a numerical approximation to the first derivative. For y tabulated at intervals of δt we have

$$\left(\frac{dy}{dt} \right) \approx \frac{y_{n+1} - y_n}{\delta t} \quad \text{or} \quad \left(\frac{dy}{dt} \right) \approx \frac{y_{n+1} - y_{n-1}}{2\delta t}.$$

The second form was found to converge more quickly so we will use it here.

The second derivative can be approximated by

$$\frac{d^2 y}{dt^2} \approx \frac{\left(\frac{dy}{dt} \right)_n - \left(\frac{dy}{dt} \right)_{n-1}}{\delta t}$$

$$\frac{d^2 y}{dt^2} \approx \frac{\left(\frac{y_{n+1} - y_n}{\delta t} \right) - \left(\frac{y_n - y_{n-1}}{\delta t} \right)}{\delta t} = \frac{y_{n+1} - 2y_n + y_{n-1}}{(\delta t)^2}.$$

Thus the equations of motion may be approximated by

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{(dt)^2} = -g \left(\frac{x_{n+1} - x_{n-1}}{2dt} \right),$$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{(dt)^2} = -g - g \left(\frac{y_{n+1} - y_{n-1}}{2dt} \right).$$

Thus we obtain

$$x_{n+1} = \frac{2x_n - x_{n-1}(1 - gdt/2)}{(1 + gdt/2)},$$

$$y_{n+1} = \frac{2y_n - y_{n-1}(1 - gdt/2) - g(dt)^2}{(1 + gdt/2)}.$$

These equations can be used to generate the positions (x_{n+1}, y_{n+1}) from (x_n, y_n) and (x_{n-1}, y_{n-1}) . Thus we can obtain positions (x_2, y_2) and onwards. However (x_1, y_1) cannot be obtained this way as it requires a position (x_{-1}, y_{-1}) before the start! The effective initial accelerations in the x and y directions are $-gv_x(0)$ and $-g - gv_y(0)$. Thus using the equation “ $s = ut + \frac{1}{2}ft^2$ ” we can approximate

$$x_1 = x_0 + v_x(0)dt - \left(\frac{1}{2}gv_x(0) \right)(dt)^2,$$

$$y_1 = y_0 + v_y(0)dt - \frac{1}{2}(g + gv_y(0))(dt)^2.$$

The task is to model the motion of the particle and to investigate some properties of it for some choice of the drag coefficient.

- Construct a spreadsheet which contains labelled cells for the quantities $v_0, \mathbf{q}, \mathbf{g}, g$ and δt . You may find it convenient to calculate the subsidiary quantities $v_x(0), v_y(0), (dt)^2, (1 + gdt/2), (1 - gdt/2)$ to avoid repeated calculation of these quantities for each position. If these are positioned near the top-left of the spreadsheet, then below them under headings n, t, x and y compute the position of the particle for integer values of n from 0 to 1500. Take an initial speed $v_0 = 100 \text{ ms}^{-1}$, $\delta t = 0.01 \text{ s}$ and initially set $\gamma = 0 \text{ s}^{-1}$, i.e. no drag. For simplicity take $g = 10 \text{ ms}^{-2}$.
- Plot y against x .
- The range (when $y = 0$) and the time of flight ($= 2T$, for this range) should agree with the values calculable from the formulae above. Do they? If not try and find the source of your error. Does the plot look like a parabola?
- If all is correct then set $\gamma = 0.4 \text{ s}^{-1}$ and see how the plot of the trajectory changes.
- For this value of γ construct a table of range R against the angle of firing θ .
- Plot these ranges against θ and determine the angle to achieve maximum range. (It is unlikely that you will have a data point at the maximum, so try fitting a polynomial trend line to the data points and deduce the maximum from the equation of this line).

Save your spreadsheet as *username-projectile.xls*.